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# ON WEAK SOLUTIONS OF A LINEAR PARABOLIC EQUATION IN A HILBERT SPACE

This paper deals with weak solutions of the linear parabolic equation

$$(0.1)\qquad Mu \equiv \frac{\partial}{\partial x_i}\left[a_{ij}(x,t)u_{x_j} + a_i(x,t)u\right] + b_i(x,t)u_{x_i} + a(x,t)u - u_t = f_o + \frac{\partial f_i}{\partial x_i}\ ^{1)}$$

in a bounded cylinder $G_T = G\times(0,T)$ with $G\subset R^n$ under the initial-boundary condition

$$(0.2)\qquad u|_{S_T} = 0, \quad u(x,0) = \psi(x), \quad x\in G,$$

where $S_T$ denotes the lateral boundary of $G_T$. In the above problem the coefficients of $M$ are real-valued functions, whereas the functions $f_i$ $(i=0,1,\dots,n)$, $\psi$ and $u$ take values in a Hilbert space $H$. We consider weak solutions of the problem (0.1), (0.2) which belong to Sobolev's space $V_2^{1,0}(G_T; H)$[2]. At first we derive some a priori estimate of this solutions in the norm of the space $V_2^{1,0}(G_T;H)$. This

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[1] Throughout this paper we shall use the summation convention.

[2] This space and the weak solution are defined in Sec.1.

estimate enables us to prove the existence and uniqueness of the solution mentioned. The last section of the paper contains a number of lemmas which have been used in the previous sections.

The results of Sections 2 and 3 of this paper constitute an extension of corresponding ones concerning the scalar case (included in [5]) to the case of any Hilbert space.

In paper [10] there was considered the problem (0.1), (0.2) in the particular case where $H$ was a Hilbert space of random variables. However in [10] the coefficients of $M$ were random functions. Therefore the results of this paper concerning the problem (0.1), (0.2) do not imply those obtained in [10].

## 1. Definitions and assumptions

In this paper $H$ denotes a real Hilbert space with a scalar product $uv$ and a norm $|u| = (uu)^{1/2} = (u^2)^{1/2}$. Let $E \subset R^k$ be a bounded domain. For any nonnegative integer $s$ we denote by $C^s(E;H)$ the set of all functions $u:E \to H$ continuous in $E$ together with all their derivatives $D^m u$, $m \leqslant s$, where

$$(1.1)\qquad D^m u(x) = \frac{\partial^m u(x)}{\partial x_1^{m_1} \dots \partial x_k^{m_k}}, \qquad m_1 + \dots + m_k = m^{3)}.$$

We abbreviate $C^0(E;H) = C(E;H)$. Let

$$C^\infty(E;H) = \bigcap_{s=1}^{\infty} C^s(E;H).$$

By $\mathring{C}^s(E;H)$ $(0 \leqslant s \leqslant \infty)$ we denote the subset of $C^s(E;H)$ consisting of those functions which have a compact support in $E$.

By $L_p(E;H)$ $(1 \leqslant p \leqslant \infty)$ we denote the set of all functions $u:E \to H$ which are Bochner measurable and have the finite norm

3) The continuity and partial derivatives are understood in the strong sense.

$$\|u\|_{p,E}=\left(\int_E |u(x)|^p dx\right)^{1/p} \text{ if } 1\leqslant p<\infty,\ \|u\|_{\infty,E}=\operatorname*{ess\,sup}_{x\in E}|u(x)|.$$

$L_p(E;H)$ is a Banach space (see Sec. IV.3 of [3][4]). Note that $L_2(E;H)$ is a Hilbert space with the scalar product

$$(u,v)_{2,E}=\int_E u(x)v(x)dx.$$

We introduce the following definition of weak derivatives of functions with values in H (cf. [1], p.179).

D e f i n i t i o n 1.1. Let functions $u,v:E\to H$ be locally integrable on E[5]. If for every function $\varphi\in\mathring{C}^m(E)$ (m being positive integer)[6] we have

$$\int_E v(x)\varphi(x)dx=(-1)^m\int_E u(x)D^m\varphi(x)dx,$$

then v is called the weak m-derivative of u in E. We write $v=D^m u$ (see (1.1)).

It easily follows from Lemma 4.12 that weak derivatives are uniquely determined. Since for any $u\in C^s(E;H)$ and $\varphi\in\mathring{C}^s(E)$ we have

$$\int_E D^m u(x)\varphi(x)dx=(-1)^m\int_E u(x)D^m\varphi(x)dx,\qquad m\leqslant s,$$

therefore the classical derivatives $D^m u$, $m\leqslant s$ of a function $u\in C^s(E;H)$ are weak m-derivatives of u in E.

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4) In this paper referring to the monographs [1]-[3], [5]-[9] we denote by a Roman numeral the chapter number.

5) Integrals of functions with values in H are always understood in the Bochner sense.

6) By $\mathring{C}^m(E)$ and $C^m(E)$ we denote the space $\mathring{C}^m(E;R)$ and $C^m(E;R)$, respectively.

Let $W_2^1(E;H)$ be the set of all functions $u \in L_2(E;H)$ which have weak derivatives $u_{x_i} \in L_2(E;H)$, $i=1,\dots,k$. By Lemma 4.14 $W_2^1(E;H)$ is a Hilbert space with the scalar product

$$(u,v)_{2,E}^{(1)} = (u,v)_{2,E} + (u_{x_i}, v_{x_i})_{2,E}.$$

It follows from the previous considerations that $C^1(E;H) \subset \subset W_2^1(E;H)$. Moreover, we introduce the space $\overset{\circ}{W}{}_2^1(E;H)$ defined as the closure of the set $\dot{C}^\infty(E;H)$ in the space $W_2^1(E;H)$.

Now we introduce various spaces of functions defined on the domain $G_T = G \times (0,T)$ with values in $H$ (cf. [10] and Sec.I.1 of [5]).

By $L_{q,r}(G_T;H)$ $(q,r \in \langle 1,\infty \rangle)$ we denote the set of all functions $u: G_T \longrightarrow H$ which are Bochner measurable and have the finite norm

$$\|u\|_{q,r,G_T} = \left\{ \int_0^T \Big[ \int_G |u(x)|^q dx \Big]^{r/q} dt \right\}^{1/r} \quad \text{if} \quad q,r \in \langle 1,\infty),$$

$$\|u\|_{\infty,r,G_T} = \left\{ \int_0^T \Big[ \operatorname*{ess\,sup}_{x \in G} |u(x,t)| \Big]^r dt \right\}^{1/r} \quad \text{if} \quad r \in \langle 1,\infty),$$

$$\|u\|_{q,\infty G_T} = \operatorname*{ess\,sup}_{t \in \langle 0,T \rangle} \Big[ \int_G |u(x,t)|^q dx \Big]^{1/q} \quad \text{if} \quad q \in \langle 1,\infty),$$

$$\|u\|_{\infty,\infty,G_T} = \operatorname*{ess\,sup}_{t \in \langle 0,T \rangle} \Big[ \operatorname*{ess\,sup}_{x \in G} |u(x,t)| \Big].$$

One can prove that if

$$\|u\|_{\infty,\infty,G_T} < \infty,$$

then

$$\|u\|_{\infty,G_T} = \operatorname*{ess\,sup}_{(x,t) \in G} |u(x,t)|$$

is finite and

$$\|u\|_{\infty,\infty,G_T} = \|u\|_{\infty,G_T}.$$

$L_{q,r}(G_T;H)$, like $L_p(E;H)$, is a Banach space.

