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ON THE DEPENDENCE OF TEMPERATURE AND DENSITY ON THE VELOCITY VECTOR FIELD IN THE MOTION OF VISCID GAS

1. Notation

Let (M, g) be an n-dimensional Riemannian manifold with boundary or not. By ϱ we denote a function $R \times M \longrightarrow R$ which is everywhere positive. This function describes the density of gas at the time t. A time dependent velocity vector field will be denoted by u, and $u_t := u(t,\cdot) \in \Gamma(TM)$ is the section of tangent bundle TM which describes the velocity of each point of gas at the time t. In the case when the boundary $\partial M \neq \emptyset$ we need u to be tangent to the boundary, i.e. $u_t(x) = u(t,x) \in T\partial M$ for any $x \in \partial M$. The distribution of temperature we denote by T $(T: R \times M \longrightarrow R)$. The pressure of gas is a function $p : R \times M \longrightarrow R$.

The presented above objects form the full couple of time dependent physical quantities characterising the ideal gas and its evolution. But they do not give full characterization of physical properties of gas such as viscosity, heat conduction and thermodynamical behaviour. We need several constants to complete the description of ideal gas. First of them it is the universal gas constant R (which is the quotient of Boltzman's constant k and the molecular mass m_1, $R := k/m_1$). By c_v we denote the specific heat at constant volume. This constant describes the heat required to rise the temperature of a unit mass of gas by one unit at constant volume, in one unit of time. There is a connection between R and c_v, namely: $c_v = \frac{n_1}{2} R$, where n_1 denotes the degrees of freedom of molecule of gas (so $\frac{2c_v}{R}$ is an integer). The constant $\varkappa$ denotes the conductivity. The constants μ and λ are the dynamical and the kinematic viscosities. For a monoatomic gas $\lambda = -\frac{2}{3}\mu$. The approximate equality holds for many kinds of gases.

2. The system of evolution equations

The equations of motion for viscid fluids are derived from physical conservation laws in several monographies. We refer to Shih-I Pai [5] and Thomas Hughes, Jerrold Marsden [2], where also fundamental results and many topics of actual interest are presented.

The full system consists of the continuity equation (C.E.), the dynamical equation (D.) - here it is the Navier-Stokes equation, the energy equation (E.) and the state equation (S.). One need also to add to this system the thermodynamical equations, which allow to express the internal energy E (which appears in the energy equation) by ϱ and T. We make use of the fact that for the ideal gas there exists a simple relation between E and other thermodynamical variables, namely: $E = c_v T$, and we obtain the following system

(C.E.) $$\frac{D\varrho}{Dt} + \varrho \operatorname{div} u = 0$$

(D.) $$\varrho \frac{Du}{Dt} = f - \operatorname{grad} p + \lambda \operatorname{grad}(\operatorname{div} u) + \mu \operatorname{div} U$$

(E.) $$\varrho\left(c_v \frac{DT}{Dt} + p\,\frac{D(1/\varrho)}{Dt}\right) = \frac{\partial Q}{\partial t} + \varkappa \Delta T + \lambda(\operatorname{div} u)^2 + \frac{1}{2}\mu U^{ij}U_{ij}$$

(S.) $$p = R\varrho T.$$

In the above system U denotes the deformation tensor (the symmetric part of the tensor field $\nabla\tilde{u}$, where ∇ is the Levi-Civitta connection defined by the Riemannian metric g and $\tilde{u}$ is a time-dependent 1-form on M corresponding u via the metric). In local coordinates

$$\tilde{u}_j = g_{ij}u^i, \qquad U_{ij} = \frac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i})$$

$$U^{ij} = g^{il}g^{jm}U_{lm}, \quad (\operatorname{div} U)^i = U^{ij}{}_{,j}.$$

The differential operators appearing on the right hand side of equations (D.) and (E.) do not involve time derivative, i.e.

$$(\operatorname{div} u)(t,x) = (\operatorname{div} u_t)(x),$$

$$(\Delta T)(t,x) = (\Delta T_t)(x), \quad T_t(x) := T(t,x),$$

and so on.

The time dependent vector field f in (D.) is given. It has an interpretation as a field of external forces acting on the points of fluid. The function Q in (E.) is also a given function, and $\frac{\partial Q}{\partial t}$ means an external heating of internal parts of gas (e.g. by a radiation).

