DEMONSTRATIO MATHEMATICA
Vol. XVI No 2 1983

Halina Brzozowska, Stanisław Fudali

HOMOTHETIC COVERINGS OF TRAPEZOIDS

Introduction

A notion of an m-th liminal function of a given figure F has been introduced in [3]. The homothetic (n+1)-covering for the family of frustums of an n-dimensional simplex is considered there and the (n+1)-th liminal function of such simplex is determined. Also in [3] the same problem is considered for the homothetic (n+2)-covering for $n = 2$ and $n = 3$. In the present paper we continue these considerations for the frustums of a 2-dimensional simplex, i.e. for the trapezoids. The object of our considerations is a homothetic r-covering of the trapezoid for $r \in \{5,6,7,8,9\}$. For every such r we determine and investigate the r-th liminal function of trapezoid. At the end of the paper we formulate some open questions, which appeared in this study.

1. Preliminaries

We recall some notions, which we will use in the sequel. By a homothetic covering of a figure F we mean each family of figures $F_1, F_2, \ldots$, homothetic to F, such that $F \subset F_1 \cup F_2 \cup \ldots$ Such covering is called essential, if a removal of some member leads to a family, which does not cover F. An essential homothetic covering of F, which consists of m members, will be called an m-covering. In this paper there are considered only the essential homothetic coverings of a given trapezoid.

For a figure F, there is defined the m-th liminal number:

D e f i n i t i o n . Let $K^m(F)$ denote the set of numbers $k \in (0,1)$, for which families $\{k_1F,\dots,k_mF\}$ with $k_j \leqslant k$ do not cover F. The least upper bound of $K^m(F)$ is called the m-th liminal number of F and denoted by $k_0^m(F)$.

By a liminal homothetic m-covering we mean a covering $\{k_1F,\dots,k_mF\}$ of F with $k_j = k_0^m(F)$; it will be denoted by Cov_0^mF.

The position of k_iF in Cov_0^rF is in general stable for each i, i.e. an arbitrary translation τ of some $k_iF \in Cov_0^rF$ in general leads to a family $\{k_1F,\dots,\tau(k_iF),\dots,k_rF\}$ not covering F. Hence we say that Cov_0^rF is rigid if for every translation τ and for any figure k_jF we have $\{k_1F,\dots,\tau(k_jF),\dots,k_rF\} \neq Cov_0^rF$. In the opposite case Cov_0^rF is called non-rigid. If Cov_0^rF is non-rigid, then by definition there exists at least one translation τ such that for some $j \in \{1,\dots,r\}$ we have $\{k_1F,\dots,\tau(k_jF),\dots,k_rF\} = Cov_0^rF$. But if there exists more than one of such translations, then, for some j, there is a number $s \in (0,1)$ such that k_jF in Cov_0^rF can be replaced by sk_jF without a breach of the covering condition.

Yu. Belousov [1] and S. Fudali [2] showed, that for the triangle T

$$(1.1)\quad \begin{cases} k_0^3(T) = \frac{2}{3}, & k_0^4(T) = \frac{4}{7}, \quad k_0^5(T) = \frac{8}{15}, \quad k_0^6(T) = \frac{1}{2}, \\ k_0^7(T) = \frac{5}{11}, & k_0^8(T) = \frac{3}{7}, \quad k_0^9(T) = \frac{2}{5}, \quad k_0^{10}(T) = \frac{11}{29}. \end{cases}$$

It is easy to see that for an n-dimensional parallelepiped R^n there exist homothetic m^n-coverings only $(m \in \{2,3,\dots\})$ and $k_0^{m^n}(R^n) = \frac{1}{m}$ (comp. (1.3) in [3]); in particular

$$(1.2)\qquad k_0^4(R^2) = \frac{1}{2} \text{ and } k_0^9(R^2) = \frac{1}{3}.$$

Let T^p denote the trapezoid $A^1A^2B^2B^1$ in which the ratio of the upper to the lower base is equal to $p\in(p<1)$. It is easy to see that T^p is a frustum of a triangle $A^1A^2A^3$ and hence

$$(1.3)\qquad A^iB^i = (1-p)A^iA^3 \quad\text{and}\quad B^iA^3 = pA^iA^3 \quad (i\in\{1,2\}).$$

The given number $p\in(0,1)$ determine a family of trapezoids. Changing p from 0 to 1 a one-parameter class of trapezoids can be received. A particular specimen of this class is a triangle T $(p=0)$ and, on the other hand, it is a parallelogram R $(p=1)$.

To every $p\in\langle 0,1\rangle$ there corresponds an r-th liminal number of T^p; for this reason we have a function $p\longmapsto k_o^r(T^p)$, which is called <u>the r-th liminal function</u> of T^p. This function (for $r\in\{3,4\}$) is determined in [3 , Theorems 1 and 2], namely

$$(1.4)\qquad k_o^3(T^p) = \max\left(\frac{2}{3}, \frac{1}{2-p}\right) = \begin{cases} \frac{2}{3} & \text{if } p\in\langle 0, \frac{1}{2}\rangle, \\ \frac{1}{2-p} & \text{if } p\in\left(\frac{1}{2}, 1\right); \end{cases}$$

$$(1.5)\qquad k_o^4(T^p) = \begin{cases} \max\left(\frac{4}{7},\frac{1}{2-p}\right) & \text{if } p\in\langle 0,\frac{3-\sqrt{5}}{2}\rangle \\ \frac{2-p}{3-p} & \text{if } p\in\left(\frac{3-\sqrt{5}}{2},1\right\rangle \end{cases} = \begin{cases} \frac{4}{7} & \text{if } p\in\langle 0,\frac{1}{4}\rangle, \\ \frac{1}{2-p} & \text{if } p\in\left(\frac{1}{4},\frac{3-\sqrt{5}}{2}\right\rangle, \\ \frac{2-p}{3-p} & \text{if } p\in\left(\frac{3-\sqrt{5}}{2},1\right\rangle. \end{cases}$$

2. <u>The homothetic r-covering of a trapezoid for $r\in\{5,\dots,9\}$</u>

T h e o r e m . If T^p is a specimen of a class of trapezoids, then the r-th liminal function of T^p, for $r\in\{5,\dots,9\}$, has the form

(i)

$$(2.1)\quad k_o^5(T^p) = \begin{cases} \max\left(\frac{8}{15}, \frac{1}{2-p}\right) & \text{for } p \in \left\langle 0, \frac{3-\sqrt{7}}{2} \right\rangle \\ \max\left(\frac{1}{2}, \frac{4-2p}{7-2p}\right) & \text{for } p \in \left(\frac{3-\sqrt{7}}{2}, 1 \right) \end{cases} = \begin{cases} \frac{8}{15} & \text{if } p \in \left\langle 0, \frac{1}{8} \right\rangle, \\ \frac{1}{2-p} & \text{if } p \in \left(\frac{1}{8}, \frac{3-\sqrt{7}}{2} \right\rangle, \\ \frac{4-2p}{7-2p} & \text{if } p \in \left(\frac{3-\sqrt{7}}{2}, \frac{1}{2} \right\rangle, \\ \frac{1}{2} & \text{if } p \in \left(\frac{1}{2}, 1 \right); \end{cases}$$

(ii)

$$(2.2)\quad k_o^6(T^p) = \frac{1}{2} \quad \text{for} \quad p \in \left\langle 0, \frac{1}{2} \right);$$

(iii)

$$(2.3)\quad k_o^7(T^p) = \begin{cases} \max\left(\frac{5}{11}, \frac{3}{7-2p}\right) & \text{if } p \in \left\langle 0, \frac{1}{4} \right\rangle \\ \max\left(\frac{6}{13}, \frac{1}{3-2p}\right) & \text{if } p \in \left\langle \frac{1}{4}, \frac{15-\sqrt{177}}{4} \right\rangle \\ \frac{6-p}{12} & \text{if } p \in \left\langle \frac{15-\sqrt{177}}{4}, \frac{11-\sqrt{73}}{4} \right\rangle \end{cases} = \begin{cases} \frac{5}{11} & \text{if } p \in \left\langle 0, \frac{1}{5} \right\rangle, \\ \frac{3}{7-2p} & \text{if } p \in \left(\frac{1}{5}, \frac{1}{4} \right\rangle, \\ \frac{6}{13} & \text{if } p \in \left(\frac{1}{4}, \frac{5}{12} \right\rangle, \\ \frac{1}{3-2p} & \text{if } p \in \left(\frac{5}{12}, \frac{15-\sqrt{177}}{4} \right\rangle, \\ \frac{6-p}{12} & \text{if } p \in \left(\frac{15-\sqrt{177}}{4}, \frac{11-\sqrt{73}}{4} \right\rangle; \end{cases}$$

