DEMONSTRATIO MATHEMATICA
Vol. XVI No 2 1983

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DIFFERENTIAL GROUPS OF CLASS $\mathcal{D}_0$ AND STANDARD CHARTS

Introduction

In this paper we consider differential groups i.e. objects which are simultaneously differential spaces and groups with the symmetry and the group operation smooth with respect to the structure of the differential space. For differential groups, which are differential spaces of class $\mathcal{D}_0$ (see [8]), we construct so called standard charts, which are very usefull in the investigation of this kind of differential groups (see [3], [4]). In Section 3 we give the following example of the application of standard charts (Th. 2.2): we prove that for every $\mathcal{D}_0$-group G the group operations can be continously extend to the group operations on the completion of G.

For all basic definitions we refer to [5], [6] and [7]. A differential space will be denoted by (M,C), its tangent space at $p \in M$ by $T_p(M,C)$ and its topology by τ_C. If $f:(M,C) \longrightarrow (N,D)$ is a smooth mapping, then its tangent mapping at $p \in M$ will be denoted by f_{*p}.

1. Differential groups and tangent mappings to the symmetry and the group operation

Assume G to be a group with the unit e.

D e f i n i t i o n 1.1. The differential structure C defined on the set of elements of G will be called a differential group structure on G if the mapping

(1) $$G \times G \in (g,h) \longmapsto Q(g,h) = gh^{-1} \in G$$

is smooth with respect to differential structures $C \times C$ and C respectively, where $C \times C$ is the canonical differential structure defined on the Cartesian product $G \times G$ (see [5]).

If C is a differential group structure on the group G, then the pair (G,C) is called a differential group.

Similary as in the Lie group theory we can prove the following theorem.

T h e o r e m 1.1. If (G,C) is a differential group, then the symmetry $inv(g) = g^{-1}$, a right translation $R_a(g) = ga$ and a left translation $L_a(g) = ag$ are diffeomorphisms of the differential space (G,C). Moreover the group operation

$$G \times G \ni (g,h) \longmapsto A(g,h) = gh \in G$$

is a smooth mapping of the differential space $(G \times G, C \times C)$ onto the differential space (G,C).

P r o o f . The following mappings

(2) $$G \ni g \longmapsto i_a(g) = (a,g) \in G \times G$$

(3) $$G \ni g \longmapsto j_a(g) = (g,a) \in G \times G$$

are smooth with respect to the differential structures C and $C \times C$, respectively (see [5]). Taking $a = e$ we obtain that $inv = Q \circ i_e$ is a smooth mapping on (G,C). On the other hand $inv = inv^{-1}$ and this implies that the symmetry is a diffeomorphism on (G,C).

Hence the mapping

$$G \times G \ni (g,h) \longmapsto B(g,h) = (g,h^{-1}) \in G \times G$$

is smooth on $(G \times G, C \times C)$ and we can conclude that the group operation $A = Q \circ B$ is smooth as a superposition of smooth mappings. The smoothnees of right and left translations follows now from equalities

$$R_a = A \circ j_a \quad \text{and} \quad L_a = A \circ i_a.$$

Taking into account that $R_a^{-1} = R_{a^{-1}}$ and $L_a^{-1} = L_{a^{-1}}$ we obtain that right and left translations are diffeomorphisms of (G,C).

From the general theory of differential spaces it follows that we can identify the tangent space $T_{(g,h)}(G \times G, C \times C)$ with the direct sum $T_g(G,C) \oplus T_h(G,C)$ of tangent spaces $T_g(G,C)$ and $T_h(G,C)$. The value of the pair $(v,w) \in \in T_g(G,C) \oplus T_h(G,C)$ on the function $f \in C \times C$ is given by the formula

$$(4) \qquad (v,w)(f) = v(f \circ i_h) + w(f \circ j_g) = (i_h)_{*g} v(f) + (j_g)_{*h} w(f)$$

(see [5], I, §17). With the aid of this formula we prove the following theorem.

