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ON SOME PROBLEM OF MARCUS

In this paper we present a solution to the following problem, raised by S. Marcus in [2]: does there exist, for any set E of class F_σ and of the first category of Baire, a function f of type α (i.e., a function whose set of points of continuity is dense and boundary) such that

$$E = \Delta_f \quad \text{and} \quad C_f - \Delta_f = \Delta_f(\infty) \cup M_f,$$

where

$$\Delta_f = \left\{x:\ |f'(x)|<\infty\right\}, \quad \Delta_f(\infty) = \left\{x:\ f'(x) = \pm\infty\right\},$$

C_f - the set of all points of continuity of the function f,
M_f - the set of all those points at which there is no one-sided (finite or infinite) derivative. Besides, in this paper we shall apply the following notation:

I - any closed interval with finite Lebesgue measure,

R - the set of real numbers,

$D_A^-f(x)$ - the set of all left-hand derivative numbers of a function $f|_{A\cup\{x\}}$ at the point x,

$D_Af(x)$ - the set of all derivative numbers of a function $f|_{A\cup\{x\}}$ at the point x,

$D_A^+f(x)$ - the set of all right-hand derivative numbers of a function $f|_{A\cup\{x\}}$ at the point x,

D_f - the set of all points of discontinuity of the function f,

$m(A)$ - the Lebesgue measure of the set A,

$m^*(A)$ - the exterior Lebesgue measure of the set A.

D e f i n i t i o n . Let a function $f : I \to R$ be given. We shall say that a set $A \subset I$ has the property B_f if

$$\bigwedge_{x_0 \in I} \bigwedge_{\varepsilon>0} \bigwedge_{\delta>0} \bigvee_{x' \in (x_0-\delta, x_0+\delta)\cap A} |f(x_0) - f(x')| < \varepsilon .$$

T h e o r e m 1. If, for a function $f : I \to R$, the set $A \subset I$ has the property B_f, then

(a) $D_I f(x) = D_A f(x)$ for $x \in I$,

(b) $D_I^- f(x) = D_A^- f(x)$ for $x \in I$,

(c) $D_I^+ f(x) = D_A^+ f(x)$ for $x \in I$.

P r o o f . It is obvious that $D_I f(x) > D_A f(x)$. In order to prove (a), it suffices to show that $D_I f(x) \subset D_A f(x)$. Let us take $x \in I$ and $\ell \in D_I f(x)$. Consider two cases. First suppose that $\ell \neq 0$ (we allow that $\ell = \pm\infty$). Then there exists a sequence $\{x_n\}$, $x_n \neq x$, $x_n \in I$, such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = \ell$. Since the set A has the property B_f, for every point x_n as well as for $\varepsilon = \frac{|f(x) - f(x_n)|}{n}$ and $\delta = \frac{|x - x_n|}{n}$ (since $\ell \neq 0$, we may assume that the sequence is chosen such that always $\varepsilon > 0$ and $\delta > 0$), there exists a point $y_n \in A$ such that

$$(1) \qquad |y_n - x_n| < \frac{|x - x_n|}{n},$$

and

$$(2) \qquad |f(x_n) - f(y_n)| < \frac{|f(x) - f(x_n)|}{n} .$$

We shall demonstrate that $\lim\limits_{n\to\infty} \frac{f(x) - f(y_n)}{x - y_n} = \ell$. To this end, it is enough to show that

$$(3) \qquad \lim_{n\to\infty} \frac{\dfrac{f(x) - f(y_n)}{x - y_n}}{\dfrac{f(x) - f(x_n)}{x - x_n}} = 1.$$

From (1) we have

$$\frac{n}{n+1} < \frac{x - x_n}{x - y_n} < \frac{n}{n-1},$$

whereas from (2)

$$0 < \frac{n-1}{n} < \frac{f(x) - f(y_n)}{f(x) - f(x_n)} < \frac{n+1}{n} \quad \text{for} \quad n \geqslant 2.$$

Consequently, from the above inequalities it follows that

$$\frac{n-1}{n+1} < \frac{\dfrac{f(x) - f(y_n)}{x - y_n}}{\dfrac{f(x) - f(x_n)}{x - x_n}} < \frac{n+1}{n-1}.$$

Hence we get (3).

Now suppose that $l = 0$. There exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$, $x_n \neq x$, and $\lim_{n\to\infty} \dfrac{f(x_n) - f(x)}{x_n - x} = 0$. Take $g(x) = f(x) + x$. It is easily seen that the set A has the property B_g. Then $\lim_{n\to\infty} \dfrac{g(x_n) - g(x)}{x_n - x} = 1$. So, for the function g and for $l = 1$, we may apply the proposition of our theorem. Consequently, there exists a sequence $\{y_n\}$, $y_n \in A$, such that $\lim_{n\to\infty} \dfrac{g(y_n) - g(x)}{y_n - x} = 1$. Hence it appears that $\lim_{n\to\infty} \dfrac{f(y_n) - f(x)}{y_n - x} = 0$. In this way, we have proved (a). The proofs of (b) and (c) are obtained by a simple modifications of the above reasoning.

From Theorem 1 a few corollaries result.

C o r o l l a r y 1. If f is a continuous function $f: I \to R$, and the set A is dense in I, then the conditions (a), (b), (c) are satisfied.

C o r o l l a r y 2. If f is a continuous function $f: I \to R$, and the set A is of full measure in I, then the proposition of Corollary 1 is true.