By $W_2^{1,0}(G_T;H)$ we denote the set of all functions $u \in L_2(G_T;H)$ which have weak derivatives $u_{x_i} \in L_2(G_T;H)$ $(i=1,\dots,n)$. Introducing in $W_2^{1,0}(G_T;H)$ the scalar product

$$(u,v)_{2,G_T}^{(1,0)} = (u,v)_{2,G_T} + (u_{x_i},v_{x_i})_{2,G_T}$$

we obtain a Hilbert space (see Lemma 4.14).

By $V_2(G_T;H)$ we denote the set of all functions $u \in W_2^{1,0}(G_T;H)$ which have the finite norm

$$|\!|\!| u |\!|\!|_{2,G_T} = \|u\|_{2,\infty,G_T} + \|u_x\|_{2,G_T},$$

where

$$u_x = (u_{x_1},\dots,u_{x_n}),\ \|u_x\|_{2,G_T}^2 = \sum_{i=1}^{n} \|u_{x_i}\|_{2,G_T}^2.$$

In virtue of Lemma 4.15 $V_2(G_T;H)$ is a Banach space.

$V_2^{1,0}(G_T;H)$ is defined as the set consisting of those functions $u \in V_2(G_T;H)$ which are continuous with respect to the variable $t \in \langle 0,T\rangle$ in the space $L_2(G;H)$, i.e.

$$\lim_{\Delta t \to 0} \|u(\cdot,t+\Delta t) - u(\cdot,t)\|_{2,G} = 0 \quad \text{for any} \quad t \in \langle 0,T\rangle .$$

Introducing in $V_2^{1,0}(G_T;H)$ the norm

$$|\!|\!| u |\!|\!|_{2,G_T}^{(1,0)} = \sup_{0\le t\le T} \|u(\cdot,t)\|_{2,G} + \|u_x\|_{2,G_T}$$

we obtain, by Lemma 4.16, a Banach space.

Finally, let us define

$$W_2^{1,1}(G_T;H) = W_2^1(G_T;H).$$

So $W_2^{1,1}(G_T;H)$ is a Hilbert space with the scalar product

$$(u,v)_{2,G_T}^{(1,1)} = (u,v)_{2,G_T}^{(1,0)} + (u_t,v_t)_{2,G_T}.$$

D e f i n i t i o n 1.2. Let us denote by $C_0^\infty(\bar{G}_T;H)$ ($\bar{G}_T$ being the closure of $G_T$) the set of all functions $u \in C^\infty(G_T;H)$ with support contained in $\bar{G}_T \setminus \bar{S}_T$, where $\bar{S}_T = S \times \langle 0,T\rangle$ and $S$ is the boundary of $G$. By $\overset{\circ}{W}{}_2^{1,0}(G_T;H)$, $\overset{\circ}{V}_2(G_T;H)$, $\overset{\circ}{V}{}_2^{1,0}(G_T;H)$ and $\overset{\circ}{W}{}_2^{1,1}(G_T;H)$ we denote the closure of $C_0^\infty(\bar{G}_T;H)$ in spaces $W_2^{1,0}(G_T;H)$, $V_2(G_T;H)$, $V_2^{1,0}(G_T;H)$ and $W_2^{1,1}(G_T;H)$, respectively.

We shall consider the problem (0.1), (0.2) under the following assumptions, denoted collectively by (A):

The coefficients $a_{ij}$ $(i,j=1,\dots,n)$ are real-valued functions defined in $G_T$ and satisfy almost everywhere in $G_T$ the conditions

$$\nu|\xi|^2 \leqslant a_{ij}(x,t)\xi_i\xi_j \leqslant \mu|\xi|^2, \qquad \xi = (\xi_1,\dots,\xi_n) \in R^n,$$
$$|\xi|^2 = \sum_{i=1}^{n} \xi_i^2,$$

where $\mu > \nu$ are positive constants. Moreover, we assume that

(1.2) $$a_i^2,\ b_i^2,\ a \in L_{q,r}(G_T) \quad (i=1,\dots,n)^{7)},$$

7) Throughout this paper we shall use the spaces $L_p(G)$, $L_{q,r}(G_T)$, $W_2^{1,1}(G_T)$, $V_2^{1,0}(G_T)$, $\overset{\circ}{W}{}_2^{1,1}(G_T)$ and $\overset{\circ}{V}{}_2^{1,0}(G_T)$ defined in [5] (p.12-15).

(1.3) $\quad f_i \in L_2(G_T;H) \quad (i=1,\dots,n), \quad f_o \in L_{q_1,r_1}(G_T;H)$

and

(1.4) $$\psi \in L_2(G;H),$$

where $q,r,q_1,r_1$ are constants satisfying the following conditions

$$\frac{1}{r} + \frac{n}{2q} = 1, \quad \frac{1}{r_1} + \frac{n}{2q_1} = 1 + \frac{n}{4},$$

$$q \in \left(\frac{n}{2},\infty\right\rangle, \quad r \in \langle 1,\infty) \quad \text{if } n \geq 2,$$

$$q \in \langle 1,\infty\rangle, \quad r \in \langle 1,2\rangle \quad \text{if } n = 1,$$

$$q_1 \in \left\langle \frac{2n}{n+2}, 2\right\rangle, \; r_1 \in \langle 1,2\rangle \quad \text{if } n \geq 3,$$

$$q_1 \in (1,2\rangle, \quad r_1 \in \langle 1,2) \quad \text{if } n = 2,$$

$$q_1 \in \langle 1,2\rangle, \quad r_1 \in \langle 1,4/3\rangle \quad \text{if } n = 1.$$

It follows from (1.2) that

(1.5) $$\Big\|\sum_{i=1}^{n} a_i^2\Big\|_{q,r,G_T}, \quad \Big\|\sum_{i=1}^{n} b_i^2\Big\|_{q,r,G_T}, \quad \|a\|_{q,r,G_T} \leq \mu_1,$$

$\mu_1$ being a positive constant.

Like in [5] (Sec.III.1) we introduce the following definition of a weak solution of the problem (0.1), (0.2).