T h e o r e m 1.2. Let (G,C) be a differential group. For any element $(g,h) \in G \times G$ and any vector $(v,w) \in \in T_g(G,C) \oplus T_h(G,C)$ the following equality is satisfied

$$(5) \qquad A_{*(g,h)}(v,w) = (R_h)_{*g} v + (L_g)_{*h} w,$$

where A is the group operation, R_h is the right translation and L_g is the left translation in the group G.

Moreover

$$(6) \qquad inv_{*g} = -(L_{g^{-1}})_{*e} \circ (L_{g^{-1}})_{*g}$$

for any $g \in G$. In particular

$$(7) \qquad inv_{*e} = -id_{T_e(G,C)},$$

where $id_{T_e(G,C)}$ is the identity mapping on $T_e(G,C)$.

P r o o f . For any function $f \in C$ we have

$$A_{*(g,h)}(v,w)(f) = (v,w)(f \circ A) = v(f \circ A \circ i_h) + w(f \circ A \circ j_g) =$$

$$= v(f \circ R_h) + w(f \circ L_g) = (R_h)_{*g}v(f) + (L_g)_{*h}w(f)$$

(see [4]). This proves the equality (5).

Let us now consider the tangent mapping to $F=(id_G, inv):G \to G \times G$ at $g \in G$. We have

(8) $$F_{*g} = (id_{T_g(G,C)}, inv_{*g}).$$

The superposition $A \circ F$ is the constant map and therefore $(A \circ F)_{*g} = A_{*F(g)} \circ F_{*g} = 0$. From this and from (5) we have for any $v \in T_g(G,C)$

$$A_{*F(g)} \circ F_{*g}v = A_{*(g,g^{-1})}(id_{T_g(G,C)}v, inv_{*g}v) =$$

$$= (R_{g^{-1}})_{*g}v + (L_g)_{*g^{-1}}(inv_{*g}v) = 0.$$

Taking into account that the map $(L_g)_{*g^{-1}}$ is invertible we conclude that for any $g \in G$

(9) $$inv_{*g} = -\left[(L_g)_{*g^{-1}}\right]^{-1} \circ (R_{g^{-1}})_{*g}.$$

On the other hand the identity $L_g \circ L_{g^{-1}} = id_G$ holds and this implies that

$$(L_g \circ L_{g^{-1}})_{*e} = (L_g)_{*g^{-1}} \circ (L_{g^{-1}})_{*e} = id_{T_e(G,C)}$$

or equivalently

(10) $$\left[(L_g)_{*g^{-1}}\right]^{-1} = (L_{g^{-1}})_{*e} .$$

Applying (10) to (9) we obtain (6).

Equality (7) follows immediately from (6) and from the equalities

$$L_{e*h} = (R_e)_{*h} = id_{T_h(G,C)}$$

for any $h \in G$.

2. Differential groups of class $\mathcal{D}_0$ and standard charts

Given a differential space (M,C), a pair (U,φ), where U is an open nonempty set in M and $\varphi: U \to R^n$ is a diffeomorphism of the differential space (U,C_U) onto the differential space $(\varphi(U), C^\infty(R^n)_{\varphi(U)})$ is a chart on (M,C). Here R^n is a real n-dimensional space.

If for any $p \in M$ there exists a chart (U,φ) with $p \in U$ then we say that (M,C) is a differential space of class $\mathcal{D}_0$ (see [2], [8]). The most important result about the spaces of class $\mathcal{D}_0$, we need, is the following

Theorem 2.1. Let (M,C) be a differential space of class $\mathcal{D}_0$, $p \in M$ and $\dim T_p(M,C) = k$. Then there exists a chart (U,φ) on (M,C) such that $p \in U$ and $\varphi : U \to R^k$. In other words the dimension of the image space R^k is a minimal possible.

For the proof we refer to [2].

It is easy to see that a differential group (G,C) is of class $\mathcal{D}_0$ iff there exists some chart on (G,C). The main result of this section is contained in the following theorem.