(iv)

$$(2.4)\quad k_o^8(T^p) = \begin{cases} \frac{3}{7} & \text{for } p \in \left\langle 0, \frac{1}{6} \right), \\ \frac{2}{5-2p} & \text{for } p \in \left\langle \frac{1}{6}, \frac{17-\sqrt{257}}{4} \right), \\ \frac{6-p}{13} & \text{for } p \in \left\langle \frac{17-\sqrt{257}}{4}, \frac{11-\sqrt{65}}{4} \right); \end{cases}$$

(v)

$$(2.5)\quad k_o^9(T^p)=\begin{cases}\max\left(\frac{2}{5},\frac{1}{3-2p}\right) & \text{if } p\in\left\langle 0,\frac{7-\sqrt{33}}{4}\right\rangle\\ \frac{5-2p}{11-2p} & \text{if } p\in\left(\frac{7-\sqrt{33}}{4},\frac{1}{2}\right\rangle\\ \max\left(\frac{9-4p}{19-4p},\frac{3-2p}{7-4p}\right) & \text{if } p\in\left(\frac{1}{2},1\right\rangle\end{cases}=\begin{cases}\frac{2}{5} & \text{if } p\in\left\langle 0,\frac{1}{4}\right\rangle,\\ \frac{1}{3-2p} & \text{if } p\in\left(\frac{1}{4},\frac{7-\sqrt{33}}{4}\right\rangle,\\ \frac{5-2p}{11-2p} & \text{if } p\in\left(\frac{7-\sqrt{33}}{4},\frac{1}{2}\right\rangle,\\ \frac{9-4p}{19-4p} & \text{if } p\in\left(\frac{1}{2},\frac{3}{4}\right\rangle,\\ \frac{3-2p}{7-4p} & \text{if } p\in\left(\frac{3}{4},1\right\rangle.\end{cases}$$

P r o o f . To simplify our considerations we assume in the sequel that all considered coefficients of homothetic transformations are equal each to other. Moreover, with regard for the shortness of the proof, we prove in complete only conclusion (i) and for the next one we omit the evident operations.

(i) The trapezoid T^p can be covered with five homothetic to T^p trapezoids $T_1^p,\ldots,T_5^p$ in two different ways, similarly as ones of the A and C type for a 4-covering of T^p (see [3]), i.e.:

1°. The centers of homothetic transformations for T_1^p and T_2^p lie at the vertices of the lower base of T^p, the center for T_3^p - at the common point of extensions of A^iB^i ($i\in\{1,2\}$), T_4^p is placed in such a way that it covers a part of the triangle F^1F^2H which is not covered with T_1^p, T_2^p, T_3^p, and T_5^p covers the rest of the triangle F^1F^2H (Fig.1). A 5-covering of T^p obtained in this way will be called of the A type.

2°. The centers of homothetic transformations for T_j^p ($j\in\{1,\ldots,4\}$) lie at the vertices of T^p and T_5^p is placed in such a way that it covers the triangle F^1F^2H which is not covered with T_j^p (Fig.2). This 5-covering will be called

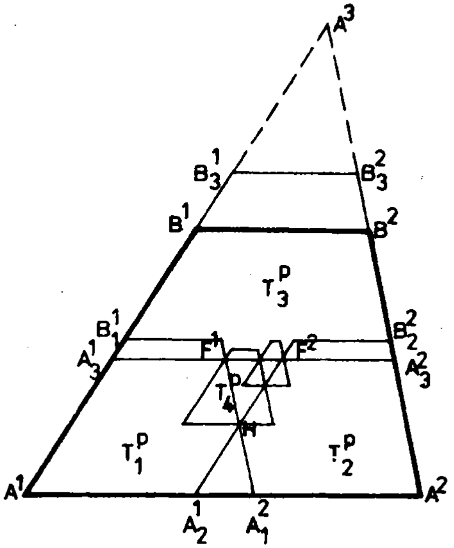


Fig.1

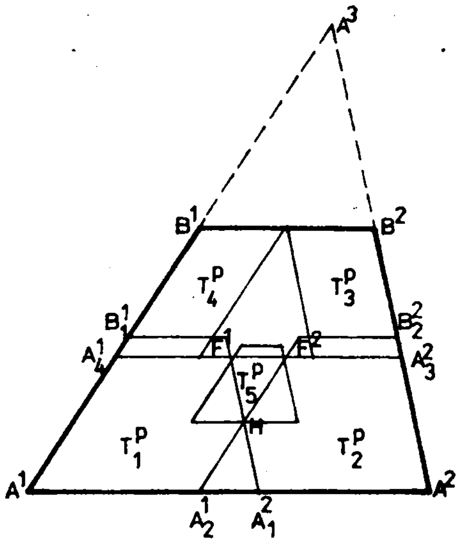


Fig.2

of the C type. (A 5-covering similar to a 4-covering of T^p of the B type is impossible; an attempt to construct such 5-covering leads to the contradiction). Five trapezoids cover T^p in the way of the A type if the inequalities

$$(2.6) \qquad A_i^iA_3^i \leqslant A_i^iB_i^i \quad \text{and} \quad 0 < A_3^1A_3^2 - (A_3^1F_o^1+F^2A_3^2) \leqslant \frac{1}{2} A_5^1A_5^2$$

are satisfied; the first - to cover the lateral edges of T^p, and the second - to cover the triangle $F_o^1F^2H_o$ (the part of the triangle F^1F^2H which is not covered with T_4^p) with T_5^p. Because

$$(2.7) \quad \begin{cases} A_{3+s}^1F_o^1 = A_{3+s}^1F^1 + F^1F_o^1, \\ A_3^iF^i = A_3^1A_3^2 - (A^1A^2-A_i^1A_i^2) = (k_3+k_i-1)A^1A^2, \\ F^1F_o^1 = HA_{4+s}^2 = A_{4+s}^1A_{4+s}^2 - (A_{3+s}^1A_3^2-A_{3+s}^1F^1-F^2A_3^2), \end{cases}$$

where s = 0, hence (2.6) is equivalent to

$$(2.8)\qquad \begin{cases} (1-p)k_i+k_3 \geqslant 1 \quad (i\in\{1,2\}), \\ 0 < 4-2(k_1+k_2+k_3)-k_4, \\ 4-2(k_1+k_2+k_3)-k_4 \leqslant \frac{1}{2}\, k_5. \end{cases}$$

The last inequalities form a necessary and sufficient condition for the existence of $Cov^5_A T^p$ (the family of 5-coverings of the A type). Setting $k_j = m$ for each $j\in\{1,\dots,5\}$ in (2.8) we get the system $m \geqslant \frac{1}{2-p}$, $m < \frac{4}{7}$, $m \geqslant \frac{8}{15}$. Then $M^5_A(T^p) = \langle \max\left(\frac{8}{15}, \frac{1}{2-p}\right), \frac{4}{7}\right)$ is a set of its solutions. There is shown in [3] that $\inf M^r_Q(F)$ is the r-th liminal number of F for any r and Q. Hence

$$(2.9)\qquad k^5_A(T^p) = \max\left(\frac{8}{15}, \frac{1}{2-p}\right) = \begin{cases} \frac{8}{15} & \text{if } p\in\langle 0, \frac{1}{8}\rangle, \\ \frac{1}{2-p} & \text{if } p\in\left(\frac{1}{8}, \frac{1}{4}\right). \end{cases}$$

To cover T^p with five homothetic to T^p trapezoids $T^p_1,\dots,T^p_5$ in the way of the C type it must be

$$A^iB^i_i \geqslant A^iA^i_{5-i} \quad (i\in\{1,2\}) \quad\text{and}\quad 0 < A^1_4A^2_3 - (A^1_4F^1+F^2A^2_3) \leqslant \frac{1}{2}\, A^1_5A^2_5.$$

