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THE SUM OF A DOUBLE SYSTEM OF GRAPHS

By a graph we shall mean a symmetrical graph i.e. $G = (V,R)$, where V is a non-empty finite set and R is a binary relation on V satisfying the following condition

$$\bigvee_{u,v\in V} \left[u\ R\ v \Rightarrow v\ R\ u \right].$$

We admit loops in G. Let $G_1 = (V_1,R_1)$ and $G_2 = (V_2,R_2)$ be graphs.

D e f i n i t i o n . A mapping φ of the graph G_1 into the graph G_2 is said to be a homomorphism if

$$\bigvee_{a,b\in V_1} \left[aR_1b \Rightarrow \varphi(a)\ R_2\varphi(b) \right].$$

The mapping φ will be called a strong homomorphism if

$$\bigvee_{a,b\in V_1} \left[aR_1b \Leftrightarrow \varphi(a)\ R_2\varphi(b) \right].$$

A quadruple $\mathcal{A} = (G_1,G_2,\varphi,\psi)$ will be called a double system of graphs $G_1 = (V_1,R_1)$ and $G_2 = (V_2,R_2)$ if φ is a strong homomorphism of G_1 into G_2, ψ is a strong homomorphism of G_2 into G_1 and $V_1 \cap V_2 = \emptyset$.

For universal algebras similar systems were introduced by E.Graczyńska in [1] . Some general constructions of this kind for relational systems were introduced by J.Płonka in [3].

A.Koźliński in [2] considered also 2-component systems of graphs, however, he used not strong but simple homomorphisms. We construct a family of graphs in the following way. We define a mapping $h: V_1 \cup V_2 \to V_2$ putting $h(v) = \varphi(v)$ for $v \in V_1$ and $h(v) = v$ if $v \in V_2$. Further we define a mapping $g: V_1 \cup V_2 \to V_1$, putting $g(u) = \psi(u)$ for $u \in V_2$ and $g(u) = u$ if $u \in V_1$.

Let $f(x_1,x_2)$ be a boolean formula constructed from the variables x_1 and x_2 by means of logical functors $\wedge, \vee, \sim$. For any such formula f we can construct a graph G_f putting $G_f(\mathcal{A}) = (V_1 \cup V_2, R_f)$, where

$$\forall_{a,b \in V_1 \cup V_2} \left[aR_f b \Leftrightarrow f(h(a)R_2h(b), g(a)R_1g(b)) \right].$$

The graph $G_f(\mathcal{A})$ is said to be the sum of a double system of the graphs G_1 and G_2 according to the formula f. We say that a graph $G = (V,R)$ is the sum of a double system of graphs $G_1 = (V_1,R_1)$ and $G_2 = (V_2,R_2)$ according to the formula f if $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ and if there exists a strong homomorphism of G_1 into G_2 and a strong homomorphism of G_2 into G_1 such that $G = G_f(\mathcal{A})$, where $\mathcal{A} = (G_1,G_2,h,g)$.

For example, the graph G in the figure 1 is the sum of a double system of graphs $G_1 = (V_1,R_1)$ and $G_2 = (V_2,R_2)$ according to the formula $f_1(x_1,x_2) = x_1 \vee x_2$, where $V_1 = \{a,b\}$, $V_2 = \{c,d\}$, $R_1 = \{(a,b),(b,a)\}$, $R_2 = \{(c,d),d,c)\}$ $h(a) = c$, $h(b) = d$, $g(c) = b$, $g(d) = a$. Observe that if f_1

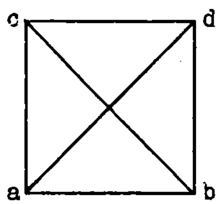

Fig.1

is equivalent to f_2, then $G_{f_1} = G_{f_2}$ (the sets of vertices of G_{f_1} and G_{f_2} are equal and $R_{f_1} = R_{f_2}$). Let $[f(x_1,x_2)]$ be the class of formulas equivalent to $f(x_1,x_2)$. Put $\chi([f]) = G_f$. It follows from the last observation that χ is well defined. Denote

$$G_{[f]} = G_f$$

hence

$$\chi([f]) = G_{[f]}.$$

Let $\mathcal{B}(x_1,x_2) = (F, \wedge, \vee, \sim)$ be a free Boolean algebra on generators x_1 and x_2 (see [4]).