Theorem 2.2. Let (G,C) be a differential group of class $\mathcal{D}_0$ and denote by k the dimension of the tangent space $T_e(G,C)$ at the unit e. Then there exists a chart (U,φ) on (G,C) satisfying the following conditions:

(SC_1) (i) U is a symmetric open neighbourhood of e;
(ii) $\varphi: U \to R^k$;
(iii) $\varphi(e) = 0$;
(SC_2) there exists a mapping $E \in C^\infty(R^k \times R^k, R^k)$ such that:
(iv) if $g,h,gh \in U$ then

$$\varphi(gh) = E(\varphi(g),\varphi(h));$$

(v) $D_1E(x,0) = D_2E(0,y) = \mathrm{id}_{R^k}$

for any $x,y \in \varphi(U)$, where by D_1E and D_2E we denote the partial derivative with respect to x and y respectively;

(SC_3) there exists a mapping $F \in C^\infty(R^k,R^k)$ such that

$$\varphi(g^{-1}) = \varphi \circ \mathrm{inv}(g) = F \circ \varphi(g)$$

for any $g \in U$;

(vii) F is a diffeomorphism of some neighbourhood of the set $\varphi(U)$ in R^k and

$$DF(0) = -\mathrm{id}_{R^k},$$

where $DF(0)$ is a total derivative of F at the point 0;

(SC_4) there exist a number $0 < m < 1$ and an open cube $P \in R^k$ such that

(viii) $\varphi(U) \subset P$;

(ix) for the mapping E and for any $x, y_1, y_2 \in \mathrm{cl}P$ (the closure of P) the following inequality is satisfied

$$\|E(x,y_2)-E(x,y_1)\| \geq m\|y_2-y_1\|,$$

where $\|\cdot\|$ is the euclidean norm in R^k;

(x) for any neighbourhood V of e in G there exists a number $r > 0$ such that for any $g \in U$

$$K(\varphi(g),r) \cap \varphi(U) \subset \varphi(gV \cap U),$$

where $K(\varphi(g),r)$ is the open ball in R^k with the center at $\varphi(g)$ and the radius r.

Definition 2.1. If (U,φ) satisfies conditions (SC_1) - (SC_4), we call it a standard chart on (G,C).

Remark. We split the proof of the above theorem into several parts. The idea of the proof is that we can obtain a standard chart from each chart by a suitable restriction.

Proposition 2.1. For any differential group (G,C) of class $\mathcal{D}_0$ there exists a chart (U,φ) on (G,C) which satisfies the condition (SC_1).

Proof. Note that if (U_1,φ_1) is a chart on (G,C) and U is an open subset of U_1, the pair $(U,\varphi|_U)$ is also a chart on (G,C).

From Theorem 2.1 there exists a chart (U_1,φ_1) on (G,C) such that $e \in U_1$ and φ_1 is a map of U_1 into R^k, where $k = \dim T_e(G,C)$. If we put $U = U_1 \cap U_1^{-1}$ $(U_1^{-1} = \mathrm{inv}(U_1))$ and for any $g \in U$ $\varphi(g) = \varphi_1(g) - \varphi_1(e)$, we obtain the chart (U,φ) satisfying (SC_1).

Proposition 2.2. For any differential group (G,C) of class $\mathcal{D}_0$ there exists a chart (U,φ) on (G,C) which satisfies conditions (SC_1) and (SC_2).

Proof. Let (U_1,φ_1) be a chart on (G,C) which satisfies (SC_1). Choose a neighbourhood U_2 of e in G such that $U_2^2 \subset U_1$. The mapping $\Phi : \varphi_1(U_2) \times \varphi_1(U_2) \longrightarrow R^k$ defined for any $x,y \in \varphi_1(U_2)$ by the formula

$$\Phi(x,y) = \varphi_1(\varphi_1^{-1}(x)\varphi_1^{-1}(y))$$

is smooth with respect to the differential structure $C^\infty(R^k \times R^k)_{\varphi_1(U_2)\times\varphi_1(U_2)}$ (see [5], I, §10) and therefore there exists an open neighbourhood B of the point $0 = \varphi(e)$ in R^k and a mapping $E \in C^\infty(R^k \times R^k, R^k)$ such that

$$\Phi|_{(B\times B)\cap(\varphi_1(U_2)\times\varphi_1(U_2))} = E|_{(B\times B)\cap(\varphi_1(U_2)\times\varphi_1(U_2))}.$$

