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ON REPRESENTATIONS OF CYCLIC n-GROUPS

1. Introduction

In the present paper we introduce the notion of a representation of an n-group, which is analogous to the notion of a group representation. For this purpose, we define a diagonal n-group, and an n-group representation in the diagonal n-group of the reversible linear transformations of a complex vector space. Our aim here is to study and describe one-dimensional representations of cyclic n-groups. We study this by means of the notion of a covering representation. We shall use the following abbreviated notation

$$f(a_1,\ldots,a_k,\underbrace{a\ldots a}_{s\text{ times}},a_{k+s+1},\ldots,a_n) = f(a_1^k,\overset{(s)}{a},a_{k+s+1}^n)$$

(a_i^j for $i>j$ and for $i>n$, $\overset{(\alpha)}{a}$ for $\alpha\leq 0$ are empty symbols).

$$\underbrace{f(f\ldots f}_{s\text{ times}}(a_1^n)\ldots)\ a_{2+(s-1)(n-1)}^{1+s(n-1)}) = f_{(s)}a_1^{1+s(n-1)},$$

$f_{(s)}$ denotes that the symbol of operation f is used s times. We shall use the notation $a^{\langle s\rangle}$ for the symbol $f_{(s)}\overset{(1+s(n-1))}{a}$ and we call it s-adic power of the element a (see [3]).

2. Diagonal n-group

Let G be a group, $\mathcal{G} = G^{n-1}$, $n \geq 2$. We define n-ary operation $\varphi: G^n \longrightarrow \mathcal{G}$ as follows. Let $\mathcal{G} \ni \underline{a}_i = (a_{i1}, a_{i2}, \ldots, a_{i\,n-1})$, $i=1,2,\ldots,n$.

$$\varphi(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) = (b_1, b_2, \ldots, b_{n-1}), \tag{1}$$

where $b_i = a_{ni} \cdots a_{n-i+1\,1} a_{n-i\,n-1} \cdots a_{2\,i+1} a_{1i}$.
It may described also by the following scheme

$$\begin{array}{cccc|cccc}
a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n-1} & a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n-1} \\
a_{2\,1} & a_{2\,2} & \cdots & a_{2\,n-1} & a_{2\,1} & a_{2\,2} & \cdots & a_{2\,n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & & & & & & \vdots \\
a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n-1} & a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n-1} \\
a_{n\,1} & a_{n\,2} & & a_{n\,n-1} & a_{n\,1} & a_{n\,2} & \cdots & a_{n\,n-1} \\
 & & & & b_1 & b_2 & & b_{n-1}
\end{array}$$

i.e. b_i is a product of n elements of G taken from the i-th diagonal. $\mathcal{G}$ equipped with the n-ary operation φ is a n-group. We call $(\mathcal{G}, \varphi)$ a diagonal n-group.

Theorem 1. Suppose that $|G| > 1$, then the diagonal n-group $(\mathcal{G}, \varphi)$ is irreducible.

Proof. An n-group $\mathfrak{A} = (A, f)$ is reducible to k-group $(n = 1+s(k-1))$ (see [2]) if and only if for every $c \in A$ there exists $d \in A$ such that

1) for every $x \in A$, $f(\overset{(k-2)s}{c}\ \overset{s}{d}, x) = x$

2) for every $x_1, x_2, \ldots, x_p$ $(p = (s-1)(k-1)+1)$

$$f(\overset{k-2}{c}, d, x_1^p) = f(d, \overset{k-2}{c}, x_1^p) = f(x_1, \overset{k-2}{c}, d, x_2^p) = f(x_1, d, \overset{k-2}{c}, x_2^p) = \ldots$$
$$\ldots = f(x_1^p, d, \overset{k-2}{c}) = f(x_1^p, \overset{k-2}{c}, d).$$

We will show that the condition 2) above is not satisfied by the n-group $(\mathcal{G}, \varphi)$.

Assumme to the contrary, that (G,φ) is reducible to a k-group. Hence for $\underline{c} = \underbrace{(e,\dots,e)}_{n-1}$ there exists a $\underline{d} = (d_1,\dots,d_{n-1})$, such that for $\underline{x}_i = (e,\dots,e)$, $(i=1,2,\dots,p)$ 2) is satisfied.