D e f i n i t i o n 1.3. A function $u \in V_2^{1,0}(G_T;H)$ is called a weak solution of the problem (0.1), (0.2) if $u \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ and if the equality

$$(1.6)\qquad -\int_{G_T} u\,\eta_t\,dxdt + \int_{G_T} M_0(u,\eta)\,dxdt = \int_G \psi(x)\eta(x,0)\,dx$$

holds for any function $\eta \in \overset{\circ}{W}{}_2^{1,1}(G_T)$ vanishing for $t = T$, where

$$(1.7)\qquad M_0(u,\eta) = (a_{ij}u_{x_j} + a_i u - f_i)\eta_{x_i} + (-b_i u_{x_i} - au + f_0)\eta.$$

Lemmas 4.20, 4.1 and assumptions (A) imply that

$$u\eta_t,\ M_0(u,\eta) \in L_1(G_T;H),\quad \psi\eta(\cdot,0) \in L_1(G;H)$$

for any $u \in \overset{\circ}{V}_2(G_T;H)$ and $\eta \in W_2^{1,1}(G_T)$. So the integrals occuring in (1.6) exist.

## 2. A priori estimate of weak solutions

In order to obtain a priori estimate of weak solutions of the problem (0.1), (0.2) in the norm $\|\cdot\|_{2,G_T}^{(1,0)}$ we need the following lemma.

L e m m a 2.1. Let assumptions (A) be satisfied and suppose that a function $u \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ is a weak solution of the problem (0.1), (0.2). Then for any $t_1, t_2 \in \langle 0,T\rangle$, $t_1 < t_2$ holds the equality

$$(2.1)\qquad \int_G u^2(x,t)\,dx\Big|_{t=t_1}^{t=t_2} + \int_{G_{t_1,t_2}} M_0(u,u)\,dxdt = 0,$$

where $M_0$ is defined by (1.7) and $G_{t_1,t_2} = G \times (t_1,t_2)$.

P r o o f . Let us put

$$(2.2)\qquad U_i = a_{ij}u_{x_j} + a_i u - f_i,\quad V = -b_i u_{x_i} - au + f_0.$$

In view of Lemmas 4.20, 4.1 and assumptions (A) we have

(2.3) $$U_i \in L_2(G_T;H),$$

(2.4) $$b_i u_{x_i},\quad au \in L_{q_2,r_2}(G_T;H),\quad f_0 \in L_{q_1,r_1}(G_T;H),$$

where $q_2 = 2q(q+1)^{-1}$, $r_2 = 2r(r+1)^{-1}$. Using Lemmas 4.21--4.23 of [10], Definition 1.3 and Green's theorem and arguing as in the proof of relation (III.2.1) of [5] (p.166, 167) we get

(2.5) $$\int_{G_{T-h}} (u_{ht}\bar{\eta} + U_{ih}\bar{\eta}_{x_i} + V_h\bar{\eta})\,dxdt = 0^{8)}\,.$$

for any function $\bar{\eta} \in W_2^{1,1}(G_{-h,T})$ vanishing for $t \leqslant 0$ and $t \geqslant T-h$[9].

Let us introduce the functions $\eta_k$ $(k > \frac{2}{t_1})$ given by formula

$$\eta_k(x,t) = \begin{cases} \eta(x,t)\chi_k(t), & (x,t) \in G_{t_1}, \\ 0, \quad x \in G, & t \in \langle -h,0\rangle \cup \langle t_1,T\rangle, \end{cases}$$

where $\eta \in \overset{\circ}{W}{}_2^{1,1}(G_{t_1})$ and

$$\chi_k(t) = \begin{cases} 0, & t \leqslant 0 \\ kt, & 0 < t < k^{-1} \\ 1, & k^{-1} < t \leqslant t_1 - k^{-1} \\ k(t_1 - t), & t_1 - k^{-1} < t < t_1, \\ 0, & t \geqslant t_1. \end{cases}$$

8) The symbols $u_h$, $U_{ih}$ and $V_h$ denote Stekloff's mollifications of functions $u$, $U_i$ and $V$, respectively (see (4.15) of [10]).

9) This argumentation is correct because Lemmas 4.6, 4.20-4.25 of paper [10] remain true if we replace $L_p(\Omega)$ and $L_2(\Omega)$ by $H$.

Taking $\bar{\eta} = \eta_k$ in (2.5) we obtain the equality

$$\int_{G_{t_1}} (u_{ht}\eta_k + U_{ih}\eta_{kx_i} + V_h\eta_k)dxdt = 0.$$

Hence, using Lemma 4.1 and Lemmas 4.6, 4.20, 4.24 of [10] (see [9]) and proceeding like in [10] (p.114-116) one can obtain the equality

$$(2.6) \qquad \int_{G_{t_1}} M_1(u,\eta)dxdt = 0, \quad \eta \in \mathring{W}_2^{1,1}(G_{t_1}),$$

where

$$(2.7) \qquad M_1(u,\eta) = u_{ht}\eta + U_{ih}\,\eta_{x_i} + V_h\eta.$$

It follows from (2.6) that

$$(2.8) \qquad \int_{G_{t_1}} M_1(u,\Phi_p)dxdt = 0$$

for any function

$$(2.9) \qquad \Phi_p = \sum_{k=1}^{p} \xi_k g_k, \quad \xi_k \in H, \quad g_k \in \mathring{W}_2^{1,1}(G_{t_1}).$$

Relations (2.2)-(2.4) and (2.6)-(2.8), Lemma 4.20 of [10] and Lemmas 4.20, 4.2 imply that for any $v \in \mathring{W}_2^{1,1}(G_{t_1};H)$ we have $M_1(u,v) \in L_1(G_{t_1})$ and

$$(2.10) \qquad \left|\int_{G_{t_1}} M_1(u,v)dxdt\right| \leqslant \|u_{ht}\|_{2,G_{t_1}} \cdot \|v - \Phi_p\|_{2,G_{t_1}} +$$
$$+ \|U_{ih}\|_{2,G_{t_1}} \cdot \|v_{x_i} - \Phi_{px_i}\|_{2,G_{t_1}} + \|f_{oh}\|_{q_1,r_1,G_{t_1}} \cdot \|v-\Phi_p\|_{q_1',r_1',G_{t_1}} +$$
$$+ \|(b_iu_{x_i}+au)_h\|_{q_2,r_2,G_{t_1}} \cdot \|v-\Phi_p\|_{q_2',r_2',G_{t_1}},$$

where $q_k^{-1} + q_k'^{-1} = 1$, $r_k^{-1} + r_k'^{-1} = 1$ for $k=1,2$. Since, by Lemma 4.17, the set of all functions (2.9) is dense in $\overset{\circ}{W}{}_2^{1,1}(G_{t_1};H)$ therefore the inequality (2.10) yields the equality

$$\int_{G_{t_1}} M_1(u,v)dxdt = 0, \quad v \in \overset{\circ}{W}{}_2^{1,1}(G_{t_1};H)$$

and consequently

$$\int_{G_{t_1}} M_1(u,u_h)dxdt = 0.$$

Proceeding further like in the proofs of relations (2.24)-(2.26) of [10] and using products of norms similar to those occurring on the right-hand side of (2.10) we conclude that

$$\frac{1}{2}\int_G u^2(x,t)dx \Big|_{t=0}^{t=t_1} + \int_{G_{t_1}} M_0(u,u)dxdt = 0.$$

Now, as in [10] (p.119), this equality implies (2.1), which completes the proof.