(the first - to cover the lateral edges of T^p, and the second - to cover the triangle F^1F^2H), which are equivalent to

$$(2.10)\qquad \begin{cases} k_i+k_{5-i} \geqslant 1, \\ 0 < 2-p-k_1-k_2-k_3(1-p), \\ 2-p-k_1-k_2-k_3(1-p) \leqslant \frac{1}{2}\, k_5, \end{cases}$$

because of $A^i_{5-i}A^3 = A^iA^3-(A^iB^i-A^i_{5-i}B^i) = \{1-[(1-p)-k_{5-i}(1-p)]\}\, A^iA^3 = [p+k_{5-i}(1-p)]\, A^iA^3$, which implies that

$$(2.11)\quad \begin{cases} A_4^1A_3^2 = [p+k_{5-i}(1-p)]\,A^1A^2 \\ A_{5-i}^iF^i = A_4^1A_3^2-(A^1A^2-A_i^1A_i^2) = [p+k_{5-i}(1-p)+k_i-1]\,A^1A^2. \end{cases}$$

The inequalities (2.10) form a necessary and sufficient condition for the existence of $\mathrm{Cov}_C^5T^p$. Setting $k_j = m$ for each j in (2.10) we get $M_C^5(T^p) = \left\langle \max\left(\frac{1}{2}, \frac{4-2p}{7-2p}\right), \frac{2-p}{3-p}\right)$ as a set of solutions of the obtained system of inequalities. Hence

$$(2.12)\qquad k_C^5(T^p) = \max\left(\frac{1}{2}, \frac{4-2p}{7-2p}\right) = \begin{cases} \frac{4-2p}{7-2p} & \text{if } p\in\left\langle 0, \frac{1}{2}\right\rangle, \\ \frac{1}{2} & \text{if } p\in\left(\frac{1}{2}, 1\right). \end{cases}$$

The 5-th liminal number of T^p is the minimum of $k_A^5(T^p)$ and $k_C^5(T^p)$. Hence we have (2.1) in view of (2.9) and (2.12).

(ii) A homothetic 6-covering of T^p can be constructed in three ways: in the way of the C type as one for $\mathrm{Cov}_C^5T^p$ and in two other ways, which will be called of the A type and the B type. In both these ways the trapezoids T_1^p, T_2^p, T_3^p are placed in this same manner as in $\mathrm{Cov}_A^5T^p$, but the placing of the rest of homothetic trapezoids in the covering of the A type is different from the one in the covering of the B type. In the A type the centers of homothetic transformations for T_4^p, T_5^p, T_6^p respectively lie at the midpoints of these parts of the lower base or the lateral edges of T^p, which are two times covered or are not covered with the trapezoids T_1^p, T_2^p, T_3^p (Fig.3). In the B type the midpoint of the lower base of T_4^p lies at the common point of the lateral sides of T_1^p and T_2^p, and the lower base of T_{7-i}^p passes through the common point of the lateral sides of T_4^p and T_i^p for each $i\in\{1,2\}$ (Fig.4).

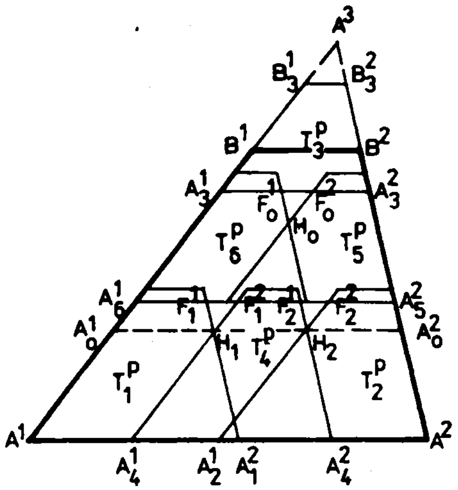


Fig.3

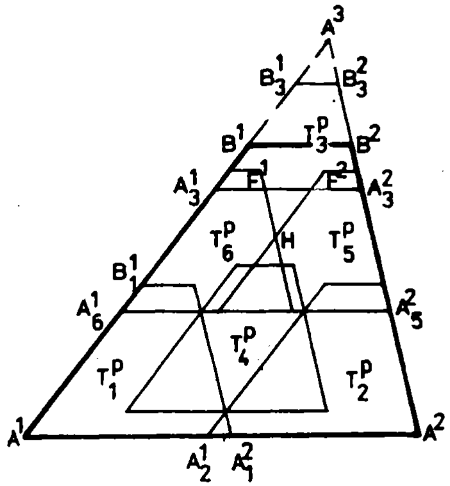


Fig.4

To cover T^p in the way of the A type it must be

$$(2.13)\begin{cases} B^1B^2 < A^1_3A^2_3, \quad A^iB^i_i+A^i_{7-i}B^i_{7-i}+A^i_3B^i_3 \geqslant A^iB^i_3 \quad (i\in\{1,2\}), \\ A^1_3F^1_0+F^2_0A^2_3-A^1_3A^2_3 \geqslant 0, \quad A^1_6F^1_1+F^2_1F^1_2+F^2_2A^2_5-A^1_6A^2_5 \geqslant 0; \end{cases}$$

the first two inequalities - to cover the upper base and the lateral edges of T^p, and the last two ones - to cover the triangles $F^1_sF^2_sH_s$ $(s\in\{0,1,2\})$. These inequalities are equivalent to

$$(2.14)\begin{cases} k_3 > p, \quad (1-p)(k_i+k_{7-i}+k_3) \geqslant 1-k_3p \quad (i\in\{1,2\}), \\ k_5+k_6+(k_i-k_{7-i})(1-p) \geqslant 1, \quad k_1+k_2+k_3+k_4-(k_i-k_{7-i})(1-p) \geqslant 2 \end{cases}$$

because of

$$(2.15)\quad \begin{cases} A^iB_3^i = A^iA^3 - B_3^iA^3 = (1-k_3p)A^iA^3, \\ A_3^iF_0^i = A_3^1A_3^2 - (A_6^1A_5^2 - A_{7-i}^1A_{7-i}^2), \\ A_6^1F_1^1 + F_1^2F_2^1 + F_2^2A_5^2 = 3A_6^1A_5^2 - 3A^1A^2 + A^1A_1^2 + A_2^1A^2 + A_4^1A_4^2, \\ A_6^1A_5^2 = \frac{1}{2}\left[(1+k_3-(k_i-k_{7-i})(1-p)\right]A^1A^2; \end{cases}$$

the last equality follows from the one of

$$A_{7-i}^iA^3 = A^iA^3 - A^iA_{7-i}^i = A^iA^3 - \left[A^iB_i^i + \tfrac{1}{2}(A^iA^3 - A^iB_i^i - A_3^iA^3) - \right.$$

$$\left. - \tfrac{1}{2}A_{7-i}^iB_{7-i}^i\right] = \tfrac{1}{2}\left[1+k_3-(k_i-k_{7-i})(1-p)\right]A^iA^3.$$

Setting $k_j = m$ for each j in (2.14) we get $M_A^6(T^p) = \left\langle \max\left(\frac{1}{2}, \frac{1}{3-2p}\right), k_0^5(T^p)\right\rangle$, because it must be $m < k_0^5(T^p)$. Then

$$(2.16)\qquad k_A^6(T^p) = \frac{1}{2} \quad \text{for} \quad p \in \left\langle 0, \frac{1}{2}\right),$$

because for $p \geq \frac{1}{2}$ there is $k_0^5(T^p) = \frac{1}{2}$ in view of (2.1).

T^p can be covered in the way of the B type (Fig.4) if (2.14) is satisfied (to cover the upper base and the lateral edges of T^p) as well as (2.13). The last is equivalent to

$$(2.17)\qquad 2(k_1+k_2)+k_3+k_4+k_5+k_6 \geq 4,$$

because in the considered case we have

$$(2.18)\qquad A_6^1A_5^2 = A^1A^2 - \frac{1}{2}A_4^1A_4^2 - A_2^1A_1^2 = \left[1 - \frac{1}{2}k_4 - (k_1+k_2-1)\right]A^1A^2.$$

The inequalities (2.14) and (2.17) form a necessary and sufficient condition for the existence of $\mathrm{Cov}_B^6T^p$. Then

$M_B^6(T^p) = \langle \max\left(\frac{1}{2}, \frac{1}{3-2p}\right), k_0^5(T^p)) = M_A^6(T^p)$. Hence $k_B^6(T^p) = k_A^6(T^p)$ for each $p \in \langle 0, \frac{1}{2})$.