So F consists of 16 classes of equivalent formulas, namely

$$[x_1 \wedge x_2],\ [x_1 \wedge \sim x_2],\ [\sim x_1 \wedge x_2],\ [\sim x_1 \wedge \sim x_2],$$

$$[x_1 \wedge x_2 \vee x_1 \wedge \sim x_2],\ [x_1 \wedge x_2 \vee \sim x_1 \wedge x_2],$$

$$[x_1 \wedge x_2 \vee \sim x_1 \wedge \sim x_2],\ [x_1 \wedge \sim x_2 \vee \sim x_1 \wedge x_2],$$

$$[x_1 \wedge \sim x_2 \vee \sim x_1 \wedge \sim x_2],\ [\sim x_1 \wedge x_2 \vee \sim x_1 \wedge \sim x_2],$$

$$[x_1 \wedge x_2 \vee x_1 \wedge \sim x_2 \vee \sim x_1 \wedge x_2],$$

$$[x_1 \wedge x_2 \vee x_1 \wedge \sim x_2 \vee \sim x_1 \wedge \sim x_2],$$

$$[x_1 \wedge x_2 \vee \sim x_1 \wedge x_2 \vee \sim x_1 \wedge \sim x_2],$$

$$[x_1 \wedge \sim x_2 \vee \sim x_1 \wedge x_2 \vee \sim x_1 \wedge \sim x_2],$$

$$[x_1 \wedge x_2 \vee x_1 \wedge \sim x_2 \vee \sim x_1 \wedge x_2 \vee \sim x_1 \wedge \sim x_2],$$

$$[x_1 \wedge \sim x_1 \vee x_2 \wedge \sim x_2].$$

Moreover, for any $[f_1], [f_2] \in F$ we have

$$[f_1] \wedge [f_2] = [f_1 \wedge f_2],\ [f_1] \vee [f_2] = [f_1 \vee f_2],\ \sim[f_1] = [\sim f_1].$$

Let $\mathcal{G} = \{ G_{[f]} : [f] \in F \}$.

We form an algebra $\mathcal{D} = (\mathcal{G}, \cap, \cup, ')$, where if $G_1 = (V_1, R_1)$, $G_2 = (V_1, R_2)$ are graphs, then

$G_1 \cap G_2 = (V_1, R_1 \cap R_2)$, $G_1 \cup G_2 = (V_1, R_1 \cup R_2)$, $G_1' = (V_1, V_1 \times V_1 \setminus R_1)$.

T h e o r e m . The mapping χ is a homomorphism of the algebra $\mathcal{B}$ onto the algebra $\mathcal{D}$.

P r o o f . Obviously χ is a function onto. We show that χ is a homomorphism. Let $[f_1]$ and $[f_2] \in F$, we have

$$\chi([f_1] \vee [f_2]) = \chi([f_1 \vee f_2]) = G_{f_1 \vee f_2} = G_{[f_1 \vee f_2]} =$$

$$= \chi([f_1]) \cup \chi([f_2]) = G_{[f_1]} \cup G_{[f_2]} = (V, R_{f_1}) \cup (V, R_{f_2}) =$$

$$= (V, R_{f_1} \cup R_{f_2}) = (V, R_{f_1 \vee f_2}) = G_{f_1 \vee f_2} = G_{[f_1 \vee f_2]}.$$

Similarly

$$\chi([f_1] \wedge [f_2]) = \chi([f_1 \wedge f_2]) = G_{f_1 \wedge f_2} = G_{[f_1 \wedge f_2]} =$$

$$= \chi([f_1]) \cap \chi([f_2]) = G_{[f_1]} \cap G_{[f_2]} = (V, R_{f_1}) \cap (V, R_{f_2}) =$$

$$= (V, R_{f_1} \cap R_{f_2}) = (V, R_{f_1 \wedge f_2}) = G_{f_1 \wedge f_2} = G_{[f_1 \wedge f_2]}$$

and

$$\chi(\sim[f]) = \chi([\sim f]) = G_{\sim f} = G_{[\sim f]},$$

$$[\chi([f])]' = (G_f)' = (V, R_f)' = (V, V \times V \setminus R_f) = (V, R_{\sim f}) = G_{\sim f} = G_{[\sim f]}.$$

So

$$\chi([f_1] \vee [f_2]) = \chi([f_1]) \cup \chi([f_2]),$$

$$\chi([f_1] \wedge [f_2]) = \chi([f_1]) \cap \chi([f_2]) \quad \text{and}$$

$$\chi(\sim[f]) = [\chi([f])]', \quad \text{concluding the proof.}$$

C o r o l l a r y 1. The algebra $\mathcal{D} = (\mathcal{G}, \cap, \cup, ')$ is a Boolean algebra.

C o r o l l a r y 2. The graphs $G_{[x_1 \wedge x_2 \vee x_1 \wedge \sim x_2]}$, $G_{[x_1 \wedge x_2 \vee \sim x_1 \wedge x_2]}$ are generators of the algebra $\mathcal{D}$.

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