Using Lemma 2.1, assumptions (A) and Lemmas 4.20, 4.1, 4.2 and arguing as in the proof of Lemma III.2.1 of [5] one can prove the following theorem.

T h e o r e m 2.1. If assumptions (A) are satisfied, then for any weak solution $u \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ of the problem (0.1), (0.2) holds the estimate

$$|u|_{2,G_T}^{(1,0)} \leq c\left[\|\psi\|_{2,G} + \left(\sum_{i=1}^{n} \|f_i\|_{2,G_T}^2\right)^{1/2} + \|f_0\|_{q_1,r_1,G_T}\right],$$

where $c$ is a positive constant depending only on $n, \nu, \mu, \mu_1$ and $q$.

3. Existence and uniqueness of weak solutions

Theorem 3.1. If assumptions (A) are satisfied, then there exists a unique solution $u \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ of the problem (0.1), (0.2).

Proof. The uniqueness of solutions is an immediate consequence of Theorem 2.1. In order to prove the existence of a solution mentioned let us first consider the equation

$$(3.1) \qquad Mu = 0 \quad \text{in} \quad G_T$$

under condition (0.2). By (1.4) and Lemma 4.7 there exists a sequence of functions

$$(3.2) \qquad \psi_k(x) = \sum_{i=1}^{i_k} d_{ki}\psi_{ki}(x), \quad d_{ki} \in H, \quad \psi_{ki} \in C(\bar{G})$$

such that

$$(3.3) \qquad \lim_{k \to \infty} \| \psi - \psi_k\|_{2,G} = 0.$$

It follows from Sec.III.4 of [5] that the scalar problem $Mu_{ki} = 0$ in $G_T$, $u_{ki}|_{S_T} = 0$, $u_{ki}(x,0) = \psi_{ki}(x)$, $x \in G$ possesses a unique solution $u_{ki} \in \overset{\circ}{V}{}_2^{1,0}(G_T)$. Hence, by (3.2), the function

$$u_k(x,t) = \sum_{i=1}^{i_k} d_{ki}u_{ki}(x,t), \quad k=1,2,\dots$$

belongs to $\overset{\circ}{V}{}_2^{1,0}(G_T;H)$ and is a solution of the problem

$$(3.4) \quad Mu_k = 0 \quad \text{in} \quad G_T, \quad u_k|_{S_T} = 0, \quad u_k(x,0) = \psi_k(x), \quad x \in G.$$

According to Theorem 2.1 we have

$$\| u_k - u_m \|_{2,G_T}^{(1,0)} \leqslant c \,\| \psi_k - \psi_m\|_{2,G}, \quad k,m=1,2,\dots$$

which, by (3.3), implies that $(u_k)$ is a Cauchy sequence in $\overset{\circ}{V}{}_2^{1,0}(G_T;H)$. In virtue of Remark 4.3 there exists a function $u \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ such that

$$\lim_{k\to\infty} \| u_k - u \|_{2,G_T}^{(1,0)} = 0. \tag{3.5}$$

Applying Definition 1.3 to the problem (3.4) we get the equality

$$-\int_{G_T} u_k \eta_t \,dxdt + \int_{G_T} \left[ (a_{ij} u_{kx_j} + a_i u_k)\eta_{x_i} + (-b_i u_{kx_i} - a u_k)\eta \right] dxdt = $$
$$= \int_G \psi_k(x)\eta(x,0)\,dx$$

for any function $\eta \in \overset{\circ}{W}{}_2^{1,1}(G_T)$ vanishing for $t = T$. Hence, using relations (3.3) and (3.5), assumptions (A) and Lemmas 4.20 and 4.1 we conclude that

$$-\int_{G_T} u\, \eta_t \,dxdt + \int_{G_T} \left[ (a_{ij} u_{x_j} + a_i u)\eta_{x_i} + (-b_i u_{x_i} - a u)\eta \right] dxdt = $$
$$= \int_G \psi(x)\eta(x,0)\,dx.$$

So $u$ is a solution of the problem (3.1), (0.2).

Proceeding similarly as above and using (1.3) and Lemma 4.10 for functions $f_m$ $(m=0,1,\dots,n)$ we obtain a solution $v \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ of the problem

$$Mv = f_o \quad \text{in} \quad G_T, \quad v|_{S_T} = 0, \quad v(x,0) = 0, \quad x \in G$$

and a solution $w_m \in \overset{\circ}{V}{}_2^{1,0}(G_T;H)$ of the problem

$$Mw_m = \frac{\partial f_m}{\partial x_m} \quad \text{in} \quad G_T, \quad w_m|_{S_T} = 0, \quad w_m(x,0) = 0, \quad x \in G.$$

It is clear that the function $u + v + \sum_{m=1}^{n} w_m$ belongs to $\overset{\circ}{V}{}_2^{1,0}(G_T;H)$ and is a solution of the problem (0.1), (0.2), which completes the proof.

4. Lemmas

In this section we state lemmas which were used in the previous sections. We retain notation concerning functional spaces introduced in Sec.1.

Lemma 4.1. Let $u_i \in L_{q_i,r_i}(G_T)$, $i=1,\dots,s-1$ $(s \geqslant 2)$ and $u_s \in L_{q_s,r_s}(G_T;H)$, where

(4.1) $$q_i \geqslant 1, \quad q^{-1} \sum_{i=1}^{s} q_i^{-1} \leqslant 1, \quad r_i \geqslant 1, \quad r^{-1} = \sum_{i=1}^{s} r_i^{-1} \leqslant 1.$$

Then

$$\prod_{i=1}^{s} u_i \in L_{q,r}(G_T;H) \quad \text{and} \quad \left\| \prod_{i=1}^{s} u_i \right\|_{q,r,G_T} \leqslant \prod_{i=1}^{s} \|u_i\|_{q_i,r_i,G_T}.$$

This lemma easily follows from inequality (II.1.8) of [5].

Lemma 4.2. Let $u_i \in L_{q_i,r_i}(G_T;H)$, $i=1,2$, where $q_i$, $r_i$ satisfy conditions (4.1) with $s = 2$. Then $u_1 u_2 \in L_{q,r}(G_T)$ and

(4.2) $$\|u_1 u_2\|_{q,r,G_T} \leqslant \|u_1\|_{q_1,r_1,G_T} \cdot \|u_2\|_{q_2,r_2,G_T}.$$

For the proof it suffices to observe that

$$|u_i| \in L_{q_i,r_i}(G_T), \quad i=1,2$$

and next to apply Schwarz inequality in $H$ and inequality (4.2) for scalar functions.