C o r o l l a r y 3. If $f: I \to R$ is an approximatively continuous function, and the set A is of full measure in I, then also conditions (a), (b), (c) are true.

C o r o l l a r y 4. If $f: I \to R$ is an approximatively continuous function, and the set A satisfies the condition

$$\bigwedge_{x\in I} \bigvee_{\eta>0} \bigvee_{\delta_0>0} \bigwedge_{\delta\in(0,\delta_0)} m^*[(x-\delta, x+\delta)\cap A] > 2\delta\eta ,$$

then the conditions (a), (b), (c) are true.

T h e o r e m 2. If a set $E \in F_\sigma$ is of the first category of Baire, then there exists $f: I \to R$ of type α, such that $E = \Delta_f$ and $C_f - \Delta_f = \Delta_f(\infty) \cup M_f$.

P r o o f . Let $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = \overline{E}_k$; the sequence $\{E_k\}$ is increasing. Since $I \setminus E$ is a residual set, there exists a set $A = \{x_1, x_2, \dots\}$ such that $\overline{A} = I$ and $A \cap E = \emptyset$.

Let us define a function g:

$$g(x) = \begin{cases} \frac{1}{i}\,\varrho(x_i, E_i) & \text{for } x = x_i,\ i = 1,2,\dots \\ 0 & \text{for } x \neq x_i,\ i = 1,2,\dots \end{cases}$$

If $E_i = \emptyset$, we assume that $\varrho(x_i, E_i) = m(I)$. The function g is continuous at points of the set $I \setminus A$. Indeed, let us take $x_0 \in I \setminus A$ and $\varepsilon > 0$. We shall find an i_0 such that $\frac{m(I)}{i_0} \leq \varepsilon$. There exists some $\delta > 0$ such that in the interval $(x_0-\delta, x_0+\delta)$

there is $0 \leqslant g(x) \leqslant \frac{m(I)}{i_0} \leqslant \varepsilon$. So, the function g is continuous in the set $I \setminus A$. Hence, if $x \in A$, then $g(x) > 0$ and there exists a sequence $\{a_n\}$ such that $a_n \to x$ and, for every n, $a_n \notin A$. Hence we get that $\lim_{n\to\infty} g(a_n) = 0$, which yields $x \in D_f$. Consequently, $A = D_g$. Therefore, the function g is of type α.

Let us take $x_0 \in E$. We shall show that $g'(x_0) = 0$. Let $\varepsilon > 0$. There exists some $j \in N$ such that $\frac{1}{j} < \varepsilon$ and $x_0 \in E_j$. Take an $h > 0$ such that in (x_0-h, x_0+h) there are no points $x_1, x_2, \ldots, x_j \in A$. Let us now consider the two cases: 1) $x_0+h \notin A$ and 2) $x_0+h = x_s \in A$, $s > j$. If $x_0+h \notin A$, then

$$\left| \frac{g(x_0+h) - g(x_0)}{h} \right| = \left| \frac{g(x_0+h)}{h} \right| = 0.$$

If $x_0+h = x_s \in A$, $s > j$, then $h = \varrho(x_0, x_s) \geqslant \varrho(x_s, E_s)$, which gives

$$\left| \frac{g(x_0+h) - g(x_0)}{h} \right| = \left| \frac{g(x_0+h)}{h} \right| =$$

$$= \frac{\varrho(x_s, E_s)}{s \cdot h} \leqslant \frac{\varrho(x_s, E_s)}{s \cdot \varrho(x_s, E_s)} = \frac{1}{s} < \frac{1}{j} < \varepsilon.$$

It follows from a theorem of Z.Zahorski in [3] that there exists a continuous function φ such that $\Delta\varphi = E$, $M_\varphi = I \setminus E$. Put

$$f(x) = g(x) + \varphi(x).$$

The function f is of type α, $D_f = A$, $E \subset \Delta_f$. Let now $x \notin E \cup A$. Consequently, $x \in M_\varphi$. So, at this point there exist at least two distinct right-hand derivative numbers l_1, l_2 and two distinct left-hand derivative numbers l'_1, l'_2. Let $\{y_n\}$, $\{z_n\}$ be sequences such that $y_n \to x$, $y_n > x$, $z_n \to x$, $z_n > x$, $y_n \notin A$, $z_n \notin A$ and

$$(4) \qquad \lim_{n\to\infty} \frac{\varphi(y_n) - \varphi(x)}{y_n - x} = l_1,$$

(5) $$\lim_{n\to\infty} \frac{\varphi(z_n) - \varphi(x)}{z_n - x} = l_2.$$

The existence of such sequences results from Corollary 1. From condition (4) we have

(6) $$\lim_{y_n\to x} \frac{f(y_n) - f(x)}{y_n - x} = \lim_{n\to\infty} \frac{\varphi(y_n) - \varphi(x)}{y_n - x} = l_1,$$

whereas from (5) we obtain

(7) $$\lim_{n\to\infty} \frac{f(z_n) - f(x)}{z_n - x} = \lim_{n\to\infty} \frac{\varphi(z_n) - \varphi(x)}{z_n - x} = l_2.$$

From (6) and (7) it follows that $l_1, l_2 \in D^+_A f(x)$. Similarly, one proves that $l'_1, l'_2 \in D^-_A f(x)$. Hence we infer that, if $x \notin E \cup A$, then $x \in M_f$. In consequence, $E = \Delta_f$ and $C_f - \Delta_f = \Delta_f(\infty) \cup M_f$. Note that $\Delta_f(\infty) = \emptyset$.

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