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CHARACTERIZATION OF POLYNOMIALS IN ALGEBRAIC OPERATORS WITH CONSTANT COEFFICIENTS

In this paper a characterization of polynomials in algebraic operators with constant coefficients is given. We solve two types of operator equations which will be called linear algebraic equations, namely equations of the form

$$P(X) = Y \quad \text{and} \quad P(A)X - XQ(B) = Y,$$

where $P(t)$ and $Q(t)$ are polynomials; A and B are algebraic elements.

1. Let X be an algebra (a linear ring) with unit I over the field of complex numbers. Let A be an algebraic element in X with the characteristic polynomial

$$(1.1) \qquad P_A(t) = \prod_{j=1}^{n} (t-t_j)^{\vartheta_j}, \quad t_i \neq t_j, \quad i \neq j,$$
$$\vartheta_1 + \vartheta_2 + \ldots + \vartheta_n = N.$$

(cf. [1]).

The element A has the following properties important in our further considerations:

Proposition 1.1. Let A be an algebraic element with the characteristic polynomial (1.1) and let $Q_i(t)$ be a polynomial in variable t with complex coefficients, satisfying the condition

(1.2) $$Q_i(t_i) \neq 0.$$

If $P(t) = (t-t_i)^{\alpha_i} Q_i(t)$, $0 \leqslant \alpha_i < \nu_i$, then $P(A) \neq 0$.

P r o o f . Suppose that there exists a polynomial $Q_{j_0}(t)$ and an integer α_{j_0} in the interval $0 \leqslant \alpha_{j_0} < \nu_{j_0}$ such that

(1.3) $$P(t) = (t-t_{j_0})^{\nu_{j_0}} Q_{j_0}(t) \quad \text{and} \quad P(A) = 0,$$

Without loss of generality, we can admit

$$Q_{j_0}(t) = \prod_{j \neq j_0} (t-t_j)^{\nu'_j} Q(t)$$

where $Q(t_j) \neq 0$, $j = 1,2,\dots,n$, $\nu'_j > \nu_j$.

Thus the element $[Q(A)]^{-1}$ exists (cf. [1]). Thus, $P(A) = 0$ if and only if $P_1(A) = 0$, where

$$P_1(t) = (t-t_{j_0})^{\alpha_{j_0}} \prod_{j \neq j_0} (t-t_j)^{\nu_j}.$$

Consider the polynomial

$$P_2(t) = P_1(t) + P_A(t).$$

We know from the above discussion that

$$P_2(A) = 0,$$

$$P_2(t) = (t-t_{j_0})^{\alpha_{j_0}} \prod_{j \neq j_0} (t-t_j)^{\nu_j} P_3(t),$$

where

$$P_3(t) = (t-t_{j_0})^{\nu_{j_0} - \alpha_{j_0}} + \prod_{j \neq j_0} (t-t_j)^{\nu'_j - \nu_j}.$$

For each $j. 1 \leq t_j \leq n$ we have $P_3(t_j) \neq 0$. Hence $P_3(A)$ is invertible. Thus $P_2(A) = 0$ if and only if

$$P_4(A) = 0$$

where

$$P_4(t) = (t-t_{j_0})^{\alpha_{j_0}} \prod_{j \neq j_0} (t-t_j)^{\nu_j}.$$

On the other hand, $\deg P_4(t) < \deg P_A(t)$, which contradicts our assumption.

L e m m a 1.1. Let A be an algebraic element with the characteristic polynomial (1.1) and let $G(t) = g_0 t^s + g_1 t^{s-1} + \ldots + g_s$ be a polynomial of degree s in t, satisfying the conditions

$$(1.4) \quad \begin{cases} G(t_i) \neq G(t_j), & i \neq j \\ G'(t_j) \neq 0, & i,j = 1,2,\ldots,n. \end{cases}$$

If $V = G(A)$ then

$$(1.5) \qquad P_V(t) = \prod_{j=1}^{n} (t - G(t_j))^{\nu_j}.$$

P r o o f . Write $P(t) = \prod_{j=1}^{n} [t - G(t_j)]^{\nu_j}$. Then

$$P(V) = \prod_{j=1}^{n} [G(A) - G(t_j)]^{\nu_j} = \prod_{j=1}^{n} (A - t_j I)^{\nu_j} \prod_{j=1}^{n} [G(A,t_j)]^{\nu_j},$$

where

$$(1.6) \quad \begin{cases} G(t,t_j) = g_0\sigma_{s-1}(t,t_j) + g_1\sigma_{s-2}(t,t_j) + \ldots + g_{s-1}\sigma_k(t,t_j) \\ \sigma_k(t,t_j) = t^k + t_j t^{k-1} + \ldots + t_j^k. \end{cases}$$

Thus the characteristic roots of the element $V = G(A)$ are $G(t_1), G(t_2), \dots, G(t_n)$.

Let $Q(t) = \left[(t - G(t_{j_0})\right]^{\alpha_{j_0}} Q_1(t)$ and $Q(V) = 0$, where $Q_1\left[G(t_{j_0})\right] \neq 0$, $\alpha_{j_0} \leqslant \nu_{j_0}$, j_0 fixed, $0 \leqslant j_0 \leqslant n$.

Without loss of generality we can admit

$$Q(t) = \left[t - G(t_{j_0})\right]^{\alpha_{j_0}} \prod_{j \neq j_0} [t - G(t_j)]^{\nu'_j}, \quad \nu'_j > \nu_j$$

then

$$Q(V) = \left[G(A) - G(t_{j_0})I\right]^{\alpha_{j_0}} \prod_{j \neq j_0} \left\{[A - t_j I] G(A, t_j)\right\}^{\nu'_j} =$$

$$= (A - t_{j_0} I)^{\alpha_{j_0}} \prod_{j \neq j_0} [A - t_j I)^{\nu_j} \left[G(A, t_{j_0})\right]^{\alpha_{j_0}} \prod_{j \neq j_0} [G(A, t_j)]^{\nu'_j}$$

where $G(t, t_j)$ are given by formula (1.6).