To cover T^p in the way of the C type the trapezoid T_5^p must cover only a part of the triangle F^1F^2H (Fig.2) and the rest of this triangle, i.e. a triangle $F_0^1F^2H_0$, ought to be covered with a trapezoid T_6^p. For this reason (2.10) must be satisfied (to cover the lateral edges of T^p) and $0 < A_4^1A_3^2 - (A_4^1F_0^1 + F^2A_3^2) \leq \frac{1}{2} A_6^1A_6^2$. The last is equivalent to

$$(2.19)\quad 0<2-p-(1-p)k_j-k_1-k_2-\tfrac{1}{2}k_5,\ 2\left[2-p-(1-p)k_j-k_1-k_2-\tfrac{1}{2}k_5\right]\leq\tfrac{1}{2}k_6$$

($j \in \{3,4\}$), because of (2.11) and (2.7) (where $s = 1$). Set $k_j = m$ for each i,j in (2.10) and (2.19). It gives $M_C^6(T^p) = \langle \max\left(\frac{1}{2}, \frac{8-4p}{15-4p}\right), \frac{4-2p}{7-2p}\right)$ and in consequence we have

$$(2.20)\qquad k_C^6(T^p) = \max\left(\frac{1}{2}, \frac{8-4p}{15-4p}\right) = \begin{cases} \frac{8-4p}{15-4p} & \text{if } p \in \langle 0, \frac{1}{4}\rangle, \\ \frac{1}{2} & \text{if } p \in \left(\frac{1}{4}, \frac{1}{2}\right). \end{cases}$$

Hence, from (2.16) and (2.20) we get (2.2) as the minimum of these functions.

(iii) A homothetic 7-covering can be obtained in three ways: in ones of the A and B type (which correspond to such ways for 6-covering) and in the way of the D type. In the last one the centers of homothetic transformations for T_1^p and T_2^p are located at the vertices A^1 and A^2 of T^p respectively, and the center for T_3^p - in the midpoint of the part of A^1A^2, which is not covered with T_1^p and T_2^p. Then the lower base of T_{6-i}^p is drawn through the common point of lateral sides of T_3^p and T_i^p (for each $i \in \{1,2\}$) and the lower bases of T_6^p and T_7^p - through the common point of lateral sides of T_4^p and T_5^p (Fig.5). (A way of covering of T^p in which the centers of homothetic transformations for T_1^p, T_2^p, T_3^p, T_4^p lie at the vertices of T^p and the centers for

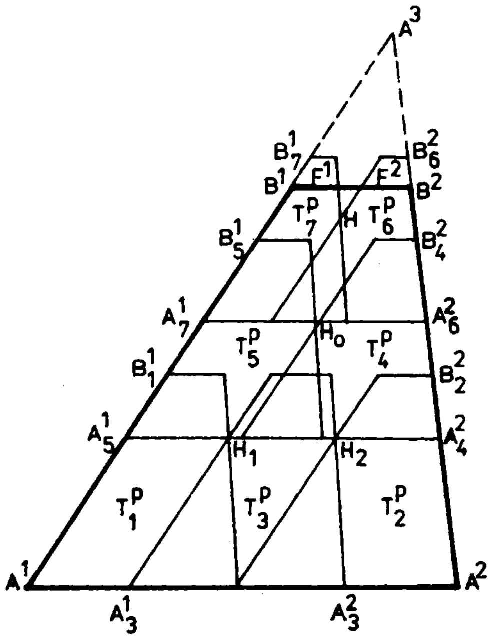


Fig.5

T_5^p, T_6^p, T_7^p - at the midpoints of the parts of the lower base and the lateral edges of T^p which are not covered with $T_1^p,\dots,T_4^p$ leads to no 7-covering but to 5-covering of T^p).

Let the triangles $F_s^1F_s^2H_s$ $(s\in\{0,1,2\})$ included in T^p be not covered with $T_1^p,\dots,T_6^p$ in the 6-covering of the A type (Fig.3). Then (2.14) is satisfied (to cover the upper base and lateral edges of T^p) and (2.14) is not satisfied, i.e. it is

(2.21) $k_5+k_6+(k_i-k_{7-i})(1-p)<1,\ k_1+k_2+k_3+k_4-(k_i-k_{7-i})(1-p)<2$

($i\in\{1,2\}$). The mentioned triangles one can cover with T_7^p, but then must be $A_0^iA_3^i \leqslant A_7^iB_7^i$ and $A_0^1A_0^2 - A_0^1A_7^1 - A_7^2A_0^2 \leqslant A_7^1A_7^2$ ($A_0^1A_0^2$ is a stright line passing through the points H_1 and H_2; Fig.3), which is equivalent to

$$k_1+k_2+2k_3+k_4+2k_7(1-p) \geqslant 3 \tag{2.22}$$

and

$$k_1+k_2+k_4+2(k_3+k_5+k_6+k_7) + (k_1+k_2-k_5-k_6)(1-p) \geqslant 5$$

because of (2.7) and $A_0^iA_3^i = A^1A^3 - A^1A_0^i - A_3^iA^3$, $A_0^iA_7^i = A_3^iF_0^i$, $A^1A_0^i = \frac{1}{2}(k_1+k_2+k_4-1)A^1A^3$. Then, from (2.14), (2.21), (2.22), we get $M_A^7(T^p) = \left\langle \max\left(\frac{5}{11}, \frac{3}{7-2p}\right), \frac{1}{2}\right)$. Hence

$$k_A^7(T^p) = \max\left(\frac{5}{11}, \frac{3}{7-2p}\right) = \begin{cases} \frac{5}{11} & \text{if } p\in\left\langle 0, \frac{1}{5}\right\rangle, \\ \frac{3}{7-2p} & \text{if } p\in\left(\frac{1}{5}, \frac{1}{2}\right)\Big|. \end{cases} \tag{2.23}$$

Consider the 7-covering of B type. For each $j\in\{1,\dots,7\}$ $k_j < k_0^6(T^p) = \frac{1}{2}$ must be here and for this reason the lower base of T_4^p is included in A^1A^2. This fact leads to the inequality $A^1A_1^2 + A_4^1A_4^2 + A_2^1A^2 \geqslant A^1A^2$ (to cover the lower base of T^p), which is equivalent to

$$k_1+k_2+k_3 \geqslant 1. \tag{2.24}$$

Let the triangle F^1F^2H be not covered with $T_1^p,\dots,T_6^p$ (Fig.4). To cover this triangle with T_7^p it must be $0 < A_3^1A_3^2 - A_3^1F^1 - F^2A_3^2 \leqslant \frac{1}{2}A_7^1A_7^2$, which is equivalent to

$$k_1+k_2+k_3+k_4+k_5+k_6 < 3, \quad 3-k_1-k_2-k_3-k_4-k_5-k_6 \leqslant \frac{1}{2}k_7 \tag{2.25}$$

in view of (2.15), where $A_6^1A_5^2$ has other form than in (2.15), namely $A_6^1A_5^2 = A^1A^2 - \frac{1}{2}(A^1A_1^2+A_4^1A_4^2+A_2^1A^2-A^1A^2)$ (it is also other form than one in (2.18)). The inequalities (2.14), (2.24) and (2.25) form a necessary and sufficient condition for the existence of $Cov_B^7T^p$. Then $M_B^7(T^p) = \langle \max(\frac{6}{13},\frac{1}{3-2p}),\frac{1}{2})$ and

$$(2.26)\qquad k_B^7(T^p) = \max(\frac{6}{13}, \frac{1}{3-2p}) = \begin{cases} \frac{6}{13} & \text{if } p\in\langle 0, \frac{5}{12}\rangle, \\ \frac{1}{3-2p} & \text{if } p\in(\frac{5}{12}, \frac{1}{2}). \end{cases}$$

T^p is covered with $T_1^p,\dots,T_7^p$ in the way of the D type (Fig.5) if (2.24) is satisfied and

$$(2.27)\qquad \begin{cases} B^1F^1+F^2B^2 \geqslant B^1B^2, & A^iB_i^i+A_{6-i}^iB_{6-i}^i+A_{8-i}^iB_{8-i}^i \geqslant A^iB_{8-i}^i, \\ A^iA_{6-i}^i \leqslant A^iB_i^i, & A_{6-i}^iA_{8-i}^i \leqslant A_{6-i}^iB_{6-i}^i \quad (i\in\{1,2\}) \end{cases}$$