So

$$\begin{aligned}
\varphi(\overset{k-2}{\underline{c}},\underline{d},\underline{x}_1^p) &= (d_{k-1},d_k,d_{k+1},\dots,d_{n-1},d_1,\dots,d_{k-2}) = \\
&= \varphi(\underline{d},\overset{k-2}{\underline{c}},\underline{x}_1^p) = (d_1,d_2,\dots,d_{n-1}) = \\
&= \varphi(\underline{x}_1,\overset{k-2}{\underline{c}},\underline{d},\underline{x}_2^p) = (d_k,d_{k+1},\dots,d_{n-1},d_1,\dots,d_{k-1}) = \dots = \\
&= \varphi(\underline{x}_1^i,\overset{k-2}{\underline{c}},\underline{d},\underline{x}_{i+1}^p) = (d_{k+i-1},d_{k+i},\dots,d_{n-1},d_1,\dots,d_{k+i-2}) = \\
&= \varphi(\underline{x}_1^i,\underline{d},\overset{k-2}{\underline{c}},\underline{x}_{i+1}^p) = (d_{i+1},\dots,d_{n-1},d_1,\dots,d_i) = \dots = \\
&= \varphi(\underline{x}_1^p,\overset{k-2}{\underline{c}},\underline{d}) = (d_1,d_2,\dots,d_{n-1}) = \\
&= \varphi(\underline{x}_1^p,\underline{d},\overset{k-2}{\underline{c}}) = (d_{p+1},\dots,d_{n-1},d_1,\dots,d_p).
\end{aligned}$$

We obtained $d_1 = d_2 = \dots = d_{n-1} = d$ i.e. $\underline{d} = (d,\dots,d)$. Now take $\underline{x}_i = (x,e,\dots,e)$, $i=1,\dots,p$ where $x \neq e$.

$$\begin{aligned}
\varphi(\overset{k-2}{\underline{c}},\underline{d},\underline{x}_1^p) &= (\underbrace{xd,xd,\dots,xd}_{p-1},\underbrace{d,\dots,d}_{n-p}) = \varphi(\underline{x}_1^p,\underline{d},\overset{k-2}{\underline{c}}) = \\
&= (\underbrace{dx,d,\dots,d}_{n-p-2},\underbrace{dx,\dots,dx}_{p}).
\end{aligned}$$

From this equality we get $xd = d$, hence $x = e$ which is impossible.

3. A notion of a representation and a covering representation of an n-group

Let V be a complex linear space, $GL(V)$ – the general linear group, let $GL(V,n)$ denote the diagonal n-group of $GL(V)$.

R e m a r k . For $n = 2$ $GL(V,2) = GL(V)$.

D e f i n i t i o n 1. A homomorphism $\varrho: G \to GL(V,n)$ is called a representation of the n-group G on V.

Let (G,f) be an n-group, $\varrho: G \to GL(V,n)$ its representation on V. Let $\tau: G \to G(2)_{(n-1)}$ be an homomorphism of the n-group G into the free covering group $G(2)$. Let $\overline{V} = \underbrace{V \oplus \dots \oplus V}_{n-1}$. Define $j: GL(V,n) \to GL(\overline{V})$ by

$$(2)\quad j(A_1,\dots,A_{n-1})(x_1,x_2,\dots,x_{n-1}) = (A_{n-1}(x_{n-1}),A_1(x_1),\dots \\ \dots,A_{n-2}(x_{n-2}))$$

for $\underline{A} \in GL(V,n)$, $\underline{A} = (A_1,\dots,A_{n-1})$.

We will show that j is a homomorphism of $GL(V,n)$ into $GL(\overline{V})_{(n-1)}$.

$$j(\varphi(\underline{A}_1,\dots,\underline{A}_n))(x_1,\dots,x_{n-1}) = j(A_{n1}A_{n-1\ n-1}\dots A_{11},\dots$$

$$\dots,A_{ni}A_{n-1\ i-1}\dots A_{1i},\dots,A_{n\ n-1}A_{n-1\ n-2}\dots A_{1\ n-1})(x_1,\dots,x_{n-1}) =$$

$$= A_{n\ n-1}A_{n-1\ n-2}\dots A_{1\ n-1}(x_{n-1}),A_{n1}A_{n-1\ n-1}\dots A_{11}(x_1),\dots),$$

$$\varphi(j(\underline{A}_1),\dots,j(\underline{A}_n)) = \varphi((A_{1\ n-1},A_{11},A_{12},\dots,A_{1\ n-1}),\dots) =$$

$$= (A_{n\ n-1}A_{n-1\ n-1}\dots A_{1\ n-1},A_{n1}A_{n-1\ n-1}\dots A_{11},\dots).$$

Hence there exists a unique homomorphism $\hat{\varrho}: G(2) \to GL(\overline{V},n)$ such that $\hat{\varrho}\tau(g) = j\varrho(g)$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\varrho"'] & G(2)_{(n-1)} \arrow[d, dashed, "\hat{\varrho}"] \\ GL(V,n) \arrow[r, "j"] & GL(\overline{V})_{(n-1)} \end{tikzcd}.$$

We call the above representation $\hat{\rho}$ a covering representation for ρ.