The unknown velocity vector field u appears in the system of evolution equations not only in visible places, but is also hidden in the so called material derivative $\frac{D}{Dt}$

$$\left.\frac{D}{Dt}\right|_{(\tau,x)} := \left.\frac{\partial}{\partial t}\right|_{(\tau,x)} + \nabla_{u_\tau(x)}.$$

It follows from the physical considerations, that if we add to the system of evolution equations the initial data: $u_0 = u(0,\cdot)$, $\varrho_0 = \varrho(0,\cdot)$ and $T_0 = T(0,\cdot)$ (sufficiently smooth) then the Cauchy problem should possess a unique solution (in the case $\partial M \neq \emptyset$ the boundary condition $u_t(x) = 0$ for all t and for any $x \in \partial M$ is required). Of course, "should possess" one have to treat rather as a wishful thinking than as a rigorous mathematical statement. Nowadays the theory of nonlinear partial differential equations seems to be too weak to manage with this Cauchy (or mixed) problem.

3. Some remarks on solving the system

First note that using (S.) one can eliminate the pressure p in equations (D.) and (E.). Thus we obtain the following system

$$(\mathrm{C.E.}) \qquad \frac{D\varrho}{Dt} + \varrho \operatorname{div} u = 0$$

$$(\mathrm{D.}') \quad \varrho \frac{Du}{Dt} = f - R\varrho \operatorname{grad} T - RT \operatorname{grad}\varrho + \lambda \operatorname{grad} \operatorname{div} u + \mu \operatorname{div} U$$

$$(\mathrm{E.}') \quad c_v \varrho \frac{DT}{Dt} - RT\frac{D\varrho}{Dt} = \frac{\partial Q}{\partial t} + \varkappa \Delta T + \lambda(\operatorname{div} u)^2 + \frac{1}{2}\mu U^{ij}U_{ij}$$

with unknown functions ϱ and T and unknown vector field u.

For this system the general theory of quasilinear equations of evolution, which was created by T.Kato (see [4]), is not applicable. The main obstructions here are the products of first order derivatives of u, which appear in (E.$'$).

Now we calculate $\frac{D\varrho}{Dt}$ from (C.E.) and substitute it in (E.$'$). Thus we obtain the following equation:

$$(\mathrm{E.}'') \qquad \frac{\partial T}{\partial t} - \frac{\varkappa}{\varrho c_v}\Delta T + \nabla_u T + \left(\frac{R}{c_v}\operatorname{div} u\right)T =$$

$$= \frac{1}{c_v\varrho}\left(\lambda(\operatorname{div} u)^2 + \frac{1}{2}\mu U^{ij}U_{ij} + \frac{\partial Q}{\partial t}\right).$$

Let us consider the case when a part of solution (u,T,ϱ) of the system ((C.E.), (D.$'$), (E.$''$)) is known, namely: we first assume that u is given (e.g. from experimental measurements). Then we can determine ϱ from the continuity equation and the initial condition $\varrho_0 = \varrho(0,\cdot)$ (see [6] for details and formulas). Hence the problem is reduced and we can regard the equation (E.$''$) as an equation with one unknown function T - all coefficients and the right hand side become known. This equation is linear and parabolic. The theory of such equations is very deep and strong nowadays. For instance, the results of L.Hörmander [1] and S.Kaplan [3] are immediately applicable to the considered equation. But we will not follow further this way, since it is possible to obtain the simpler equation for T from our system.

The structure of the system of evolution equations for ideal gas has several remarkable properties. Initially, we would like to point out the behaviour of ϱ. It appears that if T and u are a known part of solution, then we need not integrate (C.E.) to obtain the density ϱ. One can express ϱ by the 2-jet of T and 1-jet of u. Now we state precisely this result.

T h e o r e m 1. Let $u: R\times M \longrightarrow TM$ be a given time dependent vector field of class C^2 and let $T: R\times M \longrightarrow R$ be a given function of class C^3. We assume that there exists a positive function $\varrho: R\times M \longrightarrow R$ such that (u,T,ϱ) is a solution of the system ((E.E.), (D.'), (E.')). Then

$$\varrho = \frac{\frac{\partial Q}{\partial t} + \varkappa\Delta T + \lambda(\operatorname{div} u)^2 + \frac{1}{2}\mu U^{ij}U_{ij}}{c_v \frac{DT}{Dt} + RT \operatorname{div} u} .$$

P r o o f . The result follows from the previus considerations and ϱ is calculated from (E.'').

One can easily obtain a relation between p, T and u as a consequence of Theorem 1 and (S.).

C o r o l l a r y 1. The pressure p algebraically depends on the second jet of T and the first jet of u.