(the first two - to cover the upper base and the lateral edges of T^p respectively, the last two - for the existence of H_1, H_2 and H_0), which are equivalent to

$$(2.28)\qquad \begin{cases} 6-p \leqslant 2(k_1+k_2+k_3+k_4+k_5)+k_6+k_7, \\ (k_i+k_{6-i})(1-p) \geqslant k_1+k_2+k_3+k_4+k_5-2, \\ k_1+k_2+k_3 \geqslant 1+2k_i(1-p), \\ k_1+k_2+k_3+2k_4+2k_5-3 \leqslant 2k_{6-i}(1-p), \end{cases}$$

because

$$(2.29)\quad \begin{cases} A^iB^i_{8-i} = A^iA^i_{6-i}+A^i_{6-i}A^i_{8-i}+A^i_{8-i}B^i_{8-i}, \\ A^1_5A^2_4 = A^1A^2-A^1_3A^2_1 = \frac{1}{2}(3-k_1-k_2-k_3)A^1A^2, \\ A^1_7A^2_6 = A^1_5A^2_4-A^1_4A^2_5 = 2A^1_5A^2_4-A^1_4A^2_4-A^1_5A^2_5, \\ B^iF^i = B^1B^2-(A^1_7A^2_6-A^1_{8-i}A^2_{8-i}). \end{cases}$$

From (2.24) and (2.28) we get $M^7_D(T^p) = \langle\frac{6-p}{12}, \min(\frac{1}{1+2p}, \frac{3}{5+2p}, \frac{1}{2}))$ because there must be $m < k^6_0(T^p) = \frac{1}{2}$ for $p \in \langle 0, \frac{1}{2})$. Hence

$$(2.30)\qquad k^7_D(T^p) = \frac{6-p}{12} \quad \text{for} \quad p \in \langle 0, \frac{11-\sqrt{73}}{4}\rangle,$$

because

$$\min(\frac{1}{1+2p}, \frac{3}{5+2p}, \frac{1}{2}) = \begin{cases} \frac{1}{2} & \text{if } p \in \langle 0, \frac{1}{2}\rangle, \\ \frac{1}{1+2p} & \text{if } p \in (\frac{1}{2}, 1). \end{cases}$$

Then we have (2.3) in view of (2.23), (2.26) and (2.30).

(iv) A homothetic 8-covering can be constructed in three ways: in one of the A type corresponding to the A type way for 7-covering, when T^p_7 covers not all triangles $F^1_sF^2_sH_s$ ($s \in \{0,1,2\}$, Fig.3), and in the ways called of the C and B type. In the last one the trapezoids $T^p_1,\dots,T^p_5$ are placed as in 7-covering of the D type (Fig.5), the midpoint of the lower base of T^p_8 coincides with the common point of lateral sides of T^p_4 and T^p_5, and the lower base of T^p_{8-i} ($i \in \{1,2\}$) includes the common point of lateral sides of T^p_8 and T^p_{6-i} (Fig.6). The C type of 8-covering is constructed in the following way: the trapezoids $T^p_1,\dots,T^p_4$ are placed as in 6-covering of the A type (Fig.3), the midpoint of the lower base of T^p_{9-i} ($i \in \{1,2\}$) coincides with the common point of lateral sides of T^p_4 and T^p_i, a lateral side of T^p_{7-i} is included in A^iB^i and the lower base of T^p_{7-i} includes the common point of a lateral side of T^p_{9-i} and of a lateral side or the upper base of T^p_i (Fig.8).

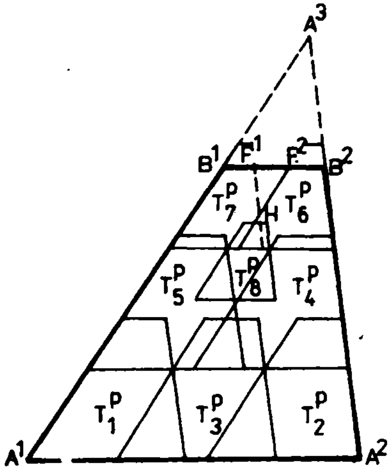
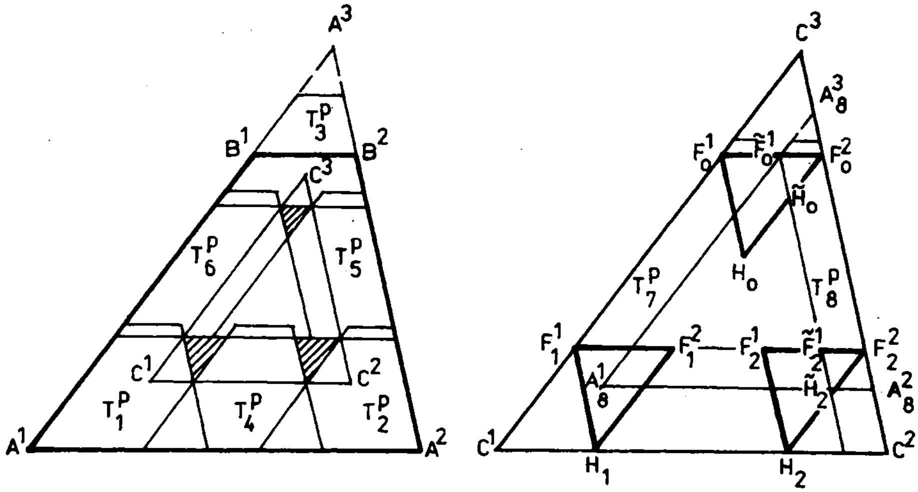

A^3
B^1
F^1
F^2
B^2
T^p_7
H
T^p_6
T^p_5
T^p_8
T^p_4
T^p_1
T^p_3
T^p_2
A^1
A^2

Fig.6


A^3
T^p_3
B^1
B^2
C^3
T^p_6
T^p_5
C^1
C^2
T^p_1
T^p_4
T^p_2
A^1
A^2
C^3
A^3_8
F^1_o
$\tilde{F}^1_o$
F^2_o
$\tilde{H}_o$
H_o
T^p_7
T^p_8
F^1_1
F^2_1
F^1_2
$\tilde{F}^1_2$
F^2_2
A^1_8
$\tilde{H}_2$
A^2_8
C^1
C^2
H_1
H_2

a) b)

Fig.7

Let T_7^p cover in the A type way only the triangle $F_1^1F_1^2H_1$ and the part of the triangle $F_0^1F_0^2H_0$ and of $F_2^1F_2^2H_2$ (Fig.3). Then the triangles $\tilde{F}_0^1F_0^2\tilde{H}_0$ and $\tilde{F}_2^1F_2^2\tilde{H}_2$ are not covered (Fig.7b). To cover them we can use T_8^p. Then (2.14) and (2.21) must be satisfied (to cover the upper base and the lateral edges of T^p and not cover all of triangles $F_s^1F_s^2H_s$). To formulate a necessary and sufficient condition for the existence of $\mathrm{Cov}_A^8T^p$ we introduce an auxiliary triangle $C^1C^2C^3$ (homothetic to the triangle $A^1A^2A^3$), which covers all $F_s^1F_s^2H_s$ ($s\in\{0,1,2\}$, Fig.7a). The trapezoid T_8^p covers $\tilde{F}_0^1F_0^2\tilde{H}_0$ and $\tilde{F}_2^1F_2^2\tilde{H}_2$ iff for each $j\in\{1,\ldots,8\}$ it is $k_j<k_0^7(T^p)$, (2.22) is satisfied and

$$A_8^2F_0^2\leqslant A_8^2B_8^2 \quad \text{and} \quad C^1C^2\leqslant A_7^1A_7^2+\tilde{F}_0^1F^2. \tag{2.31}$$

We have

$$C^1C^2 = A^1A^2-\tfrac{1}{2}(A_1^1A_1^2+A_2^1A_2^2+A_4^1A_4^2-A^1A^2)-A_3^1F_0^1-F_0^2A_3^2 = \tag{2.32}$$

$$=\left[\tfrac{5}{2}-\tfrac{1}{2}(k_1+k_2+k_4)-k_3-k_5-k_6+(k_5+k_6-k_1-k_2)(1-p)\right]A^1A^2,$$

because of (2.15). Now replenish T_8^p to the triangle $A_8^1A_8^2A_8^3$ and notice that the triangles $\tilde{F}_0^1F_0^2A_8^3$ and $\tilde{H}_2A_8^2F_2^2$ are equal and

$$F_0^2A_8^3 = A_8^2F_2^2 = \tfrac{1}{2}(A_8^2A_8^3-F_2^2F_0^2),\quad A_8^2F_0^2 = A_8^2F_2^2+F_2^2F_0^2 = \tfrac{1}{2}(A_8^2A_8^3+F_2^2F_0^2),$$

$$F_2^2F_0^2= A_5^2A_3^2 = A_5^2A^3-A_3^2A^3 =\left\{\tfrac{1}{2}\left[1+k_3-(k_2-k_5)(1-p)\right]-k_3\right\}A^2A^3.$$

