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SOME TRANSFORMATIONS IN NON-SYMMETRIC FINSLER SPACES

Various transformations in Finsler space have been obtained by Matsumoto [1]. The concept of non-symmetric Finsler space has been introduced by Upadhyay and Sharma [2]. In this paper we shall use some transformations in non-symmetric Finsler space and establish some theorems.

1. Preliminaries

Let us consider an n-dimensional non-symmetric Finsler space (NS-Fn) equipped with a non-symmetric tensor g_{ij} given by [3]

$$g_{ij}(x,\dot{x}) \overset{\text{def}}{\models} h_{ij}(x,\dot{x}) + K_{ij}(x,\dot{x}), \tag{1.1}$$

where $h_{ij}(x,\dot{x}) = \frac{1}{2}\dot{\partial}^2_{ij}F^2(x,x)^{*)}$ is a symmetric part of g_{ij} and K_{ij} is a skew symmetric tensor. It is positively homogeneous of degree zero in $\dot{x}^i$.

Let us define

$$B^i_{mp}(x,\dot{x}) \overset{\text{def}}{\models} h^{ir}(x,\dot{x})\, B_{mpr}(x,\dot{x}), \tag{1.2}$$

where

$$B_{mpr}(x,\dot{x}) \overset{\text{def}}{\models} \frac{1}{2}\dot{\partial}_r g_{mp}(x,\dot{x}). \tag{1.3}$$

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*) $\dot{\partial}_i F = \frac{\partial F}{\partial \dot{x}^i}$ and $\dot{\partial}^2_{ij}F = \frac{\partial^2 F}{\partial \dot{x}^i\,\partial \dot{x}^j}$.

R e m a r k . The conjugate tensor of h_{ij} will be denoted by h^{ij} thus $h_{ij}h^{ir} = \delta^r_j$ and

$$(1.4) \qquad A^i_{mp} \overset{def}{=} h^{ir}A_{mpr}; \quad A_{mpr} \overset{def}{=} h_{ir}A^i_{mpr}.$$

The non-symmetric coefficients D^i_{mp} and E^i_{mp}, which we shall use in the remaining part of the paper, are defined as

$$(1.5) \qquad D^i_{mp}(x,\dot{x}) \overset{def}{=} F(x,\dot{x})B^i_{mp}(x,\dot{x})$$

and

$$(1.6) \qquad E^i_{mp}(x,\dot{x}) \overset{def}{=} \frac{1}{2} h^{ir}(x,\dot{x}) [\partial_r g_{mp} + \partial_m g_{pr} - \partial_p g_{rm}].$$

2. Let $F(x,\dot{x})$ and $\tilde{F}(x,\dot{x})$ be defined over NS-Fn and NS-$\tilde{F}$n respectively. The two metric of corresponding spaces will be said conformal if there exists a factor of proportionality $\sigma(x,\dot{x})$ between the two tensors

$$(2.1) \qquad \tilde{g}_{ij}(x,\dot{x}) = \sigma(x,\dot{x})g_{ij}(x,\dot{x}),$$

$$(2.2) \qquad \tilde{h}_{ij}(x,\dot{x}) = \sigma(x,\dot{x})h_{ij}(x,\dot{x})$$

and

$$(2.3) \qquad \tilde{K}_{ij}(x,\dot{x}) = \sigma(x,\dot{x})K_{ij}(x,\dot{x}).$$

Let the distance functions Fn and $\tilde{F}$n be related as

$$(2.4) \qquad \tilde{F}(x,\dot{x}) = \sigma^{1/2}F(x,\dot{x}).$$

It has been proved by Upadhyay and Sharma [2] that the factor of proportionality $\sigma(x,\dot{x})$ is an arbitrary function of the line element and it is homogeneous of degree zero in the directional arguments.

M.Matsumoto [1] introduced the notations of transformed Finsler space $\tilde{F}n$ whose distance function $F(x,\dot{x})$ is related by the metric function $F(x,\dot{x})$ of Fn by the relation

$$\tilde{F}^2(x,\dot{x}) = F^2(x,\dot{x}) + (f_i(x)\dot{x}^i)^2, \tag{2.5}$$

where $f_i(x)$ denotes the components of a covariant vector.

In this paper we shall study this type of transformation in a non-symmetric Finsler space.

Let the distance function $F(x,\dot{x})$ of NS-Fn be transformed to $\tilde{F}(x,\dot{x})$ as follows

$$\tilde{F}^2(x,\dot{x}) = \sigma(x,\dot{x})F^2(x,\dot{x}) + (f_i(x)\dot{x}^i)^2. \tag{2.6}$$

From (2.6), it follows that transformations (2.1) and (2.3) are particular cases of the transformation (2.6).