4. One-dimensional representations of cyclic n-groups

Let (G,f) be a finite, abelian n-group, let V be one dimensional, complex vector space, and let $\rho : G \longrightarrow GL(V,n)$ be a representation of the n-group G in the space V. For an arbitrary $g \in G$ and $x \in V$

$$\rho(g)(x) = (a_1x,\dots,a_{n-1}x), \tag{3}$$

where $a_1,a_2,\dots,a_{n-1}$ are complex, non-zero numbers. Next we shall consider the free covering group $G(2)$ of the n-group G, and we shall proof the following

L e m m a 1. If G is an commutative n-group, then the free covering group $G(2)$ is commutative too.

P r o o f . G is commutative if and only if for every $g_1,g_2,\dots,g_n \in G$, $f(g_1,\dots,g_n) = f(g_{\sigma(1)}, g_{\sigma(2)},\dots,g_{\sigma(n)})$, where σ is an arbitrary permutation of the set $\{1,2,\dots,n\}$.

Let $g,h \in \tau(G)$. Suppose that $gh \neq hg$. Since $\tau: G \longrightarrow G(2)_{(n-1)}$ is a monomorphism of the n-group G into the free covering group $G(2)$ $\tau(g')\tau(h') \neq \tau(h')\tau(g')$, where $g',h' \in G$ and $\tau(g') = g$, $\tau(h') = h$. Let $\bar{h}'$ be a skew element to h'.

Since $\tau(g')\tau(h')\tau(h')^{n-3}\tau(\bar{h}') = \tau(h')\tau(g')\tau(h')^{n-3}\tau(\bar{h}')$, $(f(g', \overset{(n-2)}{h'}, \bar{h}')) \neq \tau(f(h',g', \overset{(n-3)}{h'}, \bar{h}'))$, $f(h',g', \overset{(n-3)}{h'}, \bar{h}') = f(g', \overset{(n-2)}{h'}, \bar{h}')$ because G is abelian. Hence $\tau(f(g', \overset{(n-2)}{h'}, \bar{h}')) \neq \tau(f(g', \overset{(n-2)}{h'}, \bar{h}'))$. Because $f(g', \overset{(n-2)}{h'}, \bar{h}') = g'$ (see [1]), so $g' \neq g'$ and we get contradiction. Hence $gh = hg$. Since $\tau(G)$ generates $G(2)$, for $g,h \in G(2) \setminus \tau(G)$ we have $g = g_1^{k_1}\dots g_m^{k_m}$, $h = h_1^{p_1}\dots h_s^{p_s}$ $(k_i,p_j \in \{1,-1\}$, $g_i,h_j \in \tau(G)$, $i = 1,2,\dots,m$; $j = 1,2,\dots,s)$; $gh = g_1^{k_1}\dots g_m^{k_m}h_1^{p_1}\dots h_s^{p_s} = h_1^{p_1}\dots h_s^{p_s}g_1^{k_1}\dots g_m^{k_m} = h\cdot g$, which completes the proof.

By the identity $\hat{\rho}\tau(g) = j\rho(g)$ we observe that $\hat{\rho}\tau(g)(\underline{x}) = j\rho(g)(\underline{x}) = (a_{n-1}x_{n-1}, a_1x_1, \ldots, a_{n-2}x_{n-2})$. So the matrix of the transformation $\hat{\rho}\tau(g) \in GL(\overline{V})$ is

$$
\begin{bmatrix}
0 & \cdots & \cdots & \cdots & a_{n-1} \\
a_1 & \ddots & & 0 & \vdots \\
 & \ddots & \ddots & & \vdots \\
 & 0 & \ddots & \ddots & \vdots \\
 & & & a_{n-2} & 0
\end{bmatrix}
\tag{4}
$$

where $a_i \neq 0$, $i=1,2,\ldots,n-1$.

Let $g_1, g_2 \in G$ and let $\rho(g_1) = (a_1,\ldots,a_{n-1})$, $\rho(g_2) = (b_1,\ldots,b_{n-1})$. $\hat{\rho}(\tau(g_1'))\hat{\rho}(\tau(g_2')) = \hat{\rho}(\tau(g_1')\tau(g_2')) = \hat{\rho}(\tau(g_2')\tau(g_1')) = \hat{\rho}(\tau(g_2'))\hat{\rho}(\tau(g_1'))$. So if the matrix of the transformation $\hat{\rho}(\tau(g_1'))$ is

$$
A = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & a_{n-1} \\
a_1 & \ddots & & 0 & \vdots \\
 & \ddots & \ddots & & \vdots \\
 & 0 & \ddots & \ddots & \vdots \\
 & & & a_{n-2} & 0
\end{bmatrix}
$$

