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REMARKS ON CONVERGENCE OF SEQUENCES OF POINT-TO-SET MAPS

1. In this paper two definitions about convergence of sequences of point-to-set maps introduced in [2] are confronted with well known definitions of convergence for sequences of point-to-point maps.

In [2] we considered sequences of maps which assigned to each point from topological space X exactly one subset of topological space Y and the maps were treated as point-to-set maps. The same maps in this paper are regarded as point-to-point maps, introducing the Vietoris topology or the Hausdorff metric in the family of subsets of space Y.

The results of the paper refer to connections between these two approaches to the idea of convergence of maps.

We shall make use of definitions, notions and some of the theorems included in [2].

2. We shall assume that both X and Y are topological spaces fulfilling the first axiom of countability. We shall denote by 2^X the family of all non-empty and closed subsets of the space X. We shall use the Vietoris topology in those sets. Introducing the Vietoris topology in 2^X we take as its subbase the following families of sets: 2^G and $2^X \setminus 2^{X \setminus G}$, where G is an open subset of X. We introduce the Vietoris topology in 2^Y in the same way. In that manner, we can consider each map $F: X \to 2^Y$ as a point-to-point map.

Lemmas 1 and 2 will play an important role in further considerations. They are counterparts of analogical theorems given in [3] for the case of metric spaces, for topological spaces.

L e m m a 1. Let Z be a regular topological space and let $A_n, A \in 2^Z$. If the sequence of sets $\{A_n\}$ is convergent to the set A in the space 2^Z, then $\mathrm{Lt}\, A_n = A$.

L e m m a 2. Let Z be a compact topological space and let $A_n, A \in 2^Z$. If $\mathrm{Lt}\, A_n = A$, then the sequence $\{A_n\}$ is convergent to A in the space 2^Z.

C o r o l l a r y 1. Let Y be a regular space. If for each $x \in X$ a sequence of sets $\{F_n(x)\}$ is convergent to a set $F(x)$ in the space 2^Y, then $F_n \longrightarrow F$.

C o r o l l a r y 2. Let Y be a compact space. If $F_n \longrightarrow F$, then for arbitrary $x \in X$ the sequence $\{F_n(x)\}$ is convergent to $F(x)$ in the space 2^Y.

These corollaries are the direct results of Lemmas 1 and 2.

From Corollaries 1 and 2 there follows

T h e o r e m 1. Let Y be a Hausdorff space. Then $F_n \longrightarrow F$ if and only if for each $x \in X$ the sequence $\{F_n(x)\}$ is convergent to the set $F(x)$ in the space 2^Y.

Lemmas 1 and 2 and Corollary 2 from [2] give the following two corollaries.

C o r o l l a r y 3. Let Y be a regular space. If a sequence of maps $\{F_n\}$, $F_n : X \longrightarrow 2^Y$ treated as point-to-point maps is continuously convergent to a map F, then $F_n \rightrightarrows F$.

C o r o l l a r y 4. Let Y be a compact space. If $F_n \rightrightarrows F$, then the sequence of maps $\{F_n\}$ treated as point-to-point maps is continuously convergent to the map F.

From Corollaries 3 and 4 we obtain

T h e o r e m 2. Let Y be a compact Hausdorff space. Then $F_n \rightrightarrows F$ if and only if the sequence of maps $\{F_n\}$ treated as point-to-point maps is continuously convergent to the map F.

Now, let $F_n : 2^X \longrightarrow 2^Y$. If X is a T_1 space, then we can treat the maps F_n as point-to-point maps from 2^X to 2^Y and as point-to-set maps from X to 2^Y.

The following corollary is the result of Corollary 3.

C o r o l l a r y 5. Let X be a T_1 space and Y a regular space. If a sequence of maps $\{F_n\}$ treated as point--to-point maps from 2^X to 2^Y is continuously convergent to a map F, then $F_n \rightrightarrows F$.

The following lemma quoted from [1] is necessary in order to formulate Theorem 3.

L e m m a 3. If a point-to-set map F from X to 2^Y is upper semicompaxt (u.s.c), then the image $F(K) = \bigcup_{x \in K} F(x)$ of a compact set $K \subset X$ is a compact set.

