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A CUBIC IDENTITY AND ITS CONSEQUENCES

Several authors considered the problem of representation of integers as the sum of four cubes. It was shown that many numbers have infinitely many such representations (cf. [2], chapter 21).

We show below that there are infinitely many positive integers n which have representations

$$(1)\qquad \begin{cases} n = x_1^3 + x_2^3 + x_3^3 + x_4^3, \\ (x_1 + x_2)(x_1 + x_3)(x_1 + x_4) \neq 0, \\ x_1, x_2, x_3, x_4 \in Z, \end{cases}$$

in which x_1 assumes all integer values with at most two exceptions. We require $(x_1+x_2)(x_1+x_3)(x_1+x_4) \neq 0$ to exclude the trivial representation of n being the sum of two cubes: $n = a^3 + b^3$ in the form $n = k^3 + (-k)^3 + a^3 + b^3$. For the two exceptions the product is zero.

The result follows from the identity

$$(2)\qquad 2(a^6-1) = k^3+(2a^2-k)^3 + (a^3-ak-1)^3 + (ak-a^3-1)^3.$$

In fact, for $n = 2(a^6-1)$, $a > 1$, the numbers $x_1 = k$, $x_2 = 2a^2-k$, $x_3 = a^3-ak-1$, $x_4 = ak-a^3-1$ satisfy (1), unless $k = a^2 \pm a + 1$.

Further, we have

$$(3)\qquad 2(a^6-1) = (a^2-a-1)^3 + (a^2+a-1)^3,$$

which follows from (2) for $k = a^2 + a + 1$. The identity (3) with $a = 2^m$ was given by R.Niewiadomski [3].

From (2) and (3) we obtain

$$(4)\qquad k^v + (2a^2 - k)^v + (a^3 - ak - 1)^v = (a^2 - a - 1)^v + (a^2 + a - 1)^v + (a^3 - ak + 1)^v,$$

which holds not only for $v = 3$, but also for $v = 1$. We notice that (2) may be written in the form

$$(a^2)^v + (a^2)^v + (k - 2a^2)^v + (ak - a^3 + 1)^v = 1^v + 1^v + k^v + (ak - a^3 - 1)^v \qquad (v = 1,3).$$

On substituting in (4) $a = \frac{p}{q}$, $k = \frac{r}{s}$ we obtain the identity

$$(q^3r)^v + (2p^2qs - q^3r)^v + (p^3s - pq^2r - q^3s)^v = (p^2qs - pq^2s - q^3s)^v + (p^2qs + pq^2s - q^3s)^v + (p^3s - pq^2r + q^3s)^v \qquad (v = 1,3).$$

Similarly, substituting in (2) $a = \frac{x}{y}$ and $k = \frac{z}{t}$ we obtain the identity

$$2(x^2yt)^v + (y^3z)^v + (xy^2z + x^3t + y^3t)^v = 2(y^3t)^v + (y^3z + 2x^2yt)^v + (xy^2z + x^3t - y^3t)^v \qquad (v = 1,3).$$

Putting in (4) $k = \frac{a^3-1}{a}$ we obtain

$$(a^3 - 1)^v + (a^3 + 1)^v = (a^3 - a^2 - a)^v + (a^3 + a^2 - a)^v + (2a)^v,$$

whence

$$(m^3 - n^3)^v + (m^3 + n^3)^v = (m^3 - m^2n - mn^2)^v + (m^3 + m^2n - mn^2)^v + (2mn^2)^v \qquad (v = 1,2).$$

From the last identity we deduce (cf. [1], Satz V, p.22) that

$$(m^3 - n^3 + u)^w + (m^3 + n^3 + u)^w + (u - m^3 + m^2n + mn^2)^w + (u - m^3 - m^2n + mn^2)^w + (u - 2mn^2)^2 = (n^3 - m^3 + u)^w + (u - m^3 - n^3)^w + (u + m^3 - m^2n - mn^2)^w + (u + m^3 + m^2n - mn^2)^w + (u + 2mn^2)^w \qquad (w = 1,2,3,4).$$

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