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ON FUNCTIONALLY COMPLETE MODAL ALGEBRAS RELATED TO M AND S4

The finite simple modal algebras are functionally complete, a property which they have common with the two element Boolean algebra. Hence there are sublogics of M and of S4, where every proposition corresponds to a polynomial function of a modal algebra $\mathcal{B}$. Vice versa every function of B corresponds to a proposition which may contain constant symbols. We characterize the functionally complete algebra by a fixpoint property.

1. Notions

Definition 1.1. The algebra $\mathcal{B} = (B; \wedge, \vee, `, 0, 1, \tau)$ is called a modal algebra if
1) $(B; \wedge, \vee, `, 0, 1)$ is a Boolean algebra,
2) τ is an unary operation such that
$\tau(1) = 1$,
$\tau(x \wedge y) = \tau(x) \wedge \tau(y)$.

Furthermore an modal algebra $\mathcal{B}$ is of the class M if $\tau(x) \leq x$ holds. In this case $\mathcal{B}$ is connected to the modal logic M.

The modal algebra $\mathcal{B}$ is of the class S4 if $\tau(x) \leq x$ and $\tau(x) \leq \tau^2(x)$. In this case $\mathcal{B}$ is connected to the modal logic S4.

For the class S4 we have obviously $\tau^2(x) = \tau(x)$. An equivalence relation θ on B compatible with the operations of the algebra $\mathcal{B}$ is called a congruence relation of $\mathcal{B}$.

$\mathcal{L}$ is simple if only the identity relation and the all relation are the congruence relations of $\mathcal{L}$. These congruence relations are called trivial. Grätzer [4].

Definition 1.2. The element $b \in B$ is called a fixpoint of the modal algebra $\mathcal{L}$ if $\tau(b) = b$. The fixpoint b is called nontrivial if $b \notin \{0,1\}$.

Lemma 1.3. For every modal algebra $\mathcal{L}$ with a non-trivial fixpoint b the relation θ defined by $c\,\theta\,d$ iff $c \wedge b = d \wedge b$ is a non-trivial congruence relation.

Proof. It is easy to see that θ is a lattice congruence. We show that θ is also compatible with $'$ and τ. Let us consider $c\,\theta\,d$. We have $c \wedge b = d \wedge b$ und therefore $(c \wedge b) \vee b' = (d \wedge b) \vee b'$. It follows $c \vee b' = d \vee b'$ and hence $c' \wedge b = d' \wedge b$, i.e. $c'\,\theta\,d'$. The compability with the operation τ holds by the fixpoint property. If $c\,\theta\,d$, then $c \wedge b = d \wedge b$ and $\tau(c \wedge b) = \tau(d \wedge b)$. Therefore $\tau(c) \wedge b = \tau(d) \wedge b$ and hence $\tau(c)\,\theta\,\tau(d)$. It is obvious that τ is non-trivial.

Lemma 1.4. If the modal algebra $\mathcal{L}$ is of the class M then the fixpoints of $\mathcal{L}$ form a sublattice of $(B,\wedge,\vee)$.

Proof. Let b_1, b_2 be the fixpoints. As $\tau(b_1 \wedge b_2) = \tau(b_1) \wedge \tau(b_2) = b_1 \wedge b_2$ then $b_1 \wedge b_2$ is a fixpoint. For $b_1 \vee b_2$ we have $\tau(b_1 \vee b_2) \geqq \tau(b_1) \vee \tau(b_2)$, because τ is an order preserving function. Moreover $\mathcal{L} \in M$ thus we have $\tau(b_1 \vee b_2) \leqq b_1 \vee b_2$ and hence $\tau(b_1 \vee b_2) \leqq \tau(b_1) \vee \tau(b_2)$.

Lemma 1.5. Let $\mathcal{L}$ be a finite algebra of the class M. To every congruence relation θ corresponds a fixpoint b and conversely.

Proof. By Lemma 1.3 we have that to every fixpoint b corresponds a congruence relation θ defined by $c\,\theta\,d$ iff $c \wedge b = d \wedge b$. Now we have only to show that for a congruence relation θ there exists a fixpoint b such that $c\,\theta\,d$ iff $c \wedge b = d \wedge b$. We consider $\inf\{x \mid x\,\theta\,1\} = b$. Thus we have $b\,\theta\,1$; hence $\tau(b)\,\theta\,1$ and $b \leqq \tau(b)$. As $\mathcal{L} \in M$, $\tau(b) \leqq b$ and b is a fixpoint. It follows from $c\,\theta\,d$

that $c \vee d' \theta 1$ and therefore $b \leq c \vee d'$. Similarly we obtain $b \leq d \vee c'$. Hence $(b \wedge c) \vee (b \wedge d') = (b \wedge d) \vee (b \wedge c')$. We conclude $b \wedge c \leq (b \wedge d) \vee (b \wedge c')$. Thus $b \wedge c \leq b \wedge d$, because $(b \wedge c) \wedge (b \wedge c') = 0$. Furthermore we obtain also that $b \wedge d \leq b \wedge c$. Conversely, if $c \wedge b = d \wedge b$ and $b = \inf\{x | x \theta 1\}$, then we have $b \theta 1$ and therefore $c \wedge b \theta c$ and $d \wedge b \theta d$. Thus $c \theta d$.