We put

$$G_1(t) = \left[G(t, t_{j_0})\right]^{\alpha_{j_0}} \prod_{j \neq j_0} [G(t, t_j)]^{\nu'_j}.$$

We shall show that

$$G_1(t_\mu) \neq 0, \quad \mu = 1, 2, \dots, n. \tag{1.7}$$

The proofs are based on the identity

$$\sigma_k(t_\mu, t_j) = \begin{cases} \dfrac{t_\mu^{k+1} - t_j^{k+1}}{t_\mu - t_j} & \text{when } \mu \neq j \\ (k+1) t_j & \text{when } \mu = j. \end{cases}$$

By (1.4)

$$G(t_\mu, t_j) \neq 0 \quad \text{if} \quad i \neq j$$

and

$$G(t_j, t_j) = G'(t_j) \neq 0,$$

therefore (1.7) holds.

Thus the element $G_1(A)$ is invertible and $Q(V) = 0$ (or $\neq 0$) if and only if

$$(A - t_{j_0} I)^{\alpha_{j_0}} \prod_{j \neq j_0} (A - t_j I)^{\nu_j} = 0 \quad \text{(respectively } \neq 0\text{)}.$$

By Proposition 1.1 this implies that $Q(V) = 0$, which contradicts our assumption.

L e m m a 1.2. Let A be an algebraic element with the characteristic polynomial (1.1). Let $G(t) = g_0 t^s + g_1 t^{s-1} + \dots + g_s$ be a polynomial, satisfying the conditions

$$(1.8) \qquad G(t_i) \neq G(t_j), \quad i \neq j$$

$$(1.9) \qquad G'(t_j) = \dots = G^{(s_j)}(t_j) = 0, \; G^{(s_j+1)}(t_j) \neq 0$$

$$j = 1, 2, \dots, n.$$

If $V = G(A)$, then

$$(1.10) \qquad P_V(t) = \prod_{j=1}^{n} [t - G(t_j)]^{\delta_j},$$

where

$$\delta_j = \begin{cases} \theta_j & \text{when } \theta_j \text{ is an integer} \\ [\theta_j]+1 & \text{otherwise} \end{cases}$$

and

$$\theta_j = \frac{\nu_j}{s_j + 1}, \quad j = 1,2,\dots,n.$$

P r o o f . The method of proof is similar to that of Lemma 1.1. From (1.8) - (1.9) we obtain

$$(1.11) \quad \begin{cases} G(t,t_j) = (t-t_j)^{s_j} G_j(t,t_j), & \text{where} \\ G_j(t_\mu,t_j) \neq 0, & \mu, j = 1,2,\dots,n. \end{cases}$$

Thus $G(A,t_j) = (A - t_jI)^{s_j} G_j(A,t_j)$, where elements $G_j(A,t_j)$ are invertible.

Write $P(t) = \prod_{j=1}^{n} [t - G(t_j)]^{\alpha_j}$. Then

$$P(V) = \prod_{j=1}^{n} [G(A) - G(t_j)I]^{\alpha_j} = \prod_{j=1}^{n} (A-t_jI)^{\alpha_j} \prod_{j=1}^{n} [G(A,t_j)]^{\alpha_j} =$$

$$= \prod_{j=1}^{n} (A - t_jI)^{(1+s_j)\alpha_j} \prod_{j=1}^{n} G_j(A,t_j)^{\alpha_j}.$$

By (1.11) the element $\prod_{j=1}^{n} [G_j(A,t_j)]^{\alpha_j}$ is invertible.

From the above discussion we conclude that $P(V) = 0$ if and only if

$$\prod_{j=1}^{n} (A - t_jI)^{(1+s_j)\nu_j} = 0.$$

By Proposition 1.1 we know that $(1+s_j)\alpha_j \geqslant \nu_j$; $j = 1,2,\dots,n$. Thus $\alpha_j \geqslant \delta_j$, $j = 1,2,\dots,n$. This implies formula (1.10).

C o r o l l a r y 1.1. Let $G(t_i) \neq G(t_j)$; $i \neq j$. Then

$$n \leqslant \deg P_V(t) \leqslant N, \quad V = G(A)$$

and

$$\deg P_V(t) = n \quad \text{if only if} \quad s_j+1 \geqslant \nu_j$$

$$\deg P_V(t) = N \quad \text{when} \quad \nu_j = 1,\ j = 1,2,\dots,n \quad \text{on } G'(t_j) \neq 0,\ j = 1,2,\dots,n.$$

L e m m a 1.3. Let A be an algebraic element with the characteristic polynomial (1.1) and let

$$G(t) = g_0 t^s + g_1 t^{s-1} + \dots + g_s$$

be a polynomial satisfying the conditions

$$(1.12)\qquad \begin{cases} G(t_1) = G(t_n), \\ G(t_i) \neq G(t_j) \quad \text{when} \quad i \neq j \quad \text{and} \ (i,j) \neq (1,n) \\ G'(t_j) \neq 0, \quad j = 1,2,\dots,n. \end{cases}$$

If $V = G(A)$ then

$$P_V(t) = [t - G(t_1)]^{\alpha_1} \prod_{j=2}^{n-1} [t - G(t_j)]^{\nu_j}$$

where $\alpha_1 = \max(\nu_1, \nu_n)$.

P r o o f . According to the assumption (1.12) we have

$$(1.13)\qquad G(t) - G(t_1) = (t - t_1)(t - t_n) G(t, t_1, t_n)$$

where

$$G(t,t_1,t_n) = g_0\sigma_{s-2}(t,t_1,t_n) + g_1\sigma_{s-3}(t,t_1,t_n) + \dots +$$

$$+ g_{s-2}\sigma_k(t,t_1,t_n) = \sum_{\alpha+\beta+\gamma=k} t^{\alpha} t_1^{\beta} t_n^{\gamma} = \frac{\sigma_k(t,t_1)}{t_1 - t_n} - \frac{\sigma_k(t,t_n)}{t_1 - t_n},$$

$\sigma_k(t,t_j)$ given by the formula (1.6).

Let

$$P_1(t) = [t - G(t_1)]^{\alpha_1} \prod_{j=2}^{n-1} [t - G(t_j)]^{\nu_j}.$$

From (1.13) we obtain

$$P_1(V) = [(A-t_1I)(A-t_nI)]^{\alpha_1} \prod_{j=2}^{n-1} (A-t_jI)^{\nu_j} [G(A,t_1,t_n)]^{\alpha_1} \prod_{j=2}^{n-1} [G(A,t_j)]^{\nu_j}.$$

Thus $P_1(V) = 0$. **By Proposition 1.1 we know that,** for each $P_2(t)$ with $\deg P_2(t) < \deg P_1(t)$ we have $P_2(A) \neq 0$.

