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A GENERIC PROPERTY OF VECTOR FIELDS ON OPEN SURFACES

This paper deals with the problem of structural stability of vector fields on open manifolds which is one of the central research themes in dynamical systems. There are known conditions which imply global structural stability of vector fields on open surfaces (see [3]). It is interesting to know if these conditions are also necessary. So far the answer is known only for R^2. In this paper we prove the necessity of one of these conditions for vector fields defined on some class of open surfaces.

Let M be homeomorphic to S^2 without a countable number of points which form a closed subset of S^2. By E we denote the set of "infinities" of M.
$H^r(M)$ - the space of C^r vector fields on M which generate flows endowed with C^r-Whitney (strong) topology ($r \geqslant 1$),
$\Phi_Y: M \times R \longrightarrow M$ the flow of Y.

We denote by $O_Y(x)$ an orbit of Y starting with x i.e. $\Phi_Y(x,0) = x$ and define the positive (resp. negative) semi-orbit by

$$O_Y^+(x) = \{\Phi_Y(x,t) \ : \ t > 0\},$$

$$O_Y^-(x) = \{\Phi_Y(x,t) \ : \ t < 0\}.$$

Finally we denote by $O_Y[x,y]$ the closed Y-orbit segment from x to y.

We distinguish three kinds of asymptotic behavior for each semi-orbit:

(a) $O_Y^{\pm}(x)$ is bounded if it is contained in some compact set $K \subset M$;

(b) $O_Y^{\pm}(x)$ escapes to infinity if for each compact set $K \subset M$ there exists a point $y \in O_Y^{\pm}(x)$ for which $O_Y^{\pm}(x) \cap K = \emptyset$;

(c) $O_Y^{\pm}(x)$ oscillates if it is neither bounded nor escapes to infinity.

These kinds of behavior for $O_Y^{+}(x)$ (resp. $O_Y^{-}(x)$) can be also described in terms of the ω-limit (resp. α-limit) set of $x \in M$ under Φ_Y, namely

$$\begin{cases} \omega(O_Y^{+}(x)) = \{ y \in M : \exists t_n \longrightarrow +\infty . \ni . \ \Phi_Y(x,t_n) \longrightarrow y \}, \\ \alpha(O_Y^{-}(x)) = \{ y \in M : \exists t_n \longrightarrow -\infty . \ni . \ \Phi_Y(x,t_n) \longrightarrow y \}. \end{cases}$$

It is easy to see that we can distinguish the following cases:

(a) is bounded iff $\omega(O_Y^{+}(x))$ is compact (and nonempty),

(b) escapes to infinity iff $\omega(O_Y^{+}(x)) = \emptyset$,

(c) oscillates iff $\omega(O_Y^{+}(x))$ is a noncompact subset of M.

We extend the definition of ω-limit (resp. α-limit) set of $x \in M$ to ω^*-limit (resp. α^*-limit) set

$$\begin{cases} \omega^*(O_Y^{+}(x)) = \{ y \in M \cup E : \exists t_n \longrightarrow +\infty . \ni . \ \Phi_Y(x,t_n) \longrightarrow y \}, \\ \alpha^*(O_Y^{-}(x)) = \{ y \in M \cup E : \exists t_n \longrightarrow -\infty . \ni . \ \Phi_Y(x,t_n) \longrightarrow y \}. \end{cases}$$

Thus

(a) $O_Y^{+}(x)$ escapes to infinity iff there exists $P \in E$ such that $\omega^*(O_Y^{+}(x)) = \{P\}$,

(b) $O_Y^{+}(x)$ oscillates iff $\omega(O_Y^{+}(x)) \neq \emptyset$ and there exists $P \in E$ such that $P \in \omega^*(O_Y^{+}(x))$.

Let PerY, $\Omega(Y)$ denote respectively the periodic points and the non-wandering points of Y, i.e.

$$\begin{cases} \text{Per } Y = \{ x \in M : \Phi_Y(x,t) = x \text{ for some } t > 0 \}, \\ \Omega(Y) = \{ x \in M : \exists x_n \to x,\ t_n \to +\infty \ .\ni.\ \Phi_Y(x_n,t_n) \to x \}. \end{cases}$$

The first positive (resp. negative) prolongation limit set of $x \in M$ under

Φ_Y is $J_Y^{\pm}(x) = \{ y \in M : x_n \to x,\ t_n \to \pm\infty \ .\ni.\ \Phi_Y(x_n,t_n) \to y \}$.