By Lemma 1.5 it follows

T h e o r e m 1.6. The finite modal algebra $\mathcal{L}$ of the class M is simple if and only if $\mathcal{L}$ has only trivial fixpoints.

An algebra $\mathcal{L}$ is subdirect irreducible if it has a least non-trivial congruence relation.

T h e o r e m 1.7. Let $\mathcal{L}$ be a finite modal algebra of the class M which is not simple. $\mathcal{L}$ is subdirectly irreducible if and only if there is a non-trivial fixpoint b such that for every fixpoint c, $c \neq 1$, we have $c \leq b$.

P r o o f . Let $\mathcal{L}$ be subdirectly irreducible. Then there exists a least non-trivial congruence relation θ. Let $b = \inf\{x \mid x \theta 1\}$. For every other non-trivial congruence relation η we have $\theta \leq \eta$ and $c \leq b$, where $c = \inf\{x \mid x \eta 1\}$. On the other hand, if there exists a greatest non-trivial fixpoint b then θ defined by $u \theta v$ iff $u \wedge b = v \wedge b$ is a least non-trivial congruence relation.

2. The classes of simple and of subdirect irreducible algebras of the class S4

A lattice $\mathcal{L} = (L, \wedge, \vee, 0)$ with the least element 0 is atomistic if every non-zero element of L is a join of atoms.

T h e o r e m 2.1. Let $\mathcal{L}$ be an atomistic algebra of the class S4. $\mathcal{L}$ is simple iff $\tau(a) = 0$ for every element $a \in B - \{1\}$.

P r o o f . Assume $b = \tau(a) > 0$ for an element $a \neq 1$. We have $\tau(b) = \tau^2(a) = \tau(a) = b$ and b is a non-trivial fixpoint. Then by Lemma 1.3 $\mathcal{L}$ is not simple. Now let $\tau(a) = 0$ for every element $a \in B - \{1\}$ and assume that θ is a congruence relation which is not the identity. As $\mathcal{L}$ is atomistic, then there exists an atom $e \in B$ with $e\,\theta\,0$. Therefore $e'\,\theta\,1$ and $\tau(e')\,\theta\,1$. Thus we obtain $0\,\theta\,1$ and θ is the all relation.

There are two extreme cases for finite subdirectly irreducible algebras of the class S4. Let b be the greatest non-trivial fixpoint. The first case is that every element c with $c < b$ is also a fixpoint. We call algebras of this kind of the class P. The other case is that for every element c, $c < b$, we have $\tau(c) = 0$. Algebras of this kind are called of the class D.

T h e o r e m 2.2. The algebras of the class P are subdirectly irreducible. The operation τ can be defined by $\tau(x) = x \wedge b$ for every $x \in (B-\{1\}$.

P r o o f . It follows by Theorem 1.7 that the algebras of the class P are subdirectly irreducible. If we define $\tau(x) = x \wedge b$ for $x \in B-\{1\}$, we have $\tau(b) = b$, $\tau^2(x) = x$ and $\tau(c) = c$ for every $c \leq b$.

Similarly, by Theorem 1.7 we obtain

T h e o r e m 2.3. The algebras of the class D are subdirectly irreducible.

Moreover, as $\tau^2(x) = \tau(x)$, each $\tau(x)$ have to be a fixpoint. Therefore for the algebras of the class D we have $\tau(1) = 1$, $\tau(x) = b$ for $b \leq x < 1$, and $\tau(x) = 0$ otherwise.

3. Functionally complete algebras of M and S4

The underlying structure of a modal algebra is a Boolean algebra. As the Boolean algebras form an arithmetical variety the class of modal algebras is also an arithmetical variety (Pixley [8]). In an arithmetical variety $\mathcal{K}$ the algebra $\mathcal{L}$ is functionally complete if and only if $\mathcal{L}$ is finite and simple. Furthermore, every functionally complete algebra $\mathcal{L}$ in $\mathcal{K}$ generates a subvariety which can be finitely axiomatized (Baker [1]).

We will not follow these considerations of universal algebra but prove in detail that the finite simple algebras of the class S4 are functionally complete.

A finite non-trivial algebra $\mathcal{B}$ is functionally complete, if for every $n \geq 1$, each function $f: B^n \to B$ is a polynomial function.

T h e o r e m 3.1. Every finite simple algebra $\mathcal{B}$ of S4 is functionally complete.

P r o o f . Let $\mathcal{B}$ be finite and simple of S4. Then $\tau(1) = 1$ and $\tau(u) = 0$ for every $u \in B - \{1\}$ by theorem 2.1. Consider two elements $c, d \in B$ and the function $t_{cd}(x) = \tau[(x \vee c') \wedge (x' \vee c)] \wedge d$. We have that $t_{cd}(c) = d$ and $t_{c,d}(v) = 0$ for $v \neq c$. Obviously t_{cd} is a polynomial function of B i.e. $t_{c,d}$ is composed by projections, constants and the operations $\wedge, \vee, ', \tau$. If $f: B \to B$ is a function and if we put $b_i = f(a_i)$, $B = \{a_1, \dots, a_n\}$ then we have $f(x) = \bigvee_{i=1}^{n} t_{a_i b_i}(x)$. Therefore every 1-place function is a polynomial function and hence also every n-place function.

Finally we notice that these algebras give further possibilities to find switching algebras for multivalued logic functions (Troy, Camalle, Irvin [11]).

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