Without loss of generality we can admit

$$P_2(t) = [t-G(t_1)]^{\alpha'_1} \prod_{j=2}^{n-1} [t-G(t_j)]^{\alpha_j}, \quad \alpha_j \geq \nu_j, \quad \alpha'_1 < \alpha_1.$$

Let $P_2(V) = 0$. Hence

$$0 = [G(A) - G(t_1)]^{\alpha_1} \prod_{j=2}^{n-1} [G(A) - G(t_j)I]^{\alpha_j} =$$

$$= (A-t_1I)^{\alpha_1}(A-t_nI)^{\alpha'_1} \prod_{j=2}^{n-1} (A-t_jI)^{\alpha_j} [G(A,t_1,t_n)]^{\alpha'_1} \prod_{j=2}^{n-1} [G(A,t_j)]^{\alpha_j}.$$

When $1 < j < n$ and

$$G(t_j,t_1,t_n) = \frac{1}{t_1-t_n} [G(t_j,t_1)-G(t_j,t_n)] = \frac{G(t_j)-G(t_1)}{(t_j-t_1)(t_j-t_n)} \neq 0.$$

If $j = 1$, then

$$G(t_1,t_1,t_n) = \frac{1}{t_1-t_n} [G(t_1,t_1) - G(t_1,t_n)] = \frac{G'(t_1)}{t_1-t_n} \neq 0.$$

Similarly, if $j = n$, then

$$G(t_n, t_1, t_n) = \frac{G'(t_n)}{t_1 - t_n} \neq 0.$$

Thus $G(A, t_1, t_n)$ is invertible.

According to the proof of Lemma 1.1 we find that $G(A, t_j)$, $j = 2,3,\ldots,n-1$, are invertible. Hence $P_2(A) = 0$ if and only if

$$(A - t_1 I)^{\alpha'_1}(A - t_n I)^{\alpha'_1} \prod_{j=2}^{n-1} (A - t_j I)^{\alpha_j} = 0$$

which contradicts our assumption.

L e m m a 1.4. Let A be an algebraic element with the characteristic polynomial

$$P_A(t) = \prod_{j=1}^{n} (t - t_j)^{\vartheta_j}.$$

Let $G(t) = g_0 t^s + g_1 t^{s-1} + \ldots + g_s$ be a polynomial satisfying conditions:

$$\text{(1.14)} \quad \begin{aligned} &1)\ G(t_1) = G(t_n), \\ &2)\ G(t_1) \neq G(t_j) \neq G(t_i),\ i \neq j,\ i,j = 2,3,\ldots,n-1 \\ &3)\ G'(t_j) = \ldots = G^{(s_j)}(t_j) = 0,\ G^{(s_j+1)}(t_j) \neq 0, \\ &\qquad\qquad j = 1,2,\ldots,n. \end{aligned}$$

If $V = G(A)$, then

$$\text{(1.15)} \qquad P_V(t) = [t - G(t_1)]^{\alpha_1} \prod_{j=2}^{n-1} [t - G(t_j)]^{\delta_j},$$

where

$$\alpha_1 = \begin{cases} \theta_1 & \text{when } \theta_1 \text{ is an integer} \\ [\theta_1]+1 & \text{when } \theta_1 \text{ is not an integer} \end{cases}$$

$$\theta_1 = \max\left(\frac{\nu_1}{s_1+1}, \frac{\nu_n}{s_n+1}\right)$$

and

(1.16) $$\delta_j = \begin{cases} \theta_j & \text{when } \theta_j \text{ is an integer} \\ [\theta_1]+1 & \text{when } \theta_j \text{ otherwise} \end{cases}$$

$$j = 2,3,\dots,n-1,$$

where

$$\theta_j = \frac{\nu_j}{s_j+1}.$$

P r o o f . According to Lemma 1.2, if $j \in \{2,3,\dots \dots,n-1\}$, then δ_j is given by the formula (1.16).

By Proposition 1.1 there is an integer α such that $P(V) = 0$ where

$$P(t) = [t - G(t_1)]^{\alpha} \prod_{j=2}^{n-1} [t - G(t_j)]^{\delta_j}.$$

According to the proof of the Lemma 1.3, for $1 < j < n$

$$G(t_j, t_1, t_n) \neq 0$$

and from (1.14)

$$G(t, t_1, t_n) = (t - t_1)^{s_1}(t - t_n)^{s_n} G_1(t, t_1, t_n)$$

where

$$G_1(t_j, t_1, t_n) \neq 0, \quad j = 1,2,\dots,n.$$

Thus $G_1(A, t_1, t_n)$ is invertible. This implies that $P(V) = 0$ if and only if

$$[(A-t_1I)(A-t_nI)]^{\alpha}(A-t_1I)^{\alpha s_1}(A-t_nI)^{\alpha s_n}\prod_{j=2}^{n-1}(A-t_jI)^{\nu_j} = 0.$$

Hence α satisfies the conditions

$$\alpha+\alpha s_1 \geqslant \nu_1,$$
$$\alpha+\alpha s_n \geqslant \nu_n$$

and

$$\alpha \geqslant \frac{1}{1+s_n}, \qquad \alpha \geqslant \frac{n}{1+s_n}.$$

From this $\alpha = \alpha_1$ and the proof is complete.

For the general case we prove the following theorem.

Proposition 1.2. Let A be an algebraic element with the characteristic polynomial

$$(1.17) \qquad P_A(t) = \prod_{i=1}^{m} \prod_{j_i=1}^{n_i} (t - t_{ij_i})^{\nu_{ij_i}}, \quad t_{ij} \neq t_{\nu\mu},$$

whenever $(i,j) \neq (\nu,\mu)$.

Let $G(t)$ be a polynomial in the variable t with complex coefficients satisfying the conditions

$$(1.18) \quad \begin{cases} G(t_{kj_k}) = n_k, \quad k=1,2,\dots,m, \quad j_k=1,2,\dots,n_k, \\ G'(t_{kj_k}) = \dots = G^{(skj_k)}(t_{kj_k}) = 0, \quad k=1,2,\dots,m, \\ \qquad\qquad j_k=1,2,\dots,n_k, \\ G^{(s_{kj_k}+1)}(t_{kj_k}) \neq 0. \end{cases}$$

If $V = G(A)$, then

$$(1.19) \qquad P_V(t) = \prod_{i=1}^{m} (t - r_i)^{\delta_i}$$

where

$$(1.20) \qquad \delta_i = \begin{cases} \alpha_i & \text{when } \alpha_i \text{ is an integer,} \\ [\alpha_i]+1 & \text{otherwise,} \end{cases}$$

where $\alpha_i = \max\left[\frac{\nu_{i1}}{s_{i1}+1}, \frac{\nu_{i2}}{s_{i2}+1}, \ldots, \frac{\nu_{in_i}}{s_{in_i}+1}\right]$.