Now let us put

(4.3) $$P = \langle \alpha_1, \beta_1 \rangle \times \dots \times \langle \alpha_k, \beta_k \rangle,$$

where

(4.4) $\qquad 0 < \beta_i - \alpha_i < 1, \qquad i = 1,\dots,k.$

Take a rectangle $P' = \langle \alpha_1', \beta_1' \rangle \times \dots \times \langle \alpha_k', \beta_k' \rangle \subset \text{int}\, P$, where int P denotes the interior of P. For a function $f \in C(P;H)$ we introduce Tonelli's polynomials

$$T_i(x) = \int_P f(y) t_i(y_1 - x_1) \dots t_i(y_k - x_k) dy, \qquad i=1,2,\dots$$

where

$$t_i(s) = (1-s^2)^i \left[ \int_{-1}^{1} (1-u^2)^i du \right]^{-1}.$$

Arguing like in Sec.II.2 of [6] one can prove the following two lemmas.

L e m m a 4.3. If $f \in C(P;H)$, then $\lim_{i\to\infty} |T_i(x) - f(x)| = 0$ uniformly in $P'$.

L e m m a 4.4. If $f \in C^s(P;H)$ $(s \geqslant 1)$ and

$$D^m f(x) = 0, \quad x \in \partial P, \quad m = 0,1,\dots,s-1,$$

then $D^m T_i$ $(m=0,1,\dots,s)$ is the i-th Tonelli's polynomial for $D^m f$ and

$$\lim_{i\to\infty} \left| D^m T_i(x) - D^m f(x) \right| = 0, \qquad m=0,1,\dots,s$$

uniformly in $P'$, where $D^m$ is the differential operator defined by (1.1) and $\partial P$ is the boundary of P.

L e m m a 4.5. Let $Q' \subset R^k$ be a closed rectangle with faces parallel to the coordinate planes. Then for any function $f \in C(Q';H)$ there exists a sequence of polynomials $W_i : R^k \to H$, $i=1,2,\dots$ uniformly convergent to f in $Q'$.

P r o o f . Take an arbitrary closed rectangle $Q \subset R^k$ with faces parallel to the coordinate planes such that $Q' \subset \text{int}\, Q$. By Theorem V.2.3 of [2] there exists an extension $F \in C(Q;H)$ of f. Now we introduce new variables

$$y_j = d_j x_j, \quad j=1,\dots,k \quad (d_j > 0)$$

such that the rectangle $P$ defined by (4.3), (4.4) is the image of $Q$. Then Lemma 4.5 easily follows from Lemma 4.3.

L e m m a 4.6. For an arbitrary positive integer $s$ every function $w \in C_0^\infty(\bar{G}_T;H)$ can be uniformly approximated in $C^s(\bar{G}_T;H)$ by functions

$$w_i(x,t) = \sum_{k=1}^{k_i} \sum_{j=1}^{j_0} c_{ijk} w_{ijk}(x,t), \quad i=1,2,\dots$$[10]

where $c_{ijk} \in H$, $w_{ijk} \in C_0^\infty(\bar{G}_T)$, $k_i$ is some positive integer depending only on $i$ and $j_0$ is some positive integer depending only on $G_T$ and $w$.

P r o o f . Proceeding like in the proof of Lemma I of [4], Sec.259 we extend the function $w$ to $W \in C_0^s(\bar{G}_{\alpha,\beta};H)$ where $\alpha \in (-T,0)$, $\beta \in (T,2T)$ are arbitrarily fixed numbers. Then there exist open rectangles $P_1,\dots,P_{j_0}$ such that

$1^0$ the sides of $P_j$ are parallel to the coordinate axes and their length is less than 1;

$2^0$ $\bar{P}_j \subset \bar{G}_{\alpha,\beta} \setminus \bar{S}_{\alpha,\beta}$, $P_j \cap G_T \neq \phi$, where $S_{\alpha,\beta}$ is the lateral boundary of the cylinder $G_{\alpha,\beta} = G \times (\alpha,\beta)$;

$3^0$ the system $P_1,\dots,P_{j_0}$ is a covering of the support of the function $w$ denoted by $\operatorname{supp} w$.

Now choose open rectangles $P'_j$, $j=1,\dots,j_0$ possessing properties $1^0$-$3^0$ and such that $\bar{P}'_j \subset P_j$. There are functions $\lambda_j \in C^\infty(\bar{G}_{\alpha,\beta})$ $(j=1,\dots,j_0)$ such that

$$\operatorname{supp} \lambda_j \subset P_j, \quad \lambda_j(x,t) = 1 \quad \text{for any} \quad (x,t) \in \bar{P}'_j.$$

---

10) I.e. for any $\varepsilon > 0$ there is $w_{i_0}$ such that

$$\left| D^m w(x,t) - D^m w_{i_0}(x,t) \right| < \varepsilon, \quad (x,t) \in \bar{G}_T, \quad m=0,1,\dots,s.$$

The functions $\lambda_j W|_{P_j}$ $(j=1,\dots,j_0)$ satisfy assumptions of Lemma 4.4. Thus for any $\eta > 0$ there is $i_0$ such that

$$(4.5)\qquad |D^m w(x,t) - D^m T_{i_0 j}(x,t)| < \eta$$

for $j=1,\dots,j_0$, $m=0,1,\dots,s$, $(x,t) \in \overline{P}'_j \setminus \overline{G}_T$, where $T_{i_0 j}$ is the $i_0$-th Tonelli's polynomial for the function $\lambda_j W|_{P_j}$. By theorem on the partition of unity (see e.g. Sec.XVIII.4 of [7]) there exist functions $\varphi_j \in C^\infty(R^{n+1})$, $j=1,\dots,j_0$ such that

$$0 \leq \varphi_j(x,t) \leq 1, \quad (x,t) \in R^{n+1}, \quad \operatorname{supp} \varphi_j \subset P'_j$$

and

$$\sum_{j=1}^{j_0} \varphi_j(x,t) = 1, \quad (x,t) \in \operatorname{supp} w.$$

So we have

$$(4.6)\qquad w(x,t) = \sum_{j=1}^{j_0} \varphi_j(x,t) w(x,t), \quad (x,t) \in \overline{G}_T.$$

Since

$$\sum_{j=1}^{j_0} \varphi_j(x,t) T_{ij}(x,t) = \sum_{j=1}^{j_0} \sum_{k=1}^{k_i} c_{ijk} w_{ijk}(x,t)$$

therefore relations (4.5), (4.6) imply the condition stated in the footnote[10]. This completes the proof.