Now, let both X and Y be compact Hausdorff spaces and let $\{F_n\}$ be a sequence of point-to-set maps, which are u.s.c. Due to Lemma 3 we can then consider point-to-point maps $\hat{F}_n : 2^X \longrightarrow 2^Y$ given by the formula

$$\hat{F}_n(A) = \bigcup_{x \in A} F_n(x) \quad \text{for} \quad A \in 2^X \text{ and } n=1,2,\ldots$$

From Corollary 3 in [2] it follows that if $F_n \rightrightarrows F$, then F is u.s.c. So we can consider a point-to-point map $\hat{F}$ defined by F.

T h e o r e m 3. Let us assume that both X and Y are compact Hausdorff spaces. Moreover, each map of the sequence $\{F_n\}$ is u.s.c. Then $F_n \rightrightarrows F$ is and only if the sequence of maps $\{\hat{F}_n\}$ treated as point-to-point maps is continuously convergent to the map $\hat{F}$.

P r o o f . Sufficiency follows directly from Corollary 5. Necessity. Suppose that $F_n \rightrightarrows F$, but the sequence $\{\hat{F}_n\}$, $\hat{F}_n : 2^X \longrightarrow 2^Y$ is not continuously convergent to $\hat{F}$. It means that there exists a sequence of sets $\{A_n\}$ and a set A such that A_n, $A \in 2^X$, $\{A_n\}$ is convergent to A in the space 2^X and $\{\hat{F}_n(A_n)\}$ is not convergent to $\hat{F}(A)$ in the space 2^Y. By definition of topology in 2^Y it follows that there exist open sets $U_1, U_2, \ldots, U_m$, $U_i \subset Y$, $i = 1,2,\ldots,m$ such that

$$\hat{F}(A) \cap U_i \neq \emptyset \quad \text{and} \quad \hat{F}(A) \subset \bigcup_{i=1}^{m} U_i$$

and for an infinite number of n

$$\hat{F}_n(A_n) \notin K(U_1,\dots,U_m) =$$

$$= \left\{ B \in 2^Y : B \cap U_i \neq \emptyset \ \text{ for } \ i = 1,2,\dots,m \ \text{ and } \ B \subset \bigcup_{i=1}^{m} U_i \right\}$$

Thus, we obtain that

(a) there is an index i_0 such that for an infinite number of n

$$\hat{F}_n(A_n) \cap U_{i_0} = \emptyset$$

or

(b) for an infinite number of n

$$\hat{F}_n(A_n) \not\subset \bigcup_{i=1}^{m} U_i.$$

Choosing appropriate subsequences, if needed, we can assume that (a) holds for all indices n or that (b) holds for all indices n. In case (a), by $\hat{F}(A) \cap U_{i_0} \neq \emptyset$, it follows that there is a point $x_0 \in A$ such that

(1) $$F(x_0) \cap U_{i_0} \neq \emptyset.$$

From the fact that the sequence $\{A_n\}$ is convergent to A in 2^X and from Lemma 1 and Definition 7 in [2] it follows that there exists a sequence $\{x_n\}$ such that $x_n \in A_n$ and $x_n \to x_0$.

By (a) we have

(2) $$F_n(x_n) \cap U_{i_0} = \emptyset.$$

Since $\mathrm{Lt}\, F_n(x_n) = F(x_0)$, it follows from (1) that for $n > N$ $F_n(x_n) \cap U_i \neq \emptyset$ which contradicts (2) and thus case (a) do not hold, so case (b) holds. Thus, there are sequences $\{x_n\}$ and $\{y_n\}$ such that

$$(3) \qquad x_n \in A_n,\ y_n \in F_n(x_n) \ \text{ and } \ y_n \notin \bigcup_{i=1}^{m} U_i.$$

By the compactness of the spaces X and Y it follows that there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$, of the sequences $\{x_n\}$ and $\{y_n\}$ respectively, and points $x_0 \in X$ and $y_0 \in Y$ such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$.

As the sequence $\{A_{n_k}\}$ is convergent to A (in 2^X), so by Lemma 1 and Definition 6 in [2] it follows that $x_0 \in A$. Moreover, by assumption, we have $\mathrm{Lt}\, F_{n_k}(x_{n_k}) = F(x_0)$, hence $y_0 \in F(x_0)$, which contradicts (3) because $\hat{F}(a) \subset \bigcup_{i=1}^{m} U_i$.