P r o o f . By the hypothesis (1.18) we obtain the characteristic roots of V are $r_1, r_2, \ldots, r_m$. Hence the characteristic polynomial of V is a polynomial of the form

$$P_V(t) = \prod_{i=1}^{m} (t - r_i)^{\delta'_i}.$$

According to Lemma 1.4 and Proposition 1.1 we have

$$\begin{aligned} &\delta'_i + \delta'_i s_{i1} \geqslant \nu_{i1}, \\ &\delta'_i + \delta'_i s_{i2} \geqslant \nu_{i2}, \qquad i = 1,2,\ldots,m \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\delta'_i + \delta'_i s_{in_i} \geqslant \nu_{in_i}. \end{aligned}$$

This implies

$$\begin{aligned} &\delta'_i \geqslant \frac{\nu_{i1}}{1+s_{i1}}, \\ &\delta'_i \geqslant \frac{\nu_{i2}}{1+s_{i2}}, \qquad i = 1,2,\ldots,m \\ &\vdots \\ &\delta'_i \geqslant \frac{\nu_{in_i}}{1+s_{in_i}}. \end{aligned}$$

Hence $\delta'_i = \delta_i$.

C o r o l l a r y 1.2. Let A be an algebraic element with single characteristic roots, i.e.

$$r_1 = r_2 = \ldots = r_n = 1$$

$$P_A(t) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} (t - t_{ij}).$$

Let $G(t)$ be a polynomial satisfying the conditions

$$G(t_{11}) = \dots = G(t_{1n_1}) = n_1,$$
$$G(t_{21}) = \dots = G(t_{2n_2}) = n_2,$$
$$\dots\dots\dots\dots\dots\dots$$
$$G(t_{m1}) = \dots = G(t_{mn_m}) = n_m.$$

If $V = G(A)$, then V is an algebraic element with single characteristic roots and

$$P_V(t) = \prod_{j=1}^{m} (t - t_j).$$

P r o o f . According to Proposition 1.2 $P_V(t)$ is a polynomial of the form

$$P_V(t) = \prod_{j=1}^{m} (t - r_j)^{\delta_j}.$$

From (1.20) we obtain $\delta_j = 1$, $j = 1,2,\dots,m$.

C o r o l l a r y 1.3. Let A be an algebraic element with single characteristic roots. Then there exists a polynomial $G_0(t)$ such that $V_0 = G_0(A)$ is an involution of order N.

P r o o f . Let $P_A(t) = \prod_{j=1}^{N} (t - t_j)$. Denote $\varepsilon = \exp \frac{2\pi i}{N}$ and we take

$$G_0(t) = \sum_{j=1}^{N} \prod_{\substack{\nu=1 \\ \nu \neq j}}^{N} \frac{\varepsilon^j (t - t_\nu)}{t_j - t_\nu} .$$

By the Lagrange interpolation formula

$$P_{V_0}(t) = \prod_{j=1}^{N} (t - \varepsilon^j) = t^{\bar{n}} - 1.$$

Hence $V_0^n = I$ and V_0 is an involution of order N.

Corollary 1.4. Let A be an algebraic element with the characteristic polynomial

$$P_A(t) = \prod_{j=1}^{n} (t - t_j)^{\nu_j}.$$

Let $G(t)$ be a polynomial satisfying the conditions

$$G(t_j) = n, \quad j = 1,2,\dots,n,$$
$$G'(t_j) = \dots = G^{(s_j)}(t_j) = 0, \quad G^{(s_j+1)}(t_j) \neq 0.$$

Then

$$P_V(t) = (t - r)^{\delta}, \quad V = G(A)$$

where

$$\delta = \max\left[\frac{\nu_1}{s_1 + 1}, \frac{\nu_2}{s_2 + 1}, \dots, \frac{\nu_n}{s_n + 1}\right].$$

The proof follows immediately from Proposition 1.2.

The corresponding results for an arbitrary function follow immediately by virtue of the Hermite interpolation formula.

Proposition 1.3. Let A be an algebraic element with the characteristic polynomial

$$P_A(t) = \prod_{i=1}^{m} \prod_{j_i=1}^{n_i} (t - t_{ij_i})^{\nu_{ij_i}}, \quad t_{ij} \neq t_{\nu\mu}, \ (i,j) \neq (\nu,\mu).$$

Let the function $g(t)$ has the $(\nu_{ij_i} - 1)$-th derivative in points t_{ij_i} $(i = 1,2,\dots,m,\ j = 1,2,\dots,n_i)$ and satisfies conditions

$$(1.21)\quad \begin{cases} g(t_{kj_k}) = n_k, \\ g'(t_{kj_i}) = \dots = g^{(s_{kj_k})}(t_{kj_k}) = 0 \\ g^{(s_{kj_k}+1)}(t_{kj_k}) \neq 0, \quad k=1,2,\dots,m, \ j_k=1,2,\dots,n_k, \end{cases}$$

where $0 \leqslant s_{kj_k} \leqslant \nu_{kj_k} - 1$.

If $U = g(A) \in X$, then

$$(1.22)\qquad P_U(t) = \prod_{i=1}^{m} (t - n_i)^{\delta_i}$$

where

$$\delta_i = \begin{cases} \alpha_i & \text{when } \alpha_i \text{ is an integer} \\ [\alpha_i]+1 & \text{otherwise} \end{cases}$$

where

$$\alpha_i = \max\left\{ \frac{\nu_{i1}}{s_{i1}+1}, \frac{\nu_{i2}}{s_{i2}+1}, \dots, \frac{\nu_{in_i}}{s_{in_i}+1} \right\}.$$

P r o o f . The Hermite interpolation formula (cf. [1]) and our assumptions together imply that there is a polynomial $G(t)$ such that

$$G(A) = g(A).$$

On the other hand, according to Proposition 1.2, we can admit in (1.20) without loss of generality that s_{ij} satisfy conditions

$$s_{ij_i} + 1 \leqslant \nu_{ij_i} \quad (i = 1,2,\dots,m, \ j_i = 1,2,\dots,n_i)$$

i.e. $s_{ij_i} \leq \nu_{ij_i} - 1$ (in the case where $s_{ij_i} > \nu_{ij_i} - 1$, we can admit $\frac{\nu_{ij_i}}{s_{ij_i}+1}$ equal 1).