L e m m a 4.7. If $f \in L_q(G;H)$, $1 \leq q < \infty$, then there exists a sequence of polynomials $W_m : R^n \to H$, $m=1,2,\dots$ convergent to $f$, i.e.

$$\lim_{m\to\infty} \|W_m - f\|_{q,G} = 0.$$

P r o o f . Take an arbitrary closed rectangle $P$, $G \subset P \subset R^n$ with faces parallel to the coordinate planes. The function $f_1 : P \to H$ defined by formula

$$f_1(x) = \begin{cases} f(x), & x \in G, \\ 0, & x \in P \setminus G \end{cases} \tag{4.7}$$

belongs obviously to $L_q(P;H)$. Take an arbitrary $\varepsilon > 0$. Then, by Theorem VII.4.3 of [8], there is $\delta > 0$ such that for any

$$F \subset P, \ |F|_n < \delta \tag{4.8}$$[11]

we have

$$\int_F |f_1(x)|^q dx < \frac{1}{2} 3^{-q} \varepsilon^{-q}. \tag{4.9}$$

In view of Theorem V.6.6 of [2] for this $\delta$ there exist set (4.8) and a simple function $f_2 : P \longrightarrow H$ such that

$$f_2(x) = 0, \quad x \in F, \tag{4.10}$$

$$|f_1(x) - f_2(x)| < \frac{1}{3} \varepsilon (2|P|_n)^{-1/q}, \quad x \in P \setminus F. \tag{4.11}$$

Relations (4.9)-(4.11) easily imply the inequality

$$\| f_1 - f_2 \|_{q,P} < \frac{\varepsilon}{3} . \tag{4.12}$$

Let us put $\alpha = \sup_{x \in P} |f_2(x)|$. It suffices to consider the case $\alpha > 0$. By Theorem V.6.7 of [2] there is a closed set $F_1 \subset P$ such that

$$|P \setminus F_1|_n < (\frac{1}{6} \varepsilon \alpha^{-1})^q \tag{4.13}$$

and the function $f_2|_{F_1}$ is continuous. In virtue of Theorem V.2.3 of [2] for the function $f_2|_{F_1}$ there exists a con-

[11] By $|B|_s$ we shall denote the s-dimensional Lebesque measure of a set B. We assume, of course, the measurability of B with respect to this measure.

tinuous extension $f_3:P \to H$ such that $|f_3(x)| \leq \alpha$ for $x \in P$. So we have

$$f_2(x) = f_3(x) \text{ for } x \in F_1 \quad \text{and} \quad |f_2(x)-f_3(x)| \leq 2\alpha \text{ for } x \in P \setminus F_1 .$$

Consequently, by (4.13), we get

$$(4.14) \qquad \|f_2 - f_3\|_{q,P} < \frac{\varepsilon}{3} .$$

According to Lemma 4.5 there exists a polynomial $W:R^n \to H$ satisfying the inequality

$$(4.15) \qquad \|f_3 - W\|_{q,P} < \frac{\varepsilon}{3} .$$

Relations (4.7), (4.12), (4.14) and (4.15) immediately imply that $\|f-W\|_{q,G} < \varepsilon$, which completes the proof.

L e m m a 4.8. Let $(f_m)$ be a nonnegative sequence of functions belonging to $L_{q,r}(G_T)$, $q,r \in \langle 1,\infty)$. If $(f_m)$ is convergent a.e. in $G_T$ to a function $f \in L_{q,r}(G_T)$, then

$$\lim_{m \to \infty} \|f_m - f\|_{q,r,G_T} = 0.$$

This lemma can be proved in a similar manner as Theorem XII.2.6(ii) of [9].

L e m m a 4.9. Let $f \in L_{q,r}(G_T;H)$, $q,r \in \langle 1,\infty)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\chi_F f\|_{q,r,G_T} < \varepsilon$$

for any set $F \subset G_T$, $|F|_{n+1} < \delta$, where $\chi_F$ is the characteristic function of $F$.

P r o o f . The argumentation is similar to that used for Theorem VII.4.3 of [8]. So let us take any $\varepsilon > 0$ and denote

$$f_m = \min(m,|f|), \quad m=1,2,\ldots$$

By Lemma 4.8 there exists $m_0$ such that

$$\| |f| - f_{m_0} \|_{q,r,G_T} < \frac{\varepsilon}{2} .$$

Hence, taking any set

$$F \subset G_T, \ |F|_{n+1} < \delta, \ \delta = \frac{\varepsilon}{2m_0}$$

we get

$$\| \chi_F f \|_{q,r,G_T} \leqslant \| |f| - f_{m_0} \|_{q,r,G_T} + m_0 \| \chi_F \|_{q,r,G_T} < \varepsilon .$$

Using Lemma 4.9 in tead of Theorem VII.4.3 of [8] and prceeding like in the proof of Lemma 4.7 one can prove the following lemma.

L e m m a 4.10. If $f \in L_{q,r}(G_T;H)$, $q,r \in \langle 1,\infty)$, then there exists a sequence of polynomials $W_m : R^{n+1} \to H$, $m=1,2,\ldots$ convergent to $f$, i.e.

$$\lim_{m \to \infty} \| W_m - f \|_{q,r,G_T} = 0.$$

L e m m a 4.11. The space $\dot{C}(E;H)$ is dense in $L_p(E;H)$, $1 \leqslant p < \infty$.

This lemma easily follows from Lemma 4.7 and Theorem II.2.13 of [1].

L e m m a 4.12. Let $u : E \to H$ be a locally integrable function on $E$. If for any function $\varphi \in \dot{C}^m(E)$ ($m$ being a nonnegative integer) we have

$$\int_E \varphi(x) u(x) dx = 0, \tag{4.16}$$

then $u(x) = 0$ a.e. in $E$.

P r o o f . We shall use the following remark.

R e m a r k 4.1. For functions with values in $H$ we introduce mollifications in the same manner as in Sec.II.2.17

of [1]. Using Lemma 4.11 one can extend Lemma II.2.18 of [1] to the functions with values in H.

Now take an arbitrary domain $E_\varrho$, $\varrho > 0$ such that $\overline{E}_\varrho \subset E$ and the distance from $E_\varrho$ to the boundary of $E$ is less than $\varrho^{-1}$. Retaining notation of Sec.II.2.17 of [1] and substituting

$$\varphi(y) = J_\varepsilon(x-y),\ \varepsilon \in (0,\varrho^{-1}),\quad x \in E_\varrho$$

in (4.16) we get $J_\varepsilon * u(x) = 0$. Hence, using Lemma II.2.18(c) of [1] and Remark 4.1, we find that

$$\|u\|_{1,E_\varrho} = 0, \quad \text{i.e.} \quad u(x) = 0 \text{ a.e. in } E_\varrho.$$

According to the above considerations for any positive integer $m$ there exists a set

$$F_m \subset E_m,\ |E_m \setminus F_m|_k = 0$$

such that $u(x) = 0$ for any $x \in F_m$. So we have

$$\{x \in E:\ u(x) \neq 0\} \subset \bigcup_{m=1}^{\infty} (E_m \setminus F_m).$$

This relation and the equality

$$\left|\bigcup_{m=1}^{\infty} (E_m \setminus F_m)\right|_k = 0$$

imply that $u(x) = 0$ a.e. in $E$.