In general, $\omega(O_Y^+(x)) \subset J_Y^+(x)$ and $\alpha(O_Y^-(x)) \subset J_Y^-(x)$, and one can easily see that $\Omega(Y) = \{ x \in M : x \in J_Y^{\pm}(x) \}$.

Modifying the definition of Nemytskii-Stepanov [4] we say that two unbounded semi-orbits $O_Y^+(x)$ and $O_Y^-(y)$ form a saddle at infinity if each escapes to infinity and $y \in J_Y^+(x)$ (i.e. $x \in J_Y^-(y)$). In this case we call $O_Y^+(x)$ (resp. $O_Y^-(x)$) the stable (resp. unstable) separatrix of the saddle at infinity.

By W_Y^+ (resp. W_Y^-) we denote the union of all stable (resp. unstable) separatrices of fixed saddles and at infinity. Each set is Y-invariant; it may consists of finitely or infinitely many distinct orbits. In either case, it is not generally closed, since a fixed saddle belongs to the closure of its separatrices.

At first we recall what so far is known about the sufficient and necessary conditions for global C^r-structural stability of vector fields on open surfaces.

T h e o r e m 1. If N is an open surface and Y is a complete C^r-vector field on N satisfying the following conditions:

(i) there are non-trivial minimal sets and no oscillating orbits,

(ii) every orbit in Per Y is hyperbolic,

(iii) $\mathrm{cl}W_Y^- \cap \mathrm{cl}W_Y^+ \subset \text{Per } Y$, then

(a) $\Omega(Y) = \text{Per } Y$,

(b) Y is a globally C^r-structurally stable.

Theorem 1 is proved in [3].

T h e o r e m 2. Let Y be a globally C^r-structurally stable vector field.

(a) If Y is on M then

(i) $\Omega(Y) = \text{Per } Y$,

(ii) Y has no oscillating orbits;

(b) If Y is on any open surface N then

(i) every orbit in Per Y is hyperbolic,

(ii) Y has no non-trivial minimal sets.

Part (a) of this theo em is proved in [2], (b) in [3].

It follows from Theorem 2 that so far it is not known whether condition (ii) is necessary for global C^r-structural stability of vector fields on M. The aim of this paper is to show, that some condition weaker than (iii) is necessary. First we shall describe some properties useful in the formulation of our condition.

L e m m a 1. Suppose that Y is C^r vector field on the open surface for which $\Omega(Y) = \text{Per } Y$ consists of hyperbolic orbits. If $\text{cl}W_Y^-$ intersects $\text{cl}W_Y^+$ at a nonsaddle x, then there exist C^r perturbations of Y such that $W_Z^- \cap W_Z^+ \neq \emptyset$.

By Theorem 2 and Lemma 1 which is proved in [3] it is enough to show that condition $W^- \cap W^+ = \emptyset$ is necessary for global C^r-structural stability. But when all orbits in Per Y are hyperbolic then the condition $W_Y^- \cap W_Y^+ = \emptyset$ has three distinct parts:

1. no saddle connections;
2. the stable and unstable separatrices of fixed saddles are not involved in any saddle at infinity;
3. the stable separatrices of saddles at infinity are not also the unstable separatrices of saddles at infinity.

The Kupka-Smale theorem (see [5]) shows that condition 1 is generic. In this paper we prove that condition 2 is also generic and therefore it is necessary for global C^r-structural stability (see Theorem 4 and Corollary).

At first we recall the definition and properties of a cycle. Definition 1, Remark 1 and Lemma 2 are from [2]. For $x \in M$ which is not a restpoint of $Y \in H^r(M)$ by (a,x) (resp. (x,b)) we denote the open transversal interval with left end a and right end x (resp. left end x and right end b).