2. Examples of applications

In this section we shall solve the equation

$$P(X) = V, \tag{2.1}$$

where $P(t)$ is a polynomial in the variable t with complex coefficients and the equation

$$AX - XB = C \tag{2.2}$$

in the case where A, B are algebraic operators.

The matrix equation (2.2) was solved by Rosenblum [3] (see also Bellman [4]). In the case where A and B are algebraic operators with simple characteristic roots, the equation (2.2) was solved by Przeworska-Rolewicz [1].

We generalize these results to a larger class of equations

$$f(A)X = Y \tag{2.3}$$

$$f(A)X - Xg(B) = C \tag{2.4}$$

in the case where f and g are polynomials with complex coefficients.

Write

$$\langle \theta \rangle = \begin{cases} \theta & \text{when } \theta \text{ is an integer} \\ [\theta]+1 & \text{otherwise.} \end{cases} \tag{2.5}$$

In the sequel we assume that $P(t)$ is a polynomial in variable t with complex coefficients.

Theorem 2.1. Let V be an algebraic element with the characteristic polynomial

$$P_V(t) = \prod_{j=1}^{m} (t - r_j)^{\delta_j}.$$

Let t_j, $j = 1,2,\ldots,m$, satisfy the equations

$$P(t_j) = r_j,$$
$$P'(t_j) = \ldots = P^{(s_j)}(t_j) = 0,$$
$$P^{(s_j+1)}(t_j) \neq 0.$$

Then the solution of the equation

$$P(X) = V$$

is an operator with the characteristic polynomial of the form

$$P_X(t) = \prod_{j=1}^{m} (t - t_j)^{\nu_j} \tag{2.6}$$

where ν_j is the smallest number θ, for which

$$\left\langle \frac{\theta}{s_j + 1} \right\rangle = \delta_j .$$

Proof. Let t_j be the characteristic roots of the operator X. According to the Proposition 1.2, the numbers $P(t_j)$ are the characteristic roots of the element $P(X)$. Thus

$$P(t_j) = r_j . \tag{2.7}$$

From (2.7) we have $t_i \neq t_j$ if $i \neq j$.

Applying Lemma 1.2 we obtain (2.6).

In the particular case $P(t) = t^n$, we have

Theorem 2.2. Let V be an algebraic element with the characteristic polynomial

$$P_V(t) = \prod_{j=1}^{m} (t - r_j)^{\delta_j}, \quad r_j \neq 0.$$

Let t_j, $j = 1,2,\ldots,m$, satisfy the equations

$$t_j^n = r_j, \quad r_j \text{ is an integer.}$$

Then the solution of the operator equation

$$X^r = V$$

is an algebraic operator with the characteristic polynomial of the form

$$P_X(t) = \prod_{j=1}^{m} (t - t_j)^{\delta_j}.$$

Proof. Write $Q(t) = \prod_{j=1}^{m} (t - t_j)^{\delta_j}$ and $X_j = X_0^{n-1} + t_j X_0^{n-2} + \ldots, t_j^{n-1} I$, where X_0 has the characteristic roots, $t_1, t_2, \ldots, t_m$, $X^n = V$. According to Proposition 1.2, the characteristic roots of X_j are numbers of the form

$$t_j = t^{n-1} + t_j t^{n-2} + \ldots + t_j^{n-1} =$$

$$= \begin{cases} n t_j^{n-1} \neq 0 & \text{if } \nu = j \\ \dfrac{n_\nu - n_j}{t - t_j} & \text{otherwise, where } \nu, j = 1,2,\ldots,m. \end{cases}$$

Thus X_j $(j = 1,2,\dots,m)$ are invertible. Hence $Q(X_0) = 0$ if and only if

$$Q(X_0) \prod_{j=1}^{m} X_j^{\delta_j} = 0.$$

On the other hand

$$Q(X_0) \prod_{j=1}^{m} X_j^{\delta_j} = \prod_{j=1}^{m} [(X_0 - t_j I)X_j]^{\delta_j} =$$

$$= \prod_{j=1}^{m} (X_0^n - t_j^n I)^{\delta_j} = \prod_{j=1}^{m} (V - n_j)^{\delta_j} = P_V(V).$$

Thus $Q(t) = P_{X_0}(t)$, which completes the proof.

For (2.3) we have

L e m m a 2.1. Let A be an algebraic element with the characteristic polynomial

$$P_A(t) = \prod_{j=1}^{m} (t - t_j)^{\nu_j}, \quad t_j \neq 0, \quad j = 1,2,\dots,n.$$

Then A is invertible and

$$A^{-1} = Q_A(A) \tag{2.7}$$

where

$$Q_A(t) = \frac{P_A(t) - P_A(0)}{t} . \tag{2.8}$$

P r o o f . From (2.8) we obtain $P_A(t) - P_A(0) = Q_A(t)t$. Thus

$$P_A(A) - P_A(0)I = Q_A(A)A.$$

By assumption $P_A(0) \neq 0$ and $A^{-1} = \dfrac{Q_A(A)}{P_A(0)}$.

C o r o l l a r y 2.1. Let A and $G(t)$ satisfy all assumptions of Proposition 1.2.

If $r = 0$, $j = 1,2,\ldots,m$ (cf. the formula (1.19)), then the element $V = G(A)$ is invertible and

$$(2.9) \qquad V^{-1} = Q_V(A), \quad \text{where} \quad Q_V(t) = \frac{P_V(t) - P_V(0)}{t}.$$

L e m m a 2.2. Let A be an algebraic element with the characteristic polynomial

$$(2.10) \qquad P_A(t) = \prod_{j=1}^{n-1} (t-t_j)^{\nu_j} t^{\nu_n}, \quad t_j \neq 0, \quad j=1,2,\ldots,n-1, \quad t_n=0.$$

Then a necessary condition for the equation

$$(2.11) \qquad AX = Y$$

to have a solution, is

$$(2.12) \qquad (A - t_n I)^{\nu_n - 1} P_n Y = 0.$$

P r o o f . The equation (2.11) is equivalent to the system of independent equations

$$AP_jX = P_jY, \quad j = 1,2,\ldots,n \qquad \text{(cf. [2], Theorem 5.1).}$$

Thus

$$(A - t_jI)^{\nu_j - 1} AP_jX = (A - t_jI)^{\nu_j - 1} P_jY,$$

provided that $X_j = P_jX$ is a solution of the equation $AX_j = P_jY$.