L e m m a 4.13. Let functions $u_m \in L_1(E;H)$, $m=1,2,\ldots$ posses weak derivatives $u_{mx_i} \in L_1(E;H)$ ($i \in \langle 1,k\rangle$ being arbitrarily fixed). If sequences $(u_m)$ and $(u_{mx_i})$ are convergent to functions $u$ and $v_i$, respectively, then there exists a weak derivative $u_{x_i}$ and $u_{x_i} = v_i$.

The proof is the same as that for real-valued functions.

Remark 4.2. Observe that Lemmas 4.1, 4.3-4.7, 4.9-4.13 remain true in the case where $H$ is a Banach space.

Lemma 4.14. $W_2^1(E;H)$, $W_2^{1,0}(G_T;H)$ and $W_2^{1,1}(G_T;H)$ are Hilbert spaces.

It suffices to prove completeness of this spaces. However this property easily follows from Lemma 4.13.

Lemma 4.15. $V_2(G_T;H)$ is a Banach space.

Proof. It suffices to prove the completeness. So let $(u_m)$ be a Cauchy sequence in $V_2(G_T;H)$. Therefore $(u_m)$ is a Cauchy sequence in the space $W_2^{1,0}(G_T;H)$ as well. Consequently, by Lemma 4.14 there exists a function $u \in W_2^{1,0}(G_T;H)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{2,G_T}^{(1,0)} = 0. \tag{4.17}$$

One can find that $u_m(\cdot,t)$, $u(\cdot,t) \in L_2(G;H)$ for almost all $t \in \langle 0,T\rangle$. Moreover, relation (4.17) and Theorem XII.2.5 of [9] imply the existence of a subsequence $(u_{m_k})$ such that

$$\lim_{k\to\infty} \|u_{m_k}(\cdot,t) - u(\cdot,t)\|_{2,G} = 0 \tag{4.18}$$

for almost all $t \in \langle 0,T\rangle$.

According to the above considerations there exists a set $\mathcal{T} \subset \langle 0,T\rangle$, $|\mathcal{T}|_1 = T$ such that (4.18) holds for $t \in \mathcal{T}$ and

$$\lim_{m,p\to\infty} (\sup_{t\in\mathcal{T}} \|u_m(\cdot,t) - u_p(\cdot,t)\|_{2,G}) = 0. \tag{4.19}$$

Take any $\varepsilon > 0$. Then, by (4.19), there is $k_0$ such that

$$\|u_{m_k}(\cdot,t) - u_{m_p}(\cdot,t)\|_{2,G} \leq \varepsilon, \quad t \in \mathcal{T},\ k,p > k_0.$$

Hence, in view of (4.18) for $t \in \mathcal{T}$, we have

$$\|u_{m_k}(\cdot,t) - u(\cdot,t)\|_{2,G} \leq \varepsilon, \quad t \in \mathcal{T},\ k > k_0.$$

Consequently, by (4.17), we get

$$u \in V_2(G_T;H), \quad \lim_{k\to\infty} \| u_{m_k} - u \|_{2,G_T} = 0.$$

This easily implies that

$$\lim_{m\to\infty} \| u_m - u \|_{2,G_T} = 0,$$

which completes the proof.

Lemma 4.16. $V_2^{1,0}(G_T;H)$ is a Banach space.

Proof. It suffices to prove the completeness. So let $(u_m)$ be a Cauchy sequence in $V_2^{1,0}(G_T;H)$. Consequently $(u_m)$ is a Cauchy sequence in $V_2(G_T;H)$, whence, by Lemma 4.15, there exists $u \in V_2(G_T;H)$ such that

$$\lim_{m\to\infty} \| u_m - u \|_{2,G_T} = 0. \tag{4.20}$$

This implies the existence of a set $\mathcal{T} \subset \langle 0,T\rangle$, $|\mathcal{T}|_1 = T$ such that $\sup_{t\in\mathcal{T}} \|u(\cdot,t)\|_{2,G} < \infty$ and

$$\lim_{m\to\infty} \left(\sup_{t\in\mathcal{T}} \|u_m(\cdot,t) - u(\cdot,t)\|_{2,G}\right) = 0. \tag{4.21}$$

Take an arbitrary $t_0 \in \langle 0,T\rangle \setminus \mathcal{T}$. Let $t_k \in \mathcal{T}$, $k=1,2,\dots$ be a sequence convergent to $t_0$. Taking into account (4.21) and the uniform continuity of functions $u_m$ with respect to $t$ (in the space $L_2(G;H)$) one can prove that $(u(\cdot,t_k))$ is a Cauchy sequence in $L_2(G;H)$. This implies the existence of a limit

$$\lim_{k\to\infty} u(\cdot,t_k).$$

One can prove that the above limit is independent of the sequence $(t_k)$. So there exists a limit

$$\lim_{\mathcal{T}\ni t\to t_0} u(\cdot,t).$$

Now let us put

$$(4.22)\qquad v(\cdot,t) = \begin{cases} u(\cdot,t), & t \in \mathcal{T}, \\ \lim\limits_{\mathcal{T}\ni\tau\to t} u(\cdot,\tau), & t \in \langle 0,T\rangle \setminus \mathcal{T}. \end{cases}$$

Using (4.21) and the continuity of functions $u_m$ with respect to $t$ we get

$$(4.23)\qquad \lim_{m\to\infty} \Big( \sup_{t\in\langle 0,T\rangle} \|u_m(\cdot,t) - v(\cdot,t)\|_{2,G}\Big) = 0.$$

Hence it follows the continuity of $v$ with respect to $t$ (in the space $L_2(G;H)$). This fact and relations (4.20), (4.22), (4.23) imply that $v \in V_2^{1,0}(G_T;H)$ and

$$\lim_{m\to\infty} \|u_m - u\|_{2,G_T}^{(1,0)} = 0,$$

which completes the proof.

R e m a r k 4.3. Lemma 4.16 and Definition 1.2 easily imply that $\overset{\circ}{V}{}_2^{1,0}(G_T;H)$ is a Banach space.

L e m m a 4.17. The set of all functions

$$\sum_{i=1}^{p} \xi_i g_i, \quad p=1,2,\ldots, \quad \xi_i \in H, \quad g_i \in C_0^\infty(\bar{G}_T)$$

is dense in the spaces

$$\overset{\circ}{W}{}_2^{1,0}(G_T;H), \quad \overset{\circ}{V}_2(G_T;H), \quad \overset{\circ}{V}{}_2^{1,0}(G_T;H) \text{ and } \overset{\circ}{W}{}_2^{1,1}(G_T;H).$$

This lemma follows immediately from Definition 1.2 and Lemma 4.6.

