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A FIXED POINT THEOREM FOR METRIC SPACES

1. Introduction

Let (X,d) be a metric space and let f be a mapping of X into itself. For a bounded subset A of X we denote by $\alpha(A)$ the measure of non-compactness of the set A (see Kuratowski [3], p.318). The mapping f is said to be contractive if $d(fx, fy) < d(x, y)$ for $x \neq y$. If $\alpha(f[A]) < \alpha(A)$ for any bounded subset A of X such that $\alpha(A) > 0$, then a continuous mapping f is called densifying.

In [2] Furi and Vignoli prove a fixed point theorem for contractive densifying mappings of a complete metric space into itself. It is known that a contractive mapping on a complete metric space need not have a fixed point. However, Edelstein [1] proves that if f is a contractive mapping and x_0 is a point in X such that the sequence of iterates $(f^n x_0)$ contains a subsequence convergent to a point u, then u is a fixed point of f and is unique.

The aim of this note is to prove a theorem on fixed points with two mappings. Using our theorem we give results of Endelstein and Furi and Vignoli type theorem.

2. Main result

Throughout this section, (X,d) denote a metric space, F and G two mappings of X into itself such that $d(Fx,Gy) < d(x,y)$ for all x, y in X with $x \neq y$, and

$$x_1 = Fx_0, \quad x_2 = Gx_1, \ldots, x_{2n+1} = Fx_{2n}, \quad x_{2n+2} = Gx_{2n+1}, \ldots$$

Assume, moreover, that there exists an element x_0 in X such that the sequence (x_n) contains a subsequence convergent to point u in X.

Under these hypotheses we have the following theorem.

Theorem. Let $(x_{i(n)})$ be a subsequence of (x_n) with even (resp. odd) indices $i(n)$ and let $(x_{i(n)})$ converge to a point u. Suppose that $(d(x_{i(n)}, x_{i(n)+1}))$ and $(d(x_{i(n)+2}, x_{i(n)+3}))$ are convergent sequences such that $\lim_{n\to\infty} d(x_{i(n)}, x_{i(n)+1}) \leq \lim_{n\to\infty} d(x_{i(n)+2}, x_{i(n)+3})$. If F is continuous at u and at $G(Fu)$ and G is continuous at Fu (resp. F is continuous at Gu and G is continuous at u and at $F(Gu)$), then either $Fu = u$ or $G(Fu) = Fu$ (resp. $Gu = u$ or $F(Gu) = Gu$).

Proof. Assume that the subsequence $(x_{i(n)})$ consists of elements with even indices. Since $\lim x_{i(n)} = u$, so $\lim x_{i(n)+1} = \lim Fx_{i(n)} = Fu$, $\lim x_{i(n)+2} = \lim Gx_{i(n)+1} = G(Fu)$ and $\lim x_{i(n)+3} = \lim Fx_{i(n)+2} = F(G(Fu))$. It follows that

$$d(u,Fu) = \lim d(x_{i(n)}, x_{i(n)+1}) \leq \lim d(x_{i(n)+2}, x_{i(n)+3}) =$$

$$= d(G(Fu), F(G(Fu))).$$

Now suppose that $Fu \neq u$ and $G(Fu) \neq Fu$. Then we have

$$d(u,Fu) \leq d(G(Fu), F(G(Fu))) < d(Fu, G(Fu)) < d(u,Fu).$$

So we arrive at a contradiction. Therefore $Fu = u$ or $G(Fu) = Fu$. We can show in a similar way that for odd indices $i(n)$, $Gu = u$ or $F(Gu) = Gu$. This completes the proof.

As an immediate consequence of our theorem we have the following corollary.

C o r o l l a r y . Assume that the sequence $(d(x_n,x_{n+1}))$ is a convergent sequence, F is continuous at points u, Gu and $G(Fu)$, and G is continuous at points u, Fu and $F(Gu)$. Then u is the unique common fixed point of F and G.

Let us remark that if $x_n \neq x_{n+1}$ for every n, then $(d(x_n,x_{n+1}))$ is a convergent sequence. Indeed, $d(x_{2n-1},x_{2n}) = d(Fx_{2n-2}, Gx_{2n-1}) < d(x_{2n-2},x_{2n-1})$ for $n = 1,2,\ldots$ Thus we have a monotone sequence of positive real numbers: $d(x_0,x_1) > d(x_1,x_2) > \ldots$, and we are done.

Now, for example, we consider the set $X = \{0,1\}$ with discrete metric and mappings G, F defined on X as $0 \mapsto 1$, $1 \mapsto 0$ and $x \mapsto x$, respectively. Then G and F satisfy assumptions of our theorem, but yet the condition $x_n \neq x_{n+1}$ $(n = 0,1,\ldots)$ is not satisfied.

3. Application

It is obvious that from our Corollary we obtain the following result of Edelstein type theorem: Let F, G, (x_n) and u be as in Sec.2. Assume that F, G are continuous mappings and $x_n \neq x_{n+1}$ for every n. Then u is the unique common fixed point of F and G (See [4]).

Now, let F and G be continuous mappings of a complete metric space (X,d) into itself satisfying the following conditions: (1) $d(Fx,Gy) < d(x,y)$ for $x \neq y$ and (2) $\alpha(GF[A]) < \alpha(A)$ for a bounded subset A of X with $\alpha(A) > 0$ (note that for densifying mappings F and G on X, (2) is trivialy satisfied). Assume, moreover, that for some x_0 in X the sequence (x_n) defined as in Sec. 2 is bounded and consisting of distinct points x_n. Modifying the proof of Furi and Vignoli [2], one can prove that $\{x_{2i}: i = 0,1,\ldots\}$ is a conditionally compact set. Consequently, by the corollary and remark given in Sec. 2, we conclude that F and G have a unique common fixed point. In particular, our result generalizes a theorem of Furi and Vignoli for densifying mappings.

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