It gives

$$\tilde{F}_0^1F^2 = \tfrac{1}{2}\left[k_8-k_5+\tfrac{1}{2}(k_2+k_5+k_3-1)\right]A^1A^2. \tag{2.33}$$

Using (2.32) and (2.33) in (2.31) we obtain

$$(2.34)\qquad \begin{cases} 1 \leqslant k_3+(k_2-k_5)(1-p)+k_8(2-4p), \\ \frac{5}{2} - \frac{1}{2}(k_1+k_2+k_4)-k_3-k_5-k_6+(k_5+k_6-k_1-k_2)(1-p) \leqslant \\ \qquad k_7 + \frac{1}{2}\left[k_7-k_5 + \frac{1}{2}(k_2+k_5+k_3-1)\right]. \end{cases}$$

From (2.14), (2.21), (2.22) and (2.34) we get $M_A^8(T^p) = \langle \max(\frac{11}{25}, \frac{3}{7-2p}), \frac{5}{11})$ because it must be $m < k_A^7(T^p)$. Hence

$$(2.35)\qquad k_A^8(T^p) = \max(\frac{11}{25}, \frac{3}{7-2p}) = \begin{cases} \frac{11}{25} & \text{if} \quad p \in \langle 0, \frac{1}{11}\rangle, \\ \frac{3}{7-2p} & \text{if} \quad p \in (\frac{1}{11}, \frac{1}{5}). \end{cases}$$

T^p can be covered in the way of the B type if the common point of lateral sides of T_6^p and T_7^p is placed above or on the upper base of T^p. Then (2.24) and (2.27) are satisfied. The last is equivalent to (2.28) and

$$(2.36)\qquad 6-p \leqslant 2(k_1+k_2+k_3+k_4+k_5)+k_6+k_7+k_8,$$

because in this case

$$(2.37)\qquad A_7^1A_6^2 = (3-k_1-k_2-k_3-k_4-k_5-\frac{1}{2}k_8)A^1A^2$$

for this reason that $A_7^1A^3 = A^1A_5^3-(A_4^1A_0^1+A_4^1H_0+A_8^1F_0^1)$. Then from (2.24), (2.28) and (2.36) we get $M_B^8(T^p) = \langle \frac{6-p}{13}, \min(\frac{1}{1+2p}, \frac{2}{3+2p})\rangle$ and

$$(2.38)\qquad k_B^8(T^p) = \frac{6-p}{13} \quad \text{for} \quad p \in \langle 0, \frac{11-\sqrt{65}}{4}\rangle.$$

T^p can be covered with $T_1^p,\dots,T_8^p$ in the way of the C type if the common point of lateral sides of T_6^p and T_5^p (i.e. the point H_0 in Fig.8) is placed above or on the lower base of T_3^p. This leads to the inequality

$$A_4^iH_i+A_{9-i}^iQ_i+A_{4+i}^iH_0 \geqslant A^iA_{7-i}^i. \tag{2.39}$$

Simultaneously there must be

$$B_1^1B_1^2+B_4^1B_4^2+B_2^1B_2^2 < A_6^1A_5^2 < A_5^1A_5^2+A_6^1A_6^2, \tag{2.40}$$

for the existence of the points H_0, H_1 and H_2. Also (2.24) must be satisfied.

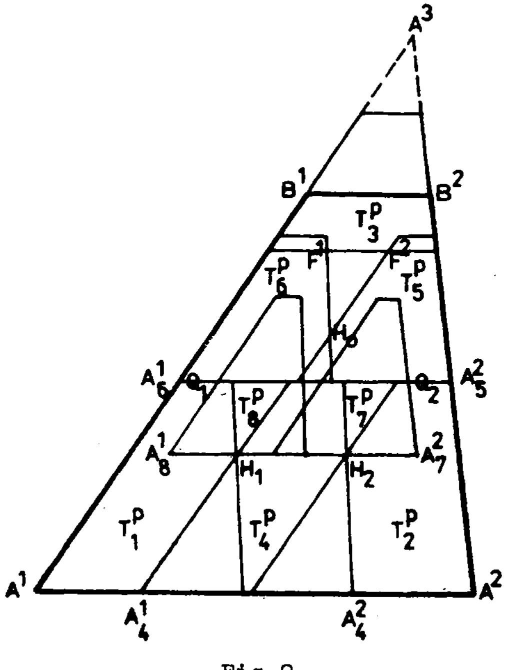


Fig.8

The inequalities (2.39) and (2.40) are equivalent to

$$\text{(2.41')}\quad \begin{cases} k_1+k_2+k_3+k_4+k_5+k_6+k_{9-i} \geqslant 3, \\ (k_1+k_2+k_4)(1+2p)+k_{9-i} < 3, \\ k_1+k_2+k_4+k_{9-i}+2(k_5+k_6) > 3, \end{cases}$$

if Q_i belongs to the lateral side $A_i^{3-i}B_i^{3-i}$ of T_i^p $(Q_i \in A_i^{3-i}B_i^{3-i})$, or to

$$(2.41'') \quad \begin{cases} k_3+k_5+k_6+2k_i(1-p) \geq 2, \\ (k_1+k_2+k_4)p+k_i(1-p) < 1, \\ 1 < k_i(1-p)+k_5+k_6, \end{cases}$$

if $Q_i \notin A_i^{3-i}B_i^{3-i}$, because

$$A_5^1A_6^2 = A_6^1A_6^2+A_5^1A_5^2-A_6^1A_5^2$$

(which implies a length of $A_{4+i}^iH_o$), and

$$A_6^1A_5^2 = \tfrac{1}{2}(3-k_1-k_2-k_4-k_{9-i})A^1A^2 \quad (\text{if } Q_i \in A_i^{3-i}B_i^{3-i})$$

or

$$A_6^1A_5^2 = [1-k_i(1-p)]\, A^1A^2 \quad (\text{if } Q_i \notin A_i^{3-i}B_i^{3-i})$$

as the base of the triangle $A^1A^2A^3$. Note also that $Q_i \in A_i^{3-i}B_i^{3-i}$ if $A_4^iH_i+A_{9-i}^iQ_i \leq A^iB_i^i$, i.e. if

$$(2.42') \quad k_1+k_2+k_4+k_{9-i}-2k_i(1-p) \leq 1,$$

and $Q_i \notin A_i^{3-i}B_i^{3-i}$ if

$$(2.42'') \quad k_1+k_2+k_4+k_{9-i}-2k_i(1-p) > 1.$$

From (2.41′) and (2.42′) we get $M_{C'}^8(T^p) = \langle \frac{3}{7}, \frac{1}{2+2p} \rangle$ and consequently

$$(2.43') \quad k_{C'}^8(T^p) = \frac{3}{7} \quad \text{for} \quad p \in \langle 0, \tfrac{1}{6} \rangle.$$

On the other hand, if $p > \frac{1}{6}$, from (2.41") and (2.42") we obtain $M_{C''}^8(T^p) = \langle \max(\frac{2}{5-2p}, \frac{1}{2+2p}), k_o^7(T^p))$, and

$$(2.43'') \quad k_{C''}^8(T^p) = \frac{2}{5-2p} \quad \text{for} \quad p \in (\tfrac{1}{6}, \tfrac{1}{3}).$$

Then, in view of (2.43′) and (2.43″), we have

$$(2.44)\qquad k_C^8(T^p) = \begin{cases} \frac{3}{7} & \text{if} \quad p \in \langle 0, \frac{1}{6}\rangle, \\ \frac{2}{5-2p} & \text{if} \quad p \in (\frac{1}{6}, \frac{1}{3}). \end{cases}$$

Taking into account (2.35), (2.38) and (2.44) we get (2.4) as the minimum of these functions.

(v) A homothetic 9-covering of T^p can be constructed in four ways: 1° in one of the A type, as for 6-covering, if every of triangles $F_s^1F_s^2H_s$ $(s \in \{0,1,2\}$, Fig.3) is covered with other homothetic trapezoid, 2° in one of the B type, as for 8-covering, if the triangle F^1F^2H is not covered (Fig.6) and 3° - 4° in ways of the C and D type. The way of the C type is following: the centers of homothetic transformations for T_1^p, T_2^p, T_6^p, T_7^p lie at the vertices of T^p, and for T_3^p and T_8^p - at the midpoints of not covered part of lower or upper base of T^p respectively. Trapezoid T_{3+i}^p is so constructed, that its lower base includes the common point of lateral sides of T_3^p and T_{3-i}^p $(i \in \{1,2\}$, Fig.9), and T_9^p - so, that its lower base includes the common points of lateral sides of T_4^p and T_5^p and of lateral sides or of the upper base of T_3^p. The way of the D type differs from the one of the C type by the placement of T_9^p only; in the D type the lower base of T_9^p is contained in this same straight line as the lower bases of T_4^p and T_5^p.