D e f i n i t i o n 1. Let the sequence $O_Y[x_n, \bar{x}_n]$ satisfy for any $n \in N$ one and only one of the following conditions:

(i_1) $x_n \in (a,b)$, $x_n \longrightarrow x$ and the first intersection $\bar{x}_n$ of $O^+_Y(x_n)$ with (a,x) lies between x_n and x_{n+1};

(ii_1) $x_n \in (a,x)$, $x_n \longrightarrow x$ and the first intersection $\bar{x}_n$ of $O^+_Y(x_n)$ with (a,x) lies between x_{n-1} and x_n;

(iii_1) $x_n \in (a,b)$, $x_n \longrightarrow x$ and the first intersection x_n of $O^+_Y(x_n)$ with (a,x) satisfy $\bar{x}_n = \bar{x}_n$;

(i_2) - (iii_2) are similar to (i_1) - (iii_1) using (x,b) instedad of (a,x) then the set $\{y \in M \cup E\colon y = \lim z_n$ and $z_n \in O_Y[x_n, \bar{x}_n]\}$, which is not a closed orbit of Y, we call the cycle of Y through x. The cycle of Y through x we will denote by $C_Y(x)$.

Let us denote by $H^r_0(M)$ the subset of $H^r(M)$ such that every element of $H^r_0(M)$ has only hyperbolic restpoints.

R e m a r k 1. If $Y \in H^r_0(M)$ and $O^+_Y(x)$ oscillates then there exists a cycle $C_Y(z)$ such that $\omega^*(O^+_Y(x)) = C_Y(z)$ $(z \in \omega(O^+_Y(x)))$.

L e m m a 2. Let there be given a compact region $K \subset M$ and $P \in E$. Then there exists an open and dense set $H^r_P(K) \subset H^r(M)$ such that if $Y \in H^r_P(K)$ and $x \in \mathrm{cl} K$ then Y has no cycl $C_Y(x)$ which contains P.

The next two lemmas are simple adaptations of the well known theory for compact manifolds (see [1]). They are formulated using the C^r compact-open topology which is weaker than Whitney topology.

L e m m a 3. Suppose x is a fixed hyperbolic sink (resp. source) for $Y \in H^r(M)$ and pick K a compact neighbourhood of x contained in its basin of attraction $W_Y^+(x)$ (resp. region of repulsion $W_Y^-(x)$) with boundary ∂K transverse to the flow ϕ_Y. There exists a compact-open C^r-neighbourhood U of Y, concentrated on K, and points $x^Z \in K$, varying C^r-continuously with $Z \in U$, such that

(i) $x^Y = x$,

(ii) x^Z is a fixed hyperbolic sink (source),

(iii) $K \subset W_Z^+(x^Z)$ $(K \subset W_Z^-(x^Z))$.

The corresponding lemma for saddle uses the fact that compact parts of stable and unstable separatrices to fixed saddles vary C^r-continuously with dynamical systems (see [1]).

For a fixed saddle x and $y \in W_Y^{\pm}(x)$ we denote by $\Sigma(x,y)$ the open segment of separatrix joining x and y.

L e m m a 5. Suppose x is a fixed hyperbolic saddle for $Y \in H^r(M)$ and $y \in W_Y^{\pm}(x)$. There exists a compact-open C^r-neighbourhood of Y concentrated on an arbitrarily given compact neighbourhood of $\mathrm{cl}\,\Sigma(x,y)$ and points x^Z, y^Z varying continuously with $Z \in U$ such that

(i) $x^Y = x$, $y^Y = y$,

(ii) x^Z is a fixed hyperbolic saddle for Z,

(iii) $y^Z \in W_Z^{\pm}(x^Z)$,

(iv) the arc $\Sigma(x^Z, y^Z)$ varies C^r-continuously with $Z \in U$.

D e f i n i t i o n 2. A flowbox for a vector field Y on a surface N is a closed quadrilateral $F \subset N$ containning no restpoints of Y, with two opposite edges $S_{\pm}$ transverse to Y and the other two edges Y-orbit segments, each joining an endpoint of S_+ to an endpoint of S_-. We call S_+ the entrance set and S_- the exit set of F.

P r o p o s i t i o n 1. Let there be given a flowbox F for $Y \in H^r(M)$, a point $p \in \mathrm{int} S_+$, and a C^r-neighbourhood U of Y. Then there exist a neighbourhood $\tilde{S}_+$ of p in S_+ and a flowbox $\tilde{F} \subset F$ with entrance set $\tilde{S}_+$ (and corresponding exist set $\tilde{S}_- \subset S_-$) such that for any pair of points $q_+ \in \tilde{S}_+$ there exists a vector field Z satisfying the conditions:

(i) $Z \in U$,

(ii) $Z = Y$ off F,

(iii) $q_- \in O_Z(q_+)$ and $O_Z[q_+,q_-] \subset F$.