and the matrix of the transformation $\hat{\rho}(\tau(g_2'))$ is

$$
B = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & b_{n-1} \\
b_1 & \ddots & & 0 & \vdots \\
 & \ddots & \ddots & & \vdots \\
 & 0 & \ddots & \ddots & \vdots \\
 & & & b_{n-2} & 0
\end{bmatrix},
$$

then $AB = BA$; clearly

$$AB = \begin{bmatrix} 0 & \cdots & \cdots & a_{n-1}b_{n-2} & 0 & \\ 0 & \ddots & \cdots & \cdots & \cdots & a_1b_{n-1} \\ a_2b_1 & \ddots & \ddots & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & \ddots & 0 & & 0 \\ & & a_{n-2}b_{n-3} & & & \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & \cdots & \cdots & b_{n-1}a_{n-2} & & \\ 0 & \ddots & \cdots & \cdots & 0 & b_1a_{n-1} \\ b_2a_1 & \ddots & \ddots & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & \ddots & 0 & & 0 \\ & & b_{n-2}a_{n-3} & & & \end{bmatrix}.$$

Comparing the coordinates of the matrices AB and BA we get

$$(5) \qquad \begin{cases} a_{n-1}b_{n-2} = b_{n-1}a_{n-2} \\ a_1b_{n-1} = b_1a_{n-1} \\ a_2b_1 = b_2a_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n-2}b_{n-3} = b_{n-2}a_{n-3} \end{cases}$$

$$(6) \qquad \begin{cases} a_{n-1} = \dfrac{a_{n-2}}{b_{n-2}}\, b_{n-1} \\ a_1 = \dfrac{a_{n-2}}{b_{n-2}}\, a_1 \\ a_2 = \dfrac{a_{n-2}}{b_{n-2}}\, b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n-2} = \dfrac{a_{n-2}}{b_{n-2}}\, b_{n-2}. \end{cases}$$

Hence there exists $\lambda = \frac{a_{n-2}}{b_{n-2}}$ such that $a_i = \lambda b_i$, for $i=1,2,\dots,n-1$. Take an arbitrary $g_0 \in G$ and let $\varrho(g_0) = (a_1,\dots,a_{n-1})$ then $\varrho(g) = (\lambda_g a_1,\dots,\lambda_g a_{n-1})$. Let $g_1,g_2,\dots,g_n \in G$, $\varrho(f(g_1^n)) = \varphi(\varrho(g_1),\dots,\varrho(g_n))$. On the other hand

$$\varrho(f(g_1^n)) = (\lambda_{f(g_1^n)} a_1,\dots$$

$$\dots,\lambda_{f(g_1^n)} a_{n-1})\cdot\varphi(\varrho(g_1),\dots,\varrho(g_n)) = (\lambda_{g_n} a_1 \lambda_{g_{n-1}} a_{n-1}\dots\lambda_{g_1},$$

$$\lambda_{g_{n-1}} a_1 \dots \lambda_{g_1} a_2,\dots,\lambda_{g_n} a_{n-1}\lambda_{g_{n-1}} a_{n-2}\dots\lambda_{g_1} a_{n-1}) = \lambda_{g_1}\lambda_{g_2}\dots$$

$$\dots\lambda_{g_n} a_1 a_2 \dots a_{n-1}(a_1,a_2,\dots,a_{n-1}).$$

Thus

$$\lambda_{f(g_1^n)} = \prod_{i=1}^{n} \lambda_{g_i} \prod_{i=1}^{n-1} a_i. \tag{7}$$

Suppose that G is a cyclic n-group of order k and $g_0 \in G$ generates $G(g_0^{\langle k\rangle} = g_0)$. Denote $g_0^{\langle j\rangle} = g_j$, $\lambda_{g_j} = \left(\prod_{i=1}^{n-1} a_i\right)^j$, $1 - \lambda_{g_0} = \lambda_{g_0^{\langle k\rangle}} = \left(\prod_{i=1}^{n-1} a_i\right)^k$. Denote $\prod_{i=1}^{n-1} a_i = \xi$, for $1 \le p \le k$ $\varrho(g_0^p) = \xi^p(a_1,\dots,a_{n-1})$. It comes to the following

T h e o r e m 2. If G is a cyclic n-group of order k, $\varrho : G \to GL(V,n)$, $\dim V = 1$, then for every $g \in G$, $\varrho(g) = \xi^p(a_1,\dots,a_{n-1})$, $(p \le k)$, where ξ is the root of the unity of the k-th degree, $a_1,a_2,\dots,a_{n-1}$ are the complex non-zero numbers.

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