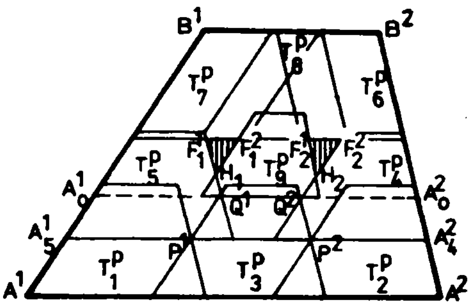


Fig.9

Let T^p_{7+s} be covering the triangle $F^1_s F^2_s H_s$ ($s \in \{0,1,2\}$, Fig.3). Then (2.14) is satisfied and $0 < F^1_s F^2_s \leqslant \frac{1}{2} A^1_{7+s} A^2_{7+s}$, which is equivalent to

$$(2.45)\quad \begin{cases} 1 > k_5+k_6+(k_i-k_{7-i})(1-p), \\ 1-k_5-k_6-(k_i-k_{7-i})(1-p) \leqslant \frac{1}{2} k_7 \quad (\text{if } s=0), \\ 3 < 3k_3-3(k_i-k_{7-i})(1-p)+2(k_1+k_2+k_4), \\ 3k_3-3(k_i-k_{7-i})(1-p)+2(k_1+k_2+k_4)-3 \leqslant 2k_{7+i} \quad (\text{if } s \in \{1,2\}) \end{cases}$$

because of (2.15). **Then, from** (2.14) and (2.45), $M^9_A(T^p) = \langle \max(\frac{2}{5}, \frac{1}{3-2p}), \frac{3}{7} \rangle$; hence

$$(2.46)\qquad k^9_A(T^p) = \max\left(\frac{2}{5}, \frac{1}{3-2p}\right) = \begin{cases} \frac{2}{5} & \text{if } p \in \langle 0, \frac{1}{4} \rangle, \\ \frac{1}{3-2p} & \text{if } p \in (\frac{1}{4}, \frac{1}{3} \rangle. \end{cases}$$

To cover the triangle F^1F^2H with T^p_9 in the way of the B type it must be satisfied (2.24), (2.28) and $0 < B^1B^2 - (B^1F^1+F^2B^2) \leqslant \frac{1}{2} A^1_9 A^2_9$. The last is equivalent to

$$(2.47)\quad \begin{gathered} 2(k_1+k_2+k_3+k_4+k_5)+k_6+k_7+k_8 < 6-p, \\ 6-p \leqslant 2(k_1+k_2+k_3+k_4+k_5)+k_6+k_7+k_8+\frac{1}{2} k_9 \end{gathered}$$

because of (2.29) and (2.37). **From** (2.24), (2.28) and (2.47) we get $M^9_B(T^p) = \langle \frac{12-2p}{27}, \min(\frac{6-p}{13}, \frac{1}{1+2p}) \rangle$. Hence

$$(2.48)\qquad k^9_B(T^p) = \frac{12-2p}{27} \quad \text{for} \quad p \in \langle 0, \frac{11-\sqrt{61}}{4}).$$

To cover the triangles $F^1_i F^2_i H_i$ ($i \in \{1,2\}$) with T^p_9 in the way of the C type (Fig.9) there must be satisfied (2.24), (2.27) and

$$(2.49)\qquad \begin{cases} A_3^iQ^i \leqslant A_3^iB_3^i, \\ A^iB^i - A_{8-i}^iB_{8-i}^i - A_3^iQ^i \leqslant A_9^iB_9^i, \\ A_7^1F_1^1 + F_1^2F_2^1 - A_7^1A_6^2 \geqslant 0. \end{cases}$$

The inequality (2.27) is equivalent to (2.28), and (2.27) and (2.49) - to

$$(2.50)\qquad \begin{cases} (k_i+k_{6-i}+k_{8-i})(1-p) \geqslant 1-p, \\ k_2+pk_3+k_{6-i} \leqslant 1, \\ 2-p \leqslant k_2+k_3+k_{6-i}+(k_{8-i}+k_9)(1-p), \\ k_1+2(k_2+k_3)+k_4+k_5+k_{6-i}+2k_8(1-p)+k_9 \geqslant 5-2p, \end{cases}$$

because

$$(2.51)\qquad A_3^iQ^i = A^iA_{6-i}^i + P^iQ^i = (k_2+k_3+k_{6-i}-1)A^1A^3$$

(P^iQ^i is an edge of a triangle, which has the base $P^iA_{6-i}^i = A_{6-i}^1A_{6-i}^2 - [A_5^1A_4^2 - (A^1A^2 - A_1^1A_1^2)]$), and

$$(2.52)\qquad \begin{cases} F_1^2F_2^1 = A_7^1A_6^2 - (A_0^1A_0^2 - A_9^1A_9^2) = A_7^1A_6^2 - [A^1A^2 - (A_3^1A_1^2 + P^1A_5^2) - A_9^1A_9^2], \\ A_7^1A_6^2 = B^1B^2 - B_8^1B_8^2 + A_8^1A_8^2 = [p+k_8(1-p)]\,A^1A^2, \\ A_{8-i}^iF_1^i = A_7^1A_6^2 - (A^1A^2 - A_3^1A_1^2 - A_{4+i}^1A_{4+i}^2). \end{cases}$$

Setting $k_j = m$ for each $j \in \{1,\ldots,9\}$ in (2.24), (2.28) and (2.50) we get $M_C^9(T^p) = \langle \frac{5-2p}{11-2p}, \frac{1}{2+p} \rangle$ as a set of solutions of the obtained system of inequalities. Hence

$$(2.53)\qquad k_C^9(T^p) = \frac{5-2p}{11-2p} \quad \text{for} \quad p \in \langle 0, \tfrac{1}{2} \rangle.$$

In the way of the D type a trapezoid T_9^p must cover both triangles $F_i^1F_i^2H_i$ ($i \in \{1,2\}$, Fig.9). Then (2.24), (2.28), (2.50'), (2.49) must be satisfied and $A_{6-i}^iA_{8-i}^i \leqslant A_9^iA_9^i$. The last two inequalities are equivalent to

$$(2.54)\qquad \begin{cases} 9-4p \leqslant 3(k_1+k_2+k_3)+2(k_4+k_5+k_9)+4k_8(1-p), \\ 3-2p \leqslant k_1+k_2+k_3+2(k_{8-i}+k_9)(1-p), \end{cases}$$

because in this case we have $F_1^2F_2^1 = A_7^1A_6^2-(A_5^1A_4^2-A_9^1A_9^2)$ (other than in (2.52')). From (2.24), (2.28'), (2.50') and (2.54) we get $M_D^9(T^p) = \langle \max(\frac{9-4p}{19-4p}, \frac{3-2p}{7-4p}), \frac{1}{1+2p}\rangle$ and we have

$$(2.55)\qquad k_D^9(T^p) = \max(\frac{9-4p}{19-4p}, \frac{3-2p}{7-4p}) = \begin{cases} \frac{9-4p}{19-4p} & \text{if} \quad p \in \langle 0, \frac{3}{4}\rangle, \\ \frac{3-2p}{7-4p} & \text{if} \quad p \in (\frac{3}{4}, 1\rangle. \end{cases}$$

In view of (2.46), (2.48), (2.53) and (2.55) we have (2.5) as the minimum of liminal functions obtained for T^p by 9-coverings of the different types.

3. Remarks and questions on liminal functions

The graphs of liminal functions $k_0^3(T^p),\ldots,k_0^9(T^p)$ are given in Fig.10. Only two from them, $k_0^4(T^p)$ and $k_0^9(T^p)$, are defined on the closed segment $\langle 0,1\rangle$, and other two, $k_0^3(T^p)$ and $k_0^5(T^p)$ - on the half-closed segment $\langle 0,1)$. Consequently, the existence of the 4-covering and 9-covering is possible for whole family of trapezoids, together with both particular cases of this family - a triangle and a parallelogram; however, the existence of 3-covering and 5-covering is possible for whole that family without a parallelogram. Remaining coverings, $Cov_0^6T^p$, $Cov_0^7T^p$, $Cov_0^8T^p$, are impossible for whole family of trapezoids because the functions $k_0^6(T^p)$, $k_0^7(T^p)$, $k_0^8(T^p)$ are defined on some proper subsegments of $\langle 0,1\rangle$. The values of all considered functions for $p = 0$ are the same as in [1] and [2] (see (1.1)). Also the ones for $p = 1$ are the same as in (1.2).