Put $U_3 = \varphi_1^{-1}(B) \cap U_2$ and $\varphi_3 = \varphi_1|_{U_3}$. Now if $g,h,gh \in U_3$, we have

$$(11) \qquad \varphi_3(gh) = \varphi_1\left[\varphi_1^{-1}(\varphi_3(g))\varphi_1^{-1}(\varphi_3(h))\right] = $$
$$= \Phi(\varphi_3(g),\varphi_3(h)) = E(\varphi_3(g),\varphi_3(h)).$$

Note that for any $g \in U_3$

$$E(\varphi_3(g),0) = E(\varphi_3(g),\varphi_3(e)) = \varphi_3(ge) = \varphi_3(g),$$

$$E(0,\varphi_3(g)) = E(\varphi_3(e),\varphi_3(g)) = \varphi_3(eg) = \varphi_3(g),$$

and therefore

$$(12) \quad \begin{cases} D_1E(\varphi_3(g),0) \circ \varphi_{3*g} = (E(\cdot,0) \circ \varphi_3)_{*g} = \varphi_{3*g}, \\ D_2E(0,\varphi_3(g)) \circ \varphi_{3*g} = (E(0,\cdot) \circ \varphi_3)_{*g} = \varphi_{3*g}. \end{cases}$$

The mapping $\varphi_3 : U_3 \to R^k$ is a diffeomorphism on the image and for any $g \in U_3$ $\dim T_g(U_3,C_{U_3}) = k$. This implies that the mapping

$$\varphi_{3*g} : T_g(U_3,C_{U_3}) \to T_{\varphi_3(g)}(R^k,C^\infty(R^k)) \cong R^k$$

is an isomorphism on R^k. Taking the superposition of both sides of equalities (12) with the mapping $(\varphi_{3*g})^{-1}$ we obtain for any $g \in U_3$

$$D_1E(\varphi_3(g),0) = id_{R^k} \quad \text{and} \quad D_2E(0,\varphi_3(g)) = id_{R^k}.$$

For $U := U_3 \cap U_3^{-1}$ and $\varphi = \varphi_3|_U$ we obtain the chart (U,φ) on (G,C) which satisfies (SC_1) and (SC_2).

P r o p o s i t i o n 2.3. For any differential group (G,C) of class $\mathcal{D}_0$ there exists a chart (U,φ) on (G,C) which satisfies conditions (SC_1) - (SC_3).

The proof of the existence of a mapping F which satisfies the condition (vi), is similar to the proof of the existence of the mapping E in the above proposition, Therefore we omit the details. Hence let us suppose that (U_1,φ_1) is a chart on (G,C), which together with some map $F\in C^{\infty}(R^k,R^k)$ satisfies conditions (i) - (vi) of Theorem 2.2. From (vi) it follows that

$$DF(0)\circ\varphi_{1*e} = \varphi_{1*e}\circ \mathrm{inv}_{*e}.$$

But $\mathrm{inv}_{*e} = -\mathrm{id}_{T_e(G,C)}$ (see Theorem 1.2) and thus we have

$$DF(0) = \varphi_{1*e}\circ(-\mathrm{id}_{T_e(G,C)})\circ(\varphi_{1*e})^{-1} = -\mathrm{id}_{R^k}.$$

This implies that there exists an open set $B\subset R^k$ such that $0\in B$ and $F_{|B}$ is a diffeomorphism. If we take now $U :=$ $:= \varphi_1^{-1}(B)\cap \mathrm{inv}(\varphi_1^{-1}(B))$ and $\varphi := \varphi_{1|U}$, we obtain the chart (U,φ) on (G,C) satisfying conditions (SC_1) - (SC_3).