C o r o l l a r y 2. The system of evolution for ideal gas can be reduced to equations containing u and T only.

This reduction we obtain by substituting ϱ in (C.E.) and (D.') by the right hand side of formula in Theorem 1. Then we obtain the equations in which third order derivatives of T and second derivatives of u appear. In these equations the derivative $\frac{\partial^2 T}{\partial t^2}$ also appears. The obtained system is rather complicated one, so there is no reason for writting it down here. The initial data for this system are: T_o, u_o and

$$\frac{\partial T}{\partial t}(0,\cdot) = \frac{1}{c_v\varrho_o}\left(\frac{\partial Q}{\partial t} + \varkappa\Delta T_o + \lambda(\operatorname{div} u_o)^2 + \frac{1}{2}\mu \overset{o}{U}{}^{ij}\overset{o}{U}_{ij}\right) + - \nabla_{u_o}T_o - \frac{R}{c_v} T_o \operatorname{div} u_o ,$$

where $U := U(\dot{0},\cdot)$ is the deformation tensor calculated by differentiating u_0 (the symmetric part of $\nabla\tilde{u}_0$).

In the case when $M = R^n$ or M is a region in R^n, a vector field u is a family of n real valued functions of $n+1$ variables. The system ((C.E.), (D'.), (E''.)) consists of $n+2$ equations for $n+2$ unknown functions. The reduced system consists of $n+1$ equations for $n+1$ unknown functions.

The formula given in Theorem 1 is rather of some practical than a theoretical importance. It allows to determine the density of viscous ideal gas from measurements of velocity and temperature.

4. The dependence of T on u

Now we assume that u is a vector field of class C^3 such that there exists ϱ and T satisfying together with u the system ((C.E.), (D.), (E.)). This assumption holds throughout this chapter. Our purpose here is to give the formula which expresses T by u and ϱ. If u and v are vector fields on M then $g(u,v)$ is a function. To simplify the notation we will write $(u,v) := g(u,v)$. For instance:

$$\nabla_u T = (u, \operatorname{grad} T).$$

Let

$$q := \frac{1}{c_v\varrho}\left(\frac{\partial Q}{\partial t} + \lambda(\operatorname{div} u)^2 + \frac{1}{2}\mu U^{ij}U_{ij}\right),$$

$$P_1 := \varrho^{-2}\left(f - \varrho\frac{Du}{Dt} + \lambda \operatorname{grad}\operatorname{div} u + \mu \operatorname{div} U\right),$$

$$P_2 := \varrho^{-2}(f - \varrho\nabla_u u + \lambda \operatorname{grad}\operatorname{div} u + \mu \operatorname{div} U).$$

We point out that under our assumption, the function q and the vector fields P_1, P_2 may be regarded as the known quantities (since u and ϱ_0 uniquely determine ϱ).

T h e o r e m 2. Let (u,T,ϱ) be a solution of the system ((C.E.), (D$'$.), (E$''$.)) such that u is of class C^3. Then T satisfies the following equation

$$\frac{\partial T}{\partial t} + aT = b,$$

where the functions a and b are given by the following formulae

$$a = \frac{R}{c_v} \operatorname{div} u - \frac{\varkappa}{c_v} \Delta \frac{1}{\varrho} - (u, \operatorname{grad} \ln \varrho)$$

$$b = q + \frac{\varkappa}{c_v R} \operatorname{div} P_1 - \frac{\varrho}{R} (u, P_1).$$

P r o o f . We first calculate grad T from the equation (D$'$.)

$$\operatorname{grad} T = \frac{\varrho}{R} P_1 - T \operatorname{grad} \ln \varrho .$$

We put grad T into (E$''$.)

$$\frac{\partial T}{\partial t} - \frac{\varkappa}{c_v \varrho} \operatorname{div} \frac{\varrho}{R} P_1 + \frac{\varkappa}{c_v \varrho} \operatorname{div}[T \operatorname{grad} \ln \varrho] + \left(\frac{R}{c_v} \operatorname{div} u\right) T +$$

$$+ (u, \frac{\varrho}{R} P_1 - T \operatorname{grad} \ln \varrho) = q.$$

Now we calculate the second divergence on the left hand side

$$\frac{\partial T}{\partial t} + \frac{\varkappa}{c_v \varrho} (\operatorname{grad} T, \operatorname{grad} \ln \varrho) + T\left[\frac{\varkappa}{c_v \varrho} \Delta \ln \varrho - (u, \operatorname{grad} \ln \varrho) +\right.$$

$$\left. + \frac{R}{c_v} \operatorname{div} u\right] = q + \frac{\varkappa}{c_v \varrho} \operatorname{div} \frac{\varrho}{R} P_1 - \frac{\varrho}{R} (u, P_1).$$