For $j = n$

$$A^{\nu_n}P_nX = A^{\nu_n - 1} P_nY = 0 \qquad \text{(cf. [1] and [2]).}$$

Hence the condition (2.12) is necessary.

L e m m a 2.3. Let conditions (2.10) and (2.12) be satisfied. Then the equation

$$AX = Y$$

has a solution X if and only if

$$(2.13)\quad \begin{cases} (A-t_iI)^{k_i}P_iX = \sum_{j=1}^{\vartheta_i-k_i} (-t_i)^{j-\vartheta_i}(A-t_iI)^{j+k_i-1}P_iY, \\ (i = 1,2,\dots,n-1, \quad k_i = 0,1,\dots,\vartheta_i-1) \\ (A - t_nI)P_nX = P_nY. \end{cases}$$

P r o o f . The equation (2.11) is equivalent to the system of the independent equations

$$AP_jX = P_jY, \quad X_j = P_jX, \quad i = 1,2,\dots,n \quad \text{(cf. [2] Theorem 5.1)}.$$

Let i be an arbitrary fixed integer in the interval $0 < i < n$. Then

$$AP_iX = (A - t_iI)P_iX + t_iP_iX.$$

Applying the operators $(A - t_iI)^{k_i}$ $(k=0,1,\dots,\vartheta_i-1)$ to both sides of the equation

$$(A - t_iI)P_iX + t_iP_iX = P_iY$$

we obtain the following system of the equations

$$(2.14)\qquad \lambda_i(A)X^{(i)} = Y^{(i)},$$

where

$$X^{(i)} = \begin{bmatrix} X_0^{(i)} \\ \vdots \\ X_{\vartheta_i-1}^{(i)} \end{bmatrix}, \qquad Y^{(i)} = \begin{bmatrix} Y_0^{(i)} \\ \vdots \\ Y_{\vartheta_i-1}^{(i)} \end{bmatrix},$$

$$X_{\mu}^{(i)} = (A - t_iI)^{\mu} P_i X, \quad Y_{\mu}^{(i)} = (A - t_iI)^{\mu} P_i Y$$

and

$$\lambda_i(A) = \begin{bmatrix} t_i & 1 & 0 & \dots & 0 \\ 0 & t_i & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_i \end{bmatrix}.$$

The solution of the system (2.14) assumes the form

$$X^{(i)} = [\lambda_i(A)]^{-1} Y^{(i)}, \tag{2.15}$$

where

$$[\lambda_i(A)]^{-1} = \begin{bmatrix} \frac{1}{t_i} & -\frac{1}{t_i^2} & \frac{1}{t_i^3} & \dots & \frac{(-1)^{\nu_i - 1}}{t_i^{\nu_i}} \\ 0 & \frac{1}{t_i} & -\frac{1}{t_i^2} & \dots & \frac{(-1)^{\nu_i - 2}}{t_i^{\nu_i - 1}} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{t_i} \end{bmatrix}.$$

Hence conditions (2.13) follow immediately.

Conversely, suppose that there exists an element X satisfying conditions (2.13). Hence $X^{(i)}$ satisfy the system (2.15). From (2.15) we obtain (2.14). In particular for $k_i = 0$ the equation

$$(A - t_iI)P_iX + t_iP_iX = P_iY, \quad i = 1,2,\dots,n$$

is satisfied. But we have supposed that the condition (2.12) is satisfied. Hence X is a solution of the equation (2.11).

For the general case we prove the following

Theorem 2.3. Let A be an algebraic element with the characteristic polynomial

$$P_A(t) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} (t - t_{ij_i})^{\,ij_i}, \quad t_{ij} \neq t_{\vartheta\mu}, \ (i,j) \neq (\vartheta,\mu).$$

Let $G(t)$ be a polynomial satisfying conditions

$$G(t_{kj_k}) = n_k, \ G'(t_{kj_k}) = \ldots = G^{(s_{kj_k})}(t_{kj_k}) = 0,$$

and

$$G^{(s_{kj_k}+1)}(t_{kj_k}) \neq 0, \quad k = 1,2,\ldots,m, \ j_k = 1,2,\ldots,n_k,$$

where $n_k \neq n_l$ whenever $k \neq l$.

If $r_m = 0$ and the condition $[G(A)]^{\delta_m - 1} R_m Y = 0$ is satisfied then the equation

$$G(A)X = Y \tag{2.16}$$

has a solution X if and only if

$$(G(A) - n_i I)^{k_i} R_i X = \sum_{j=1}^{\delta_i - k_i} (-n_i)^{j-\delta_i} [G(A) - n_i I]^{j+k_i-1} R_i Y;$$

$$i = 1,2,\ldots,m-1, \ k_i = 0,1,\ldots,\delta_i - 1,$$

and

$$G(A) R_n X = R_n Y,$$

where $R_1, R_2, \ldots, R_m$ are the projectors associated with $G(A)$.

The proof is immediate if we apply Proposition 1.2 and Lemma 2.3.

Remark. The equation of the type (2.16) has been considered by Przeworska-Rolewicz in [2]. The method here is different.

We shall have a similar result for equations (2.2) and (2.4).

L e m m a 2.4. Let A and B be algebraic operators with the characteristic polynomials

$$(2.17)\qquad P_A(t) = \prod_{j=1}^{n} (t - t_j)^{\nu_j}, \quad P_B(t) = \prod_{k=1}^{m} (t - t_k)^{\mu_k}.$$

Denote by $P_1, P_2, \dots, P_n$ the projectors associated with A. The corresponding projectors for B will be denoted by $Q_1, Q_2, \dots, Q_m$. Then a necessary condition for the equation

$$(2.18)\qquad AX - XB = Y$$

to have a solution is

$$(2.19)\qquad (A - t_i I)^{\nu_i - 1} P_i Y Q_k (B - \tau_k I)^{\mu_k - 1} = 0$$

for any i and k such that $\alpha_{ik} = t_i - \tau_k = 0$.