L e m m a 2.1. If $E\in C^{\infty}(R^k\times R^k,R^k)$ and $D_2E(0,0) =$ $= \mathrm{id}_{R^k}$, then for any number $m<1$ there exists an open cube P in R^k such that $0\in P$ and

$$\|E(x,y_2)-E(x,y_1)\| \geqslant m\|y_2 - y_1\| \tag{13}$$

for any $x,y_1,y_2\in \mathrm{cl}P$.

P r o o f . Let for any $x,y\in R^k$ $A(x,y) := D_2E(x,y)-\mathrm{id}_{R^k}$. If $L(R^k)$ is the vector space of linear operators on R^k, the mapping $A : R^k\times R^k \longrightarrow L(R^k)$ is continuous with respect to the norm in $L(R^k)$. Moreover $A(0,0) = 0$. Now if $m<1$, then there exists an open cube $P\subset R^k$ such that $0\in P$ and for any $x,y\in \mathrm{cl}P$ $\|A(x,y)\| < 1 - m$. Then for any $x,y_1,y_2\in \mathrm{cl}P$ we have

$$E(x,y_2) - E(x,y_1) = \int_0^1 D_2E(x,y_1+t(y_2-y_1))(y_2-y_1)dt =$$

$$= y_2-y_1 + \int_0^1 A(x,y_1+t(y_2-y_1))(y_2-y_1)dt$$

and, as a consequence,

$$\| E(x,y_2)-E(x,y_1)\| \geqslant \| y_2-y_1\| - \int_0^1 (1-m)\| y_2-y_1\| dt = m\| y_2-y_1\| .$$

L e m m a 2.2. If $E \in C^{\infty}(R^k \times R^k, R^k)$ and P is an open cube in R^k such that $0 \in P$ and for some $0 < m < 1$ and any $x, y_1, y_2 \in \mathrm{cl}P$ the inequality (13) holds, then

(a) for any $x \in \mathrm{cl}P$ the mapping $E(x,\cdot)_{|P}$ is a diffeomorphism;

(b) for any neighbourhood Y of 0 in R^k there exists $r > 0$ such that for any $x \in \mathrm{cl}P$

$$K(E(x,0),r) \subset E(x,Y),$$

where $K(E(x,0),r)$ is the open ball in R^k with the center at $E(x,0)$ and the radius r.

P r o o f . The statement (a) follows immediately from the inequality (13). For the proof of (b) let us consider the number $\varepsilon > 0$ such that $\mathrm{cl}K(0,\varepsilon) \subset P$. The mapping $E(x,\cdot)_{|P}$ is a diffeomorphism for any $x \in \mathrm{cl}P$ and therefore $E(x,K(0,\varepsilon))$ is a bounded region in R^k. Moreover $\partial[E(x,K(0,\varepsilon))] = E(x,\partial K(0,\varepsilon))$ (∂B is a boundary of B). From (13), for any $z \in \partial[E(x,K(0,\varepsilon))]$

$$\| z - E(x,0)\| \geqslant m\varepsilon .$$

Then for any $x \in \mathrm{cl}P$

$$K(E(x,0),m\varepsilon) \subset E(x,K(0,\varepsilon)).$$

This gives the proof of (b) for $Y = K(0,\varepsilon)$.

As an immediate consequence we obtain the proof in the general case.

P r o p o s i t i o n 2.4. For any differential group (G,C) of class $\mathcal{D}_0$ there exists a chart (U,φ) on (G,C) which satisfies the conditions (SC_1) - (SC_4).