Under transformation (2.6), the symmetric tensor h_{ij} and the skew symmetric tensor K_{ij} are transformed into $\tilde{h}_{ij}$ and $\tilde{K}_{ij}$ given by

$$\tilde{h}_{ij} = \sigma h_{ij} + \frac{1}{2} F^2\dot{\sigma}_{ij} + f_i f_j \quad (\dot{\sigma}_k = \dot{\partial}_k\sigma) \tag{2.7}$$

and

$$\tilde{K}_{ij} = \sigma K_{ij} \tag{2.8}$$

respectively.

By virtue of (1.1), (2.7) and (2.8), we get

$$\tilde{g}_{ij} = \sigma g_{ij} + \frac{1}{2} F^2\dot{\sigma}_{ij} + f_i f_j. \tag{2.9}$$

Thus we have

Theorem 2.1. If NS-Fn is Riemannian then the transformed NS-$\tilde{F}n$ is also Riemannian if and only if the factor of proportionality σ is a function of positional coordinates only.

Differentiating (2.9) partially with respect to $\dot{x}^k$ and using (1.3) we get

$$(2.10)\qquad \tilde{B}_{ijk} = \sigma B_{ijk} + M_{ijk},$$

where

$$(2.11)\qquad M_{ijk} \overset{\text{def}}{=} \frac{1}{2} g_{ij}\dot{\sigma}_k + \frac{1}{4}\dot{\sigma}_{ijk}F^2 + \frac{1}{4}\dot{\sigma}_{ij}(\dot{\partial}_k F^2)$$

and

$$\dot{\sigma}_{ijk} = \dot{\partial}_k(\dot{\sigma}_{ij}) = \dot{\partial}_k(\dot{\partial}_i(\dot{\partial}_j\sigma)).$$

In view of (2.10) we get the following theorem

Theorem 2.2. Under transformation (2.6), the transformation of B_{ijk} is given by $\tilde{B}_{ijk} = \sigma B_{ijk}$ if and only if $M_{ijk} = 0$.

Transvection of (2.7) with $\tilde{h}^{ip}$ yields the following

$$(2.12)\qquad \tilde{h}^{ip}(\sigma h_{ij} + \frac{1}{2}F^2\dot{\sigma}_{ij} + f_i f_j) = \delta^p_j.$$

Transvecting the both sides of the above result with $h^{jk}f_k h^{mn}f_n$, we get

$$(2.13)\qquad f_i f^m \tilde{h}^{ip} = \frac{f^p f^m}{\sigma + f^2 + \alpha F^2},$$

where we have put

$$f^i = h^{im}f_m, \qquad f^2 = h_{ij}f^i f^j$$

and

$$\alpha \overset{\text{def}}{=} \frac{1}{2} h^{ij}\dot{\sigma}_{ij}.$$

Contracting (2.12) with h^{jm} and applying (2.13), we obtain

$$(2.14)\qquad \tilde{h}^{mp} = \beta h^{mp} - \frac{\beta^2 f^m f^p}{1 + f^2\beta},$$

where we have put $\beta = \dfrac{1}{\sigma + \alpha F^2}$.

Transvecting both sides of (2.10) by $\tilde{h}^{mk}$ and using the relation (2.14) we get

$$(2.15)\qquad \tilde{B}^m_{ij} = \sigma\beta B^m_{ij} + \beta M^m_{ij} - \frac{\beta^2 f^m}{1+f^2\beta}(\sigma B_{ij*} + M_{ij*});$$

where

$$B_{ij*} \overset{def}{=} B_{ijm}f^m{}^{*)}.$$

In view of (2.15), we obtain

$$(2.16)\quad \tilde{B}^m_{(ij)} = \beta(\sigma B^m_{(ij)} + M^m_{(ij)}) - \frac{\beta^2 f^m}{1+f^2\beta}(\sigma B_{(ij)*} + M_{(ij)*}),$$

where

$$B^i_{(jh)} \overset{def}{=} \frac{1}{2}(B^i_{jh} + B^i_{hj}).$$

Let us contract (2.16) by putting $m = i$, to get the following

$$(2.17)\quad \tilde{B}_j = \beta(\sigma B_j + M_j) - \frac{\beta^2}{1+\beta f^2}(\sigma(B_{(*j)*} + M_{(*j)*}),$$

where $B^m_{(mj)} \overset{def}{=} B_j$ and $M^m_{(mj)} \overset{def}{=} M_j$.

This leads to the following theorem

T h e o r e m 2.3. In order that the quantities $B_j \overset{def}{=} B^i_{(ij)}$ satisfy the relation

$$\tilde{B}_j = \sigma(\beta B_j - \frac{\beta^2}{f^2\beta+1}B_{(j*)*}),$$

it is necessary and sufficient that the quantities

$$M_j \overset{def}{=} M^i_{(ij)} \quad \text{satisfy} \quad M_j - \frac{\beta}{f^2\beta+1}M_{(*j)*} = 0.$$

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*) The star denotes the position of an index related to contraction.