We substitute grad T in the above equation once more

$$\frac{\partial T}{\partial t} + T \left[\frac{\varkappa}{c_v \varrho} \Delta \ln \varrho - (u, \operatorname{grad} \ln \varrho) - \frac{\varkappa}{c_v \varrho^3} (\operatorname{grad} \varrho, \operatorname{grad} \varrho) +\right.$$

$$\left. + \frac{R}{c_v} \operatorname{div} u\right] = q + \frac{\varkappa}{c_v \varrho} \operatorname{div} \frac{\varrho}{R} P_1 - \frac{\varrho}{R} (u, P_1) - \frac{\varkappa}{c_v R \varrho} (P_1, \operatorname{grad} \varrho).$$

The obtained above equation has the form given in statement. It is not hard to verify the formulae for a and b by standard calculus. It concludes the proof.

From the continuity equation it follows that in our case the density function ϱ is of class C^2. Hence the functions a and b are continuous. The obtained first order partial differential equation on T is elementary and can be pretty easy integrated:

$$T(t,x) = \left\{ T_0(x) + \int_0^t b(s,x) \exp \int_0^s a(\tau,x)d\tau ds \right\} \exp - \int_0^t a(s,x)ds.$$

The above formula allows to eliminate T form the equation (D_1'). However this formula is not good enough for reducting the system, since the derivative $\frac{\partial u}{\partial t}$ is involved in the coefficient b (recall that P_1 includes $\frac{\partial u}{\partial t}$). But the formula on T can be **converted** in such way that the dependence on $\frac{\partial u}{\partial t}$ will disappear.

Now let

$$b_i := \frac{\varkappa}{c_v R} \operatorname{div} P_i - \frac{\varrho}{R} (u, P_i), \quad i = 1,2.$$

We known that P_2 do not include $\frac{\partial u}{\partial t}$ and the relation between P_1 and P_2 is as follows

$$P_1 = P_2 - \varrho^{-1} \frac{\partial u}{\partial t} .$$

Hence

$$b = q+b_1 = q+b_2 - \frac{\varkappa}{c_v R} \operatorname{div} \varrho^{-1} \frac{\partial u}{\partial t} + \frac{1}{R} \left(u, \frac{\partial u}{\partial t}\right).$$

Thus

$$b = q+b_2 - \frac{\varkappa}{c_v R} \operatorname{div} \varrho^{-2} \frac{\partial \varrho}{\partial t} u - \frac{\partial}{\partial t}\left\{ \frac{\varkappa}{c_v R} \operatorname{div} \varrho^{-1} u - \frac{1}{2R} (u,u)\right\}$$

The continuity equation allows to eliminate $\frac{\partial \varrho}{\partial t}$ from the above equality

$$b = q + b_2 + \frac{\varkappa}{c_v R}\,\mathrm{div}\,\frac{\mathrm{div}\,\varrho u}{\varrho^2}\,u - \frac{\partial}{\partial t}\left\{\frac{\varkappa}{c_v R}\,\mathrm{div}\,\frac{u}{\varrho} - \frac{1}{2R}(u,u)\right\}.$$

Now we put the right hand side of the above equality under the integral in the formula for T and next we integrate the last term by parts. It leads to the following result.

T h e o r e m 3. Under the assumptions of Theorem 2 we have the following relation between T and u

$$T = T_1 + T_2,$$

where

$$T_1 = \frac{1}{2R}(u,u) - \frac{\varkappa}{c_v R}\,\mathrm{div}\,\frac{u}{\varrho}$$

and

$$T_2(t,x) = \left\{T_0(x) - \frac{1}{2R}(u_0,u_0) + \frac{\varkappa}{c_v R}\,\mathrm{div}\,\frac{u_0}{\varrho_0} + \right.$$

$$\left. + \int_0^t \left(q + b_2 + \frac{\varkappa}{c_v R}\,\mathrm{div}\,\frac{\mathrm{div}\,\varrho u}{\varrho^2}\,u - aT_1\right)(s,x)\exp\int_0^s a(\tau,x)\,d\tau\,ds\right\}\times$$

$$\times \exp - \int_0^t a(s,x)\,ds.$$

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