P r o o f . Fix a number $0<m<1$ and choose a chart (U_1,φ_1) on (G,C), which satisfies conditions (SC_1) - (SC_3). Let $E\in C^{\infty}(R^k\times R^k,R^k)$ be such as in (SC_2). We have $D_2E(0,0) = id_{R^k}$ and therefore we may choose an open cube $P\subset R^k$ containing 0 such that inequality (13) is satisfied for any $x,y_1,y_2\in clP$. Let U be a symmetric neighbourhood of the unit e in G such that $U^2\subset\varphi_1^{-1}(P)$. Moreover, take $\varphi = \varphi_{1|U}$. It is easy to see that (U,φ) is a chart on (G,C) satisfying (SC_1) - (SC_3) and the points (viii) and (ix) of (SC_4). Then it remains to prove that (U,φ) satisfies the condition (x).

Let V be an arbitrary neighbourhood of e in G. Choose the number $\varepsilon>0$ such that $\varphi_1^{-1}(K(0,\varepsilon))\subset V\cap U$ and $clK(0,\varepsilon)\subset P$. Then from Lemma 2.2 there exists $r>0$ such that for any $x\in clP$

$$K(E(x,0),r)\subset E(x,K(0,\varepsilon)).$$

For any $g\in U$ we have

$$(14)\qquad K(\varphi(g),r) = K(E(\varphi(g),0),r)\subset E(\varphi(g),K(0,\varepsilon)).$$

Let us fix $g\in U$ and let y be an arbitrary element of $K(\varphi(g),r)\cap\varphi(U)$. From (14) there exists $z\in K(0,\varepsilon)$ such that $y = E(\varphi(g),z)$. Moreover note that $\varphi(U) = E(\varphi(g),\varphi_1(g^{-1}U))$ and $\varphi_1(g^{-1}U)\subset\varphi_1(U^2)\subset P$. From this and taking into account that $E(\varphi(g),\cdot)_{|P}$ is an one to one map we obtain that $z\in\varphi_1(g^{-1}U)$. As a consequence we have that $z\in\varphi_1(g^{-1}U)\cap \cap K(0,\varepsilon)\subset\varphi_1(g^{-1}U\cap V\cap U)$ and then $z = \varphi_1(h_1)$, where $h_1\in V\cap g^{-1}U\cap U$. This implies that $y = E(\varphi(g),z) =$

$= E(\varphi_1(g), \varphi_1(h_1)) = \varphi_1(gh_1)$. But $gh_1 \in gV \cap U$ and therefore $y \in \varphi(gV \cap U)$. This completes the proof of this proposition as well as the proof of Theorem 2.2.

Let us note that if (U, φ) is a standard chart on a differential group (G,C) and $U_1 \subset U$ is an open symmetric neighbourhood of the unit e in G, then $(U_1, \varphi|_{U_1})$ is also a standard chart on (G,C).

The standard charts are very convenient in the investigations of differential groups of class $\mathcal{D}_0$ (see [3], [4]). In the next section we give an example of the application of the notion of the standard chart.

3. The completion of a differential group of class $\mathcal{D}_0$

Given a topological group G and neighbourhood of the unit $V \subset G$, two elements $g, h \in G$ are said to be closed of order V with respect to the left (right) uniform structure on G if $g^{-1}h \in V$ $(hg^{-1} \in V)$. It is well known, that if G is a Hausdorff space, then there exists a complete uniform space $\hat{G}_s$ $(\hat{G}_r)$ such that G is a dense subspace of $\hat{G}_s$ $(\hat{G}_r)$ with respect to the left (right) uniform structure on G (see [1], II, III).

On the uniform space $\hat{G}_s$ $(\hat{G}_r)$ there exists a group structure consistent with the uniform structure on $\hat{G}_s$ $(\hat{G}_r)$ and with the group structure on G iff the following condition is satisfied:

(C_1) for any left Cauchy filter $\mathcal{F}$ on G, the image $inv(\mathcal{F}) = \{inv(A) \subset G : A \in \mathcal{F}\}$ is also a left Cauchy filter on G (see [1] III).

Here we shall prove that any differential group of class $\mathcal{D}_0$ fulfils the above condition.

Let U be an open subset of the topological group G. By $\hat{U}_s$ we denote the set of all left Cauchy filters $\mathcal{F}$ on G such that $U \in \mathcal{F}$.