P r o o f . Let X be a solution of the equation (2.18) and let $\alpha_{ik} = t_i - \tau_k = 0$. Multiplaying both sides of the equation (2.18) by $(A - t_i I)^{\nu_i - 1} P_i$ from the left and by $Q_k (B - \tau_k I)^{\mu_k - 1}$ from the right we obtain the equality

$$A(A-t_iI)^{\nu_i-1} P_i Y(B-\tau_kI)^{\mu_i-1} Q_k - (A-t_iI)^{\nu_i-1} P_i XB(B-\tau_kI)^{\mu_k-1} Q_k =$$

$$= (A-t_iI)^{\nu_i-1} P_i Y(B-\tau_kI)^{\mu_i-1} Q_k$$

which along with the equalities

$$(A - t_i I)^{\nu_i} P_i = 0, \qquad (B - \tau_k I)^{\mu_k} Q_k = 0$$

imply that

$$\alpha_{ik}(A-t_iI)^{\nu_i-1}P_iXQ_k(B-\tau_kI)^{\mu_k-1} = (A-t_iI)^{\nu_i-1}P_iYQ_k(B-\tau_iI)^{\mu_k-1}$$

which proves the necessity of the condition (2.19).

Theorem 2.4. Let the conditions (2.17) and (2.19) be satisfied. Then the equation (2.18) has a solution X if and only if

$$(2.20)\quad \begin{cases} (A-t_iI)^{j_i}P_iXQ_k(B-\tau_kI)^{l_k} = \\ = \sum\limits_{j=1}^{\nu_i-j_i}\sum\limits_{l=1}^{\mu_i-l_k}(-\alpha_{ik})^{j+l-(\nu_i+\mu_k)}(A-t_iI)^{j+j_i-1} \times \\ \times P_iYQ_k(B-\tau_kI)^{l+l_k-1} \\ (i = 1,2,\dots,n,\ k = 1,2,\dots,m,\ j_i = 0,1,\dots,\nu_i-1, \\ l_k = 0,1,\dots,\mu_k-1) \quad \text{when} \quad \alpha_{ik} = t_i - \tau_k \neq 0 \\ (A-t_iI)P_iXQ_k - P_iXQ_k(B-\tau_kI) = P_iYQ_k, \quad \text{when} \quad \alpha_{ik} = 0. \end{cases}$$

We shall prove Theorem 2.4 by means of the additional lemmas:

Lemma 2.5. Let ν_i be positive integers and let A_{ik} be $\nu_i \times \nu_i$ matrices

$$(2.21)\qquad A_{ik} = \begin{bmatrix} \alpha_{ik} & 1 & 0 & \dots & 0 \\ 0 & \alpha_{ik} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_{ik} \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

If $\alpha_{ik} \neq 0$, then

$$(2.22)\quad A_{ik}^{-s} = \begin{bmatrix} \alpha_{ik}^{-s} & -\frac{s}{1!}\alpha_{ik}^{-s-1} & \dots & (-1)^{\nu_i-1}\binom{s+\nu_i-1}{\nu_i}\alpha_{ik}^{1-s-\nu_i} \\ 0 & -\alpha_{ik}^{-s} & \dots & (-1)^{\nu_i-2}\binom{s+\nu_i-1}{\nu_i}\alpha_{ik}^{-s-\nu_i+2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & \alpha_{ik}^{-s} \end{bmatrix}.$$

P r o o f . From (2.21) we have $(A_{ik} - \alpha_{ik}I)^{\nu_i} = 0$. Thus

$$A_{ik}^{-s} = \alpha_{ik}^{-s}I - \frac{s}{1!}\alpha_{ik}^{-s-1}(A - \alpha_{ik}I) + \dots$$

$$\dots + (-1)^{\nu_i}\binom{s+\nu_i-1}{\nu_i}\alpha_{ik}^{-s-\nu_i-1}(A_{ik} - \alpha_{ik}I)^{\nu_i-1}.$$

This implies (2.22).

L e m m a 2.6. Let ν_i and μ_k be positive integers and $\alpha(A,B)$ a $(\nu_i\mu_i)\times(\nu_i\mu_{i_k})$ matrix:

$$(2.23)\qquad \alpha(A,B) = \begin{bmatrix} A_{ik} & I & 0 & \dots & 0 \\ 0 & A_{ik} & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{ik} \end{bmatrix},$$

where A_{ik} are given by the formula (2.21).

If $\alpha_{ik} \neq 0$ then

$$[\alpha(A,B)]^{-1} = \begin{bmatrix} A_{ik}^{-1} & -A_{ik}^{-2} & \dots & (-1)^{\mu_i-1}A^{-\mu_k} \\ 0 & A_{ik}^{-1} & \dots & (-1)^{\mu_k-2}A^{-\mu_k+1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A^{-1} \end{bmatrix}.$$

P r o o f . According to Lemma 2.5 the matrix A_{ik} is invertible. Denote the matrix (2.24) by $\beta(A,B)$; then

$$\alpha(A,B)\beta(A,B) = \beta(A,B)\alpha(A,B) = I.$$

Thus $\beta(A,B) = [\alpha(A,B)]^{-1}$.

P r o o f of Theorem 2.4. The equation (2.18) is equivalent to the system of independent equations

$$AP_iXQ_k - P_iXQ_kB = P_iYQ_k$$

($X_{ik} = P_iXQ_k$, $i = 1,2,\dots,n$, $k = 1,2,\dots,m$). Let i and k be arbitrarily fixed integers, such that

$$\alpha_{ik} = t_i - \tau_k \neq 0.$$

Then $AP_iXQ_k = (A-t_iI)P_iXQ_k + t_iP_iXQ_k$ and $P_iXQ_kB = P_iXQ_k(B-\tau_kI) + \tau_kP_iYQ_k$.