L e m m a 4.18. If $u \in \overset{\circ}{W}{}_2^1(E;H)$, then $u \in L_p(E;H)$ and

$$(4.23)\qquad \|u\|_{p,E} \le \beta \|u_x\|_{2,E}^{\alpha} \cdot \|u\|_{2,E}^{1-\alpha},$$

where

$$u_x = (u_{x_1},\dots,u_{x_k}),\quad \|u_x\|^2_{2,E} = \sum_{i=1}^{k} \|u_{x_i}\|^2_{2,E},$$

$p \in \langle 2, 2k(k-2)^{-1}\rangle$ if $k \geqslant 3$, $p \in \langle 2,\infty)$ if $k=1$, $\alpha = \frac{k}{2} - \frac{k}{p}$ and $\beta$ is a constant depending only on $p$ and $k$.

P r o o f . Take any function $u \in \mathring{C}^{\infty}(E;H)$ and denote

$$v_m(x) = |u(x)|^{(m+1)/m} = (u^2(x))^{(m+1)/2m},\quad m=1,2,\dots$$

One can easily check that $v_m$ have continuous derivatives $v_{mx_i}$ $(i=1,\dots,k)$ defined by formula

$$v_{mx_i}(x) = \begin{cases} (1+m^{-1})|u(x)|^{m^{-1}-1}u(x)u_{x_i}(x) & \text{if } u(x) \neq 0, \\ 0 \text{ if } u(x) = 0. \end{cases}$$

Now consider the function $v$ defined by formula

$$v(x) = |u(x)|,\quad x \in E.$$

Observing that $\lim_{m\to\infty} v_m(x) = v(x)$ for $x \in \mathring{E}$ and

$|v_m(x)| \leqslant K^2$ for $x \in E$, $m=1,2,\dots$, where $K = \max(1, \sup_{x\in E} |u(x)|)$

and using Theorem XII.2.6(i) of [9] we get

$$\lim_{m\to\infty} \|v_m - v\|_{1,E} = 0. \tag{4.24}$$

Like as above, introducing the functions $w_i$ $(i=1,\dots,k)$ given by

$$w_i(x) = \begin{cases} |u(x)|^{-1}u(x)u_{x_i}(x) & \text{if } u(x) \neq 0, \\ 0 \text{ if } u(x) = 0, \end{cases}$$

we find that

$$\lim_{m\to\infty} \|v_{mx_i} - w_i\|_{1,E} = 0,\quad i = 1,\dots,k. \tag{4.25}$$

In virtue of Lemma 4.13 in the scalar case it follows from relations (4.24) and (4.25) that $w_i$ are weak derivatives of function $v$ and $v_{x_i} = w_i$ $(i=1,\dots,k)$. Thus we have proved that $v \in \mathring{W}_2^1(E)$. This implies, by Theorem II.2.2 of [5], the inequality

$$\|v\|_{p,E} \leq \beta \|v_x\|_{2,E}^{\alpha} \|v\|_{2,E}^{1-\alpha}.$$

Hence, in view of relations

$$\|v\|_{p,E} = \|u\|_{p,E}, \quad \|v_x\|_{2,E} \leq \|u_x\|_{2,E},$$

the inequality (4.23) holds for the function $u$.

Now take any function $u \in \mathring{W}_2^1(E;H)$. Then there exists a sequence of functions $u_m \in \mathring{C}^{\infty}(E;H)$, $m=1,2,\dots$ such that

$$(4.26) \qquad \lim_{m\to\infty} \|u_m-u\|_{2,E} = 0, \quad \lim_{m\to\infty} \|u_{mx}-u_x\|_{2,E} = 0.$$

Hence, taking into account the inequality (4.23) for functions $u_m-u_l$, we conclude that $(u_m)$ is a Cauchy sequence in $L_p(E;H)$. So there exists a function $w \in L_p(E;H)$ such that

$$\lim_{m\to\infty} \|u_m-w\|_{p,E} = 0.$$

This implies, by (4.26), that $w = u$, i.e. $u \in L_p(E;H)$ and

$$(4.27) \qquad \lim_{m\to\infty} \|u_m-u\|_{p,E} = 0.$$

Using inequality (4.23) for functions $u_m$ and relations (4.26), (4.27) we conclude that (4.23) holds true for the function $u$ as well. This completes the proof.

Lemma 4.19. If $u \in \mathring{W}_2^{1,0}(G_T;H)$, then $u(\cdot,t) \in \mathring{W}_2^1(G;H)$ for almost all $t \in \langle 0,T\rangle$.

Proof. One can easily find that for any function $u \in \mathring{W}_2^{1,0}(G_T;H)$ we have

(4.28) $$u(\cdot,t),\ u_{x_i}(\cdot,t) \in L_2(G;H), \quad i=1,\dots,n$$

for almost all $t \in \langle 0,T\rangle$.

Now let $u \in \overset{\circ}{W}{}_2^{1,0}(G_T;H)$. Then, by Definition 1.2, there exists a sequence $u_m \in C_0^\infty(\bar{G}_T;H)$, $m=1,2,\dots$ such that

$$\lim_{m\to\infty} \int_0^T dt \int_G |u_m(x,t)-u(x,t)|^2 dx = 0,$$

$$\lim_{m\to\infty} \int_0^T dt \int_G |u_{mx_i}(x,t) - u_{x_i}(x,t)|^2 dx = 0, \quad i=1,\dots,n.$$

Hence, by Theorem XII.2.5 of [9], it follows the existence of a subsequence $(u_{m_s})$ such that for almost all $t \in \langle 0,T\rangle$ we have

(4.29) $$\lim_{s\to\infty} \|u_{m_s}(\cdot,t) - u(\cdot,t)\|_{2,G} = 0,$$

(4.30) $$\lim_{s\to\infty} \|u_{m_s x_i}(\cdot,t) - u_{x_i}(\cdot,t)\|_{2,G} = 0, \quad i=1,\dots,n.$$

Therefore, by (4.28) and Lemma 4.13, the functions $u_{x_i}(\cdot,t)$ are weak derivatives of the function $u(\cdot,t)$ for almost all $t \in \langle 0,T\rangle$. Moreover, relations (4.28)-(4.30) imply the assertion of the lemma.

Proceeding as in [5], p.89 and using Lemmas 4.18 and 4.19 one can prove the following lemma.

L e m m a 4.20. If $u \in \overset{\circ}{V}_2(G_T;H)$, then $u \in L_{q,r}(G_T;H)$, where $\frac{1}{r} + \frac{n}{2q} = \frac{n}{4}$,

$$r \in \langle 2,\infty\rangle, \quad q \in \langle 2, 2n(n-2)^{-1}\rangle \quad \text{if } n>2,$$

$$r \in (2,\infty\rangle, \quad q \in \langle 2,\infty) \quad \text{if } n=2,$$

$$r \in \langle 4,\infty\rangle, \quad q \in \langle 2,\infty\rangle \quad \text{if } n=1.$$

Moreover,

$$\|u\|_{q,r,G_T} \leqslant \beta \|u\|_{2,G_T},$$

$\beta$ being a positive constant depending only on $n$ and $q$.

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