P r o p o s i t i o n 3.1. Let G be a topological group. Then the condition (C_1) is equivalent to the following condition

(C_2) there exists an open symmetric neighbourhood U of the unit e in G such that for any left Cauchy filter $\mathcal{F} \in \hat{U}_s$ the image $inv(\mathcal{F})$ is also a left Cauchy filter on G.

P r o o f . The implication $(C_1) \Rightarrow (C_2)$ is obvious. Suppose that a topological group G fulfils the condition (C_2) and let $U \subset G$ be such, as in (C_2). If $\mathcal{F}$ is a left Cauchy filter on G, then there exists a set $A \in \mathcal{F}$ such that for any two elements $g,h \in A$ $g^{-1}h \in U$. If we fix $g \in A$, then $g^{-1}A = L_{g^{-1}}(A) \subset U$ and the left Cauchy filter $L_{g^{-1}}(\mathcal{F})$ is an element of $\hat{U}_s$. From (C_2) $inv(L_{g^{-1}}(\mathcal{F})) = R_g(inv(\mathcal{F}))$ is a left Cauchy filter on G. Then $inv(\mathcal{F})$ is also a left Cauchy filter on G.

Now let us consider a differential group (G,C) of class $\mathcal{D}_0$. The group G with the topology τ_C is a Hausdorff space because the intersection of all neighbourhoods of the unit e is equal to the single point set $\{e\}$ (see [1], III). So G can be identified with the dense subspace of some complete uniform space $\hat{G}_s$.

T h e o r e m 3.1. If (U,φ) is a standard chart on a differential group (G,C), then the mappings φ and φ^{-1} are uniformly continuous with respect to the left uniform structure on U and the usual uniform structure on R^k.

P r o o f . Let a mapping $E \in C^\infty(R^k \times R^k, R^k)$ and a cube $P \subset R^k$ be such as in (SC_2) and (SC_4). Choose for a given $\varepsilon > 0$ a number $\delta > 0$ such that for any $x, y_1, y_2 \in clP$, if $\|y_2 - y_1\| < \delta$, then $\|E(x,y_2) - E(x,y_1)\| < \varepsilon$. It is possible because of the uniform continuity of the mapping E on $clP \times clP$. Now choose a neighbourhood of the unit $V \subset U$ such that for any $z \in V$ $\|\varphi(z)\| < \delta$. If $h,g \in U$ and $h \in gV$, then

$$\|\varphi(h) - \varphi(g)\| = \|\varphi(gz) - \varphi(g)\| = \|E(\varphi(g),\varphi(z)) - E(\varphi(g),\varphi(e))\| < \varepsilon.$$

This proves the uniform continuity of φ. The uniform continuity of φ^{-1} follows immediately from the point (x) of Theorem 2.2.

T h e o r e m 3.2. If (G,C) is a differential group of class $\mathcal{D}_0$, then there exists a group structure on $\hat{G}_s$ consistent with the uniform structure on $\hat{G}_s$ and with the group structure on G.

P r o o f . It suffices to prove that G fulfils the condition (C_2). Let us choose a standard chart (U,φ) and a mapping $F \in C^{\infty}(R^k,R^k)$ such as in Theorem 2.2. Note that F is uniformly continuous on $\varphi(U)$ ($\varphi(U)$ is a bounded set in R^k) and from Theorem 3.1 it follows that the symmetry $inv_{|U}$ is uniformly continuous with respect to the left uniform structure. Now if a filter $\mathcal{F} \in \hat{U}_s$, then $\mathcal{B} = \{A \cap U \subset U : A \in \mathcal{F}\}$ is a left Cauchy filter on U and $inv(\mathcal{B})$ is a left Cauchy filter on U. On the other hand $inv(\mathcal{B})$ is a basis of the filter $inv(\mathcal{F})$. This implies that $inv(\mathcal{F})$ is a left Cauchy filter on G (see [1], II).

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Received January 22, 1982.