Multiplaying both sides of the equation

$$(A-t_iI)P_iXQ_k - P_iXQ_k(B-\tau_kI) + \alpha_{ik}P_iXQ_k = P_iYQ_k$$

by $(A-t_iI)^{j_i}$ ($j_i = 0,1,\dots,\nu_i-1$) from the left and by $(B-\tau_jI)^{l_k}$ ($l_k = 0,1,\dots,\mu_k-1$) from the right we obtain the following system of equations

$$\alpha(A,B)X^{(i,k)} = Y^{(i,k)},$$

where

$$X^{(i,k)} = \begin{bmatrix} X_{00}^{(i,k)} \\ X_{10}^{(i,k)} \\ \vdots \\ X_{\nu_i-1\,0}^{(i,k)} \\ X_{01}^{(i,k)} \\ \vdots \\ X_{\nu_i-1,\mu_i-1}^{(i,k)} \end{bmatrix}, \qquad Y^{(i,k)} = \begin{bmatrix} Y_{00}^{(i,k)} \\ Y_{10}^{(i,k)} \\ \vdots \\ Y_{\nu_i-1\,0}^{(i,k)} \\ Y_{01}^{(i,k)} \\ \vdots \\ Y_{\nu_i-1,\mu_k-1}^{(i,k)} \end{bmatrix}$$

and

$$X^{(i,k)}_{j_i,l_k} = (A-t_iI)^{j_i}P_iXQ_k(B-\tau_kI)^{l_k},$$

$$Y^{(i,k)} = (A-t_iI)^{j_i}P_iYQ_k(B-\tau_kI)^{l_k},$$

$(j_i = 0,1,\dots,\nu_i-1,\ \ l_k = 0,1,\dots,\mu_k-1)$.

The solution of (2.25) is assumed to be

$$X^{(i,k)} = [\alpha(A,B)]^{-1}Y^{(i,k)}, \tag{2.26}$$

where $[\alpha(A,B)]^{-1}$ is given by the formula (2.24). Hence the conditions (2.20) follow immediately.

Conversely, suppos that there exists an element X satisfying conditions (2.20). Hence $X^{(i,k)}$ satisfy the system (2.26). From (2.26) we obtain (2.25). For $j = l_k = 0$ the equation

$$(A-t_iI)P_iXQ_k - P_iXQ_k(B-\tau_kI) + (t_i-\tau_k)P_iXQ_k = P_iXQ_k,$$
$$(i = 1,2,\dots,n,\ k = 1,2,\dots,m)$$

is satisfied. But we have supposed that the condition (2.19) is satisfied. Hence X is a solution of the equation (2.18).

T h e o r e m 2.5. Let A and B satisfy the condition (2.17). Then the equation

$$AX = XB \tag{2.27}$$

has a solution of the form

$$X = a_0\sigma_s(A,X_0,B) + a_1\sigma_{s-1}(A,X_0,B) + \dots + a_sX_0, \quad X_0 \in X, \tag{2.28}$$

where

$$\sigma_k(A,X_0,B) = A^kX_0 + A^{k-1}X_0B + \dots + X_0B^k \tag{2.29}$$

and

$$P(t) = a_0 t^{s+1} + a_1 t^s + \dots + a_{s+1}$$

is an arbitrary polynomial satisfying the conditions

$$P(A) = 0, \quad P(B) = 0.$$

P r o o f . From (2.29) we obtain

$$A\sigma_k(A,X_0,B) - \sigma_k(A,X_0,B)B = A^{s+1}X_0 - X_0B^{k+1}.$$

Denote the sum (2.28) by $P(A,X_0,B)$. Then

$$AP(A,X_0,B) - P(A,X_0,B)B = P(A)X_0 - X_0P(B) = 0.$$

Thus $X = P(A,X_0,B)$ satisfies the equation (2.27) which completes the proof.

Let B be an algebraic element with the characteristic polynomial

$$P_B(t) = \prod_{i=1}^{s} \prod_{j_i=1}^{m_i} (t - \tau_{ij_i})^{\mu_{ij_i}} \tag{2.30}$$

$$(\tau_{ij} \neq \tau_{\nu\mu} \quad \text{whenever } (i,j) \neq (\nu,\mu))$$

and let $F(t)$ be a polynomial in variable t with complex coefficients satisfying the conditions:

$$F(\tau_{kj_k}) = \theta_k,$$
$$F'(\tau_{kj_k}) = \dots = F^{(n_{kj_k})}(\tau_{kj_k}) = 0,$$
$$F^{(n_{kj_k}+1)}(\tau_{kj_k}) \neq 0, \quad k = 1,2,\dots,s; \; j_k = 1,2,\dots,m_k.$$

Then, according to Proposition 1.2, we obtain

$$P_U(t) = \prod_{k=1}^{s} (t - \theta_k)^{\sigma_k}, \quad U = F(B),$$

where $\theta_k = \langle \beta_k \rangle$ ($\langle\cdot\rangle$ is defined as in (2.5)) and

$$\beta_k = \max\left[\frac{\mu_{k1}}{r_{k1}+1}, \frac{\mu_{k2}}{r_{k2}+1}, \dots, \frac{\mu_{km_k}}{r_{km_k}+1}\right].$$

Denote by $D_1, D_2, \dots, D_m$ the projectors associated with $V = G(A)$. The corresponding projector for $U = F(B)$ will be denoted by $R_1, R_2, \dots, R_s$.

We have a similar result for the equation

$$G(A)X - XF(B) = Y. \tag{2.32}$$

T h e o r e m 2.6. Let A and B be algebraic elements with the characteristic polynomials (1.17) and (2.30), resp. Let $G(t)$ and $F(t)$ be polynomials satisfying the conditions (1.18) and (2.31), resp. Then the equation (2.32) has a solution X if and only if

$$(V - n_iI)^{j_i} D_iXR_k(U - \theta_kI)^{l_k} =$$
$$= \sum_{j=1}^{\delta_i - j_i} \sum_{l=1}^{\sigma_k - l_k} (-n_i + \theta_k)^{j+l-(\sigma_i-\sigma_k)} (V - n_iI)^{j+j_i-1} D_iYR_k(U - \theta_kI)^{l+l_k-1},$$

$i = 1,2,\dots,m$, $k = 1,2,\dots,s$, $j_i = 0,1,\dots,\delta_i - 1$, $l_k = 0,1,\dots,\sigma_k - 1$, if $n_i \neq \theta_k$,

$$(V - n_iI)D_iXR_k - D_iXR_k(U - \theta_kI) = D_iYQ_k \quad \text{if } n_i = \theta_k,$$

where $V = G(A)$, $U = F(B)$.

The proof is immediate if we apply Theorem 2.4 to the elements $V = G(A)$ and $U = F(B)$.

BIBLIOGRAPHY

[1] D. Przeworska-Rolewicz: Equations with transformed argument. An algebraic approach. Amsterdam-Warszawa 1973.

[2] D. Przeworska-Rolewicz, S. Rolewicz: Equations in linear spaces. Warszawa 1968.

[3] M. Rosenbloom: On the operator equations $BX - XA = Q$, Duke Math. J. 23 (1956) 263-269.

[4] R. Bellman: Introduction to matrix analysis. New York-Toronto-London 1960.

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