Therefore, neither (a) nor (b) holds and the proof is concluded.

3. Now, let us assume that both X and Y are metric spaces. We shall denote by ϱ a metric in the space Y. Let $\mathcal{B}(Y)$ be the family of all non-empty, closed and bounded subsets of the space Y. We introduce in $\mathcal{B}(Y)$ the Hausdorff metric dist letting $\mathrm{dist}(C,D) = \max\left[\sup_{y\in C} d(y,D),\ \sup_{z\in D} d(z,C)\right]$ for $C,D \in \mathcal{B}(Y)$ where $d(y,D) = \inf_{z\in D} \varrho(y,z)$ (see [3], [4], [5]), For each $A \subset Y$ and $\varepsilon > 0$, we let $S(A,\varepsilon) = \{y \in Y: d(y,A) < \varepsilon\}$. Let F_n, F be point-to-set maps such that $F_n, F: X \to \mathcal{B}(Y)$ $n = 1,2,\ldots$ Since $\mathcal{B}(Y)$ is a metric space, so we can consider F_n and F as point-to-point maps from X to $\mathcal{B}(Y)$.

Lemma 4. If a sequence of maps $\{F_n\}$, $F_n: X \to B(Y)$ is uniformly convergent to a map F in the Hausdorff metric, then the following properties take place:

(i) for arbitrary $\varepsilon > 0$ there is an index N such that for $n > N$ and $x \in X$ $F_n(x) \subset S(F(x),\varepsilon)$

(ii) for arbitrary $\varepsilon > 0$ there is an index N such that for $n > N$ and $x \in X$ $F(x) \subset S(F_n(x),\varepsilon)$

(iii) $F_n \longrightarrow F$.

P r o o f . Properties (i) and (ii) are evident.

To prove (iii) we shall show that $F^*(x) \subset F(x) \subset F_*(x)$ for each $x \in X$ (see [2]). Let $y \in F^*(x)$. By the definition of F^* it follows that there is a sequence $\{y_n\}$, $y_n \in F_n(x)$ such that for a certain subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ we have $y_{n_k} \longrightarrow y$. By assumption, $d(y_{n_k},F(x)) \longrightarrow 0$, so $d(y,F(x)) = 0$. So, due to closity of the set $F(x)$, $y \in F(x)$. In order to show the second inclusion it is worth noticing that, by assumption, $d(z,F_n(x)) \longrightarrow 0$ for $z \in F(x)$. Thus, there is a sequence $\{y_n\}$, $y_n \in F_n(x)$ fulfilling the condition $\varrho(z,y_n) < \frac{1}{n}$. From this it follows that $y_n \to z$, so $z \in F_*(x)$. Therefore, $F^*(x) \subset F(x) \subset F_*(x)$ and the proof is concluded.

From Theorems 8 and 9 in [2] and from Lemma 4 we directly achieve the following corollaries

C o r o l l a r y 6. If maps F_n are lower semicontinuous (l.s.c) and a sequence $\{F_n\}$ is uniformly convergent to a map F, then F is l.s.c.

C o r o l l a r y 7. If Y is a compact space, maps F_n are u.s.c and the sequence $\{F_n\}$ is uniformly convergent to a map F, then F is u.s.c.

C o r o l l a r y 8. If Y is a compact space and a sequence of continuous maps $\{F_n\}$ is uniformly convergent to a map F, then F is a continuous map.

It is well known that if Y is a compact metric space, then the space 2^Y with the Vietoris topology is compact and its topology is equivalent with topology that is generated by the Hausdorff metric.Due to well known theorem concerning the continuous and the uniform convergence and due to Theorem 3 and Corollary 3 in [2] we obtain

C o r o l l a r y 9. Let both X and Y be compact metric spaces and let $\{F_n\}$ be a sequence of continuous maps. Then $F_n \rightrightarrows F$ if and only if the sequence $\{\hat{F}_n\}$ is uniformly convergent to $\hat{F}$ in the Hausdorff metric.

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