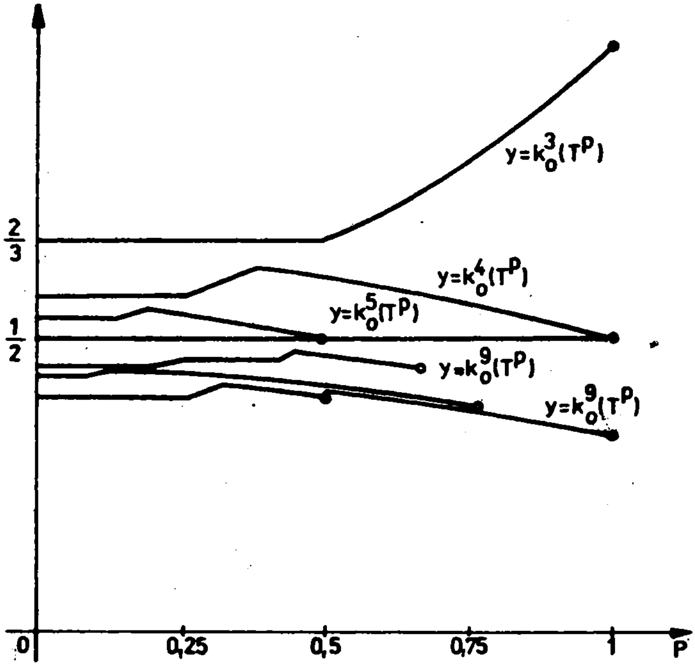


Fig.10

The analysis of the considered liminal functions leads to some questions:

1°. Does there exist an n^2-covering for the whole family of trapezoids, together with both its particular cases, for each $n \in \{2,3,\ldots\}$, i.e. is it possible to define a function $k_0^{n^2}(T^p)$ on $\langle 0,1\rangle$ for each $n \in \{2,3,\ldots\}$? If so, is the function $k_0^s(T^p)$ meaningful on the whole segment $\langle 0,1\rangle$ only in the case $s = n^2$?

The authors conjecture that the answer to the latter question is affirmative. The conjecture is supported on the fact, that there exist n^2-coverings of parallelogram only and these coverings do exist for each $n \geq 2$.

2^o. Among the liminal functions considered here only $k_0^6(T^p)$ is constant. Is there an $s > 6$, such that $k_0^s(T^p)$ is constant?

The authors conjecture that such functions do exist (if $s = m(2m-1)$) and they are defined over half-closed segments $\langle 0, s_m)$, where $s_m \to 0$, when $m \to \infty$. The authors suppose also that then it will be $k_0^{m(2m-1)}(T^p) = \frac{1}{m}$ for each $p \in \langle 0, s_m)$.

3^o. Is there an $m \in \{10, 11, \ldots\}$ such that a function $k_0^m(T^p)$ is defined for $p = 0$ only?

The liminal functions considered here are not differentiable at some points, even not all are continuous (see $k_0^9(T^p)$). This is associated with the change of a covering-way of T^p. However the function $k_0^6(T^p)$ is continuous and differentiable on the whole its domain.

4^o. Are there non-constant liminal functions, which are differentiable on the whole their domains?

5^o. What are geometrical reasons of the fact that the change of a covering-way of T^p involves the non-differentiability, even non-continuity of liminal functions at some points?

A sequence of $k_0^n(T^p)$ for fixed $p \in \langle 0, 1\rangle$ is decreasing.

6^o. Is this sequence convergent? If it is, then what is its limit?

A liminal m-covering of the class of trapezoids for $m \in \{4, 5, 7, 8, 9\}$ is non-rigid for some p, namely $\mathrm{Cov}_0^4 T^p$ is such for $p \in (\frac{1}{4}, \frac{3-\sqrt{5}}{2})$, $\mathrm{Cov}_0^5 T^p$ - for $p \in (\frac{1}{8}, \frac{3-\sqrt{7}}{2}) \cup (\frac{1}{2}, 1)$, $\mathrm{Cov}_0^7 T^p$ - for $p \in (\frac{1}{5}, \frac{1}{4}) \cup (\frac{5}{12}, \frac{15-\sqrt{177}}{4})$, $\mathrm{Cov}_0^8 T^p$ - for $p \in (\frac{1}{6}, \frac{17-\sqrt{257}}{4})$ and $\mathrm{Cov}_0^9 T^p$ - for $p \in (0, \frac{7-\sqrt{33}}{4})$. This can be observed by investigation of corresponding inequalities - (3.1) (in [3]) for $\mathrm{Cov}_0^4 T^p$, (2.8) and (2.10) for $\mathrm{Cov}_0^5 T^p$, (2.22) and (2.25) for $Cov_0^7T^p$, (2.42″) for $Cov_2^8T^p$ and (2.45) for $Cov_0^9T^p$. Substituting $k_0^m(T^p)$ in these inequalities instead of $k_1,\dots,k_{m-1}$ we always have $k_m < k_0^m(T^p)$ for the indicated p. Moreover, we can observe in $Cov_0^5T^p$ that k_5 decreases (from $\frac{8}{15}$ to $\frac{19-7\sqrt{7}}{6}$ for $p \in (\frac{1}{8}, \frac{3-\sqrt{7}}{2})$ and from $\frac{1}{2}$ to 0 for $p \in (\frac{1}{2}, 1)$), also k_7 in $Cov_0^7T^p$ decreases for $p \in (\frac{1}{5}, \frac{1}{4})$ (from $\frac{5}{11}$ to $\frac{9}{26}$) but on the interval $(\frac{5}{12}, \frac{15-\sqrt{177}}{4})$ the last increases (from $\frac{6}{13}$ to $\frac{9-\sqrt{177}}{8}$). Also k_8 in $Cov_0^8T^p$ and k_9 in $Cov_0^9T^p$ increase (the last on $(\frac{1}{4}, \frac{7-\sqrt{33}}{4})$ from $\frac{3}{10}$ to $\frac{3(20-3\sqrt{33})}{32}$). Simultaneously k_9 in $Cov_0^9T^p$ is constant on $(0, \frac{1}{4})$ and can have the value of $\frac{3}{10}$ (the liminal value is $\frac{2}{5}$ there).

7°. What are geometrical reasons that k_m increases in some non-rigid m-coverings and decreases in other ones and is constant in another?

An m-th liminal function of the class of trapezoids in general is increasing on the non-rigidity intervals of $Cov_0^mT^p$, but it sometimes occurs that it is constant on such ones. It is so for $k_0^5(T^p)$, on $(\frac{1}{2}, 1)$ and for $k_0^9(T^p)$ on $(0, \frac{1}{4})$. On the other hand, the same function $k_0^5(T^p)$ is also constant on $(0, \frac{1}{8})$, but $Cov_0^5T^p$ is rigid for $p \in (0, \frac{1}{8})$. This means that constancy of the m-th liminal function (or its increasity) has no effect on the rigidity of the corresponding liminal m-covering. However, we observe that the liminal m-covering is rigid on the invervals, on which the m-th liminal function is decreasing.

8°. Is a liminal m-covering rigid on every interval, on which the m-th liminal function decreases (for $m > 9$)?

9°. What are geometrical reasons of the non-rigidity of the liminal m-covering on some intervals?

10°. What are geometrical reasons of constancy of the m-th liminal function on some intervals?

REFERENCES

[1] Ю.Ф. Б е л о у с о в : Теоремы о покрытии плоских фигур, Ukrain. Geometr. Sb. 20 (1977) 10-17.
[2] S. F u d a l i : Homotetyczne pokrycie trójkąta, Matematyka 35 (1982) 94-109.
[3] S. F u d a l i : Homothetic coverings of frustums of simplicies, Demonstratio Math. 16 (1983) 133-150.

DEPARTMENT OF MATHEMATICS, J.JASINSKI MILITARY ENGINEERS COLLEGE,
50-961 WROCŁAW, POLAND
INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW,
50-137 WROCŁAW, POLAND
Received August 14, 1981.