By virtue of (1.6), (2.7) and (2.9) we get

$$(2.18)\quad \tilde{E}^i_{mp} = \sigma\left(\beta E^i_{mp} - \frac{f^i\beta^2}{f^2\beta+1} E_{*mp}\right) + \left(\beta P^i_{mp} - \frac{f^i\beta^2}{f^2\beta+1} P_{*mp}\right),$$

where

$$P_{rmp} = \frac{1}{2}\Big[\partial_r(f_m f_p) + \partial_m(f_p f_p) - \partial_p(f_r f_m) + F^2(\partial_r \dot{\sigma}_{mp} + \partial_m\dot{\sigma}_{pr} - \partial_p\dot{\sigma}_{rm}) + \frac{1}{2}\left\{(\partial_r F^2)\dot{\sigma}_{mp} + (\partial_m F^2)\dot{\sigma}_{pr} - (\partial_p F^2)\dot{\sigma}_{rm}\right\}\Big].$$

The symmetric and skew symmetric parts of the $P^i_{mp} \overset{def}{=} h^{ir}P_{rmp}$ are given by

$$(2.19)\quad P^i_{(mp)} = \frac{1}{2} h^{ir}\left[\partial_m(f_p f_r) + F^2\partial_r\dot{\sigma}_{mp} + \frac{1}{2}\partial_r(f_m f_p)\right],$$

where $P^i_{(mp)} \overset{def}{=} \frac{1}{2}(P^i_{mp} + P^i_{pm})$ and

$$(2.20)\quad P^i_{[mp]} = \frac{1}{2} h^{ir}\left[\partial_{[m}f_{p]}f_r + \partial_{[m}\dot{\sigma}_{p]r} + \partial_{[m}F^2\dot{\sigma}_{p]r}\right],$$

where $P^i_{[mp]} \overset{def}{=} \frac{1}{2}(P^i_{mp} - P^i_{pm})$.

Using (2.18) and (2.20), with notations $E_j \overset{def}{=} E^i_{[ij]}$ we get

$$\tilde{E}_p = \sigma\left(\beta E_p - \frac{\beta^2}{f^2\beta+1} E_{*[*p]}\right) + \left(\beta P_p - \frac{\beta^2}{f^2\beta+1} P_{*[*p]}\right).$$

Thus we have

T h e o r e m 2.4. In order ~~that the~~ quantities $E_j \overset{def}{=} E^i_{[ij]}$ satisfy the relation $\tilde{E}_p = \sigma\left(\beta E_p - \frac{\beta^2}{f^2\beta+1} E_{*[*p]}\right)$, it is necessary and sufficient that the quantities $P_p \overset{def}{=} P^i_{[ip]}$ satisfies $P_p - \frac{\beta}{f^2\beta+1} P_{*[*p]} = 0.$

Let us obtain a relation between $\tilde{D}^i_{mp}$ and D^i_{mp} with the help of (1.5), (2.4) and (2.15) as follows

$$(2.21)\quad \tilde{D}^m_{ij} = F\sigma^{1/2}\left[\beta(\sigma B^m_{ij} + M^m_{ij}) - \frac{f^m\beta^2F^2\sigma^{1/2}}{f^2\beta+1}(\sigma B_{ij*} + M_{ij*})\right].$$

Again in view of relation (1.5), we get

$$\tilde{D}^m_{ij} = \sigma^{1/2}\beta(\sigma D^m_{ij} + F\, M^m_{ij}) - \frac{f^m\beta^2\sigma^{1/2}}{f^2\beta+1}(\sigma D_{ij*} + F\, M_{ij*}),$$

which on simple arrangement of terms becomes

$$(2.22)\qquad \tilde{D}^m_{ij} = \sigma^{3/2}\Big(\beta\, D^m_{ij} - \frac{f^m\beta^2}{f^2\beta+1}D_{ij*}\Big) + F\,\sigma^{1/2}\Big(M^m_{ij} - \frac{f^m\beta^2}{f^2\beta+1}M_{ij*}\Big).$$

Thus we have

T h e o r e m 2.5. Under the transformations (2.6) the relation between the tensors $\tilde{D}^m_{ij}$ and D^m_{ij} is given by

$$\tilde{D}^m_{ij} = \sigma^{3/2}\left(\beta D^m_{ij} - \frac{f^m\beta^2}{f^2\beta+1}D_{ij*}\right) + F\,\sigma^{1/2}\left(M^m_{ij} - \frac{f^m\beta^2}{f^2\beta+1}M_{ij*}\right).$$

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