P r o o f . Pick a C^∞ function $f : R^2 \to [0,1]$ which vanishes off F and is positive at every point of int$F$. For $u \in R$ let $Y_u(x) = Y(x) + uf(x)V(x)$ where V is a vector field perpendicular to Y on F. There exists $\varepsilon > 0$ such that $Y_u \in U$ for all $u \in [-\varepsilon,\varepsilon]$, and Y_u satisfies (ii). Furthermore, for $u \neq 0$, the vector field Y_u is transverse to Y on int$F$. For $q \in \mathrm{int}S_+$ and $|u|$ small we define $g(u,q)$ as the first point of $O^+_Y(q)$ on S_- (i.e. $g(u,\cdot)$ is the Poincaré map of Y_u). It is easy to see that g is a continuous function of two variables and for q fixed $g(u,q)$ is strictly monotone with respect to u. Thus, for $\delta > 0$ small and q near p, the set $G(\delta,q) = \{g(u,q) : |u| < \delta\}$ varies continuously with q, so that for given $0 < \delta < \varepsilon$ we can find $q_1 < g(0,p) < q_2$ and a neighbourhood V of p in S_+ such that for every $q \in V$, $G(\delta,q)$ includes $[q_1,q_2]$. If we pick $p_i = S_+ \cap O^-_Y(q_i)$, $i=1,2$ then the flowbox $\widetilde{F}$ defined by $\widetilde{S}_+ = S_+[p_1,p_2]$, $\widetilde{S}_- = S_-[q_1,q_2]$ satisfies (iii) for an appropriate $Y_u \in U$.

P r o p o s i t i o n 2. Suppose that $O^+_Y(p_1)$, $O^-_Y(p_2)$ form a saddle at infinity of $Y \in H^r_0(M)$. Then for any disjoint neighbourhoods V_i of p_i, $i = 1,2$ and C^r-neighbourhood U of Y there exists $Z \in U$ such that:

(i) $Z = Y$ off $V_1 \cup V_2$,

(ii) $p_2 \in O^+_Z(p_1)$.

P r o o f . Pick $S_i \subset V_i$ compact transverse sections at p_i, $i = 1,2$. Because $O^+_Y(p_1)$, $O^-_Y(p_2)$ escape to infinity, S_1,S_2 can be chosen such that $\mathrm{cl}O^+_Y(p_1) \cap S_2 = \emptyset$, $\mathrm{cl}O^-_Y(p_2) \cap S_1 = \emptyset$ and $O_Y(p_i)$ has exactly one common point with S_i. Pick $F_i \subset V_i$ flowboxes with S_1 (resp. S_2) the entrance (resp. exit) set of F_1 (resp. F_2) and apply Proposition 1 to obtain $\widetilde{F}_i$ with $p_i \in \widetilde{S}_i \subset S_i$. Let S^*_i denote the opposite transverse edge of $\widetilde{F}_i$, p^*_i the intersection of $O^+_Y(p_i)$ and S^*_i. Let S^{**}_2 be compact transverse interval with $p^*_2 \in S^{**}_2 \subset \mathrm{int}S^*_2$. For S^{**}_2 there exist transverse interval $S_1^{**} \subset S_1^{*}$ and points $x_k \longrightarrow p_1^{*}$ in S_1^{**} such that the first intersection x_k' of $O_Y^+(x_k)$ with S_2^{**} satisfy $x_k' \longrightarrow p_2' \in J_Y^+(p_1^*) \cap S_2^{**}$ and $\mathrm{cl}\,O_Y^-(p_2') \cap S_1^{**} = \emptyset$. By Proposition 1 there exist $Z_i \in U$, $i=1,2$ such that $Z_i = Y$ off F_i, $x_k \in O_{Z_1}^+(p_1)$, $x_k' \in O_{Z_2}^-(p_2)$ for some $x_k \in S_1^{**}$ and $O_{Z_1}[p_1, x_k] \subset F_1$, $O_{Z_2}[x_k', p_2] \subset F_2$. But then the equations $Z(x) = Z_i(x)$ for $x \in V_i$ and $Z(x) = Y(x)$ for $x \notin V_1 \cup V_2$ define a vector field $Z \in U$ with $p_2 \in O_Z^+(p_1)$.

P r o p o s i t i o n 3. For any open set $U \subset H^r(M)$ and a compact region $K \subset M$ there exists an open set $V \subset U$ and $k \in \{0,1,\dots\}$ such that if $Y \in V$ then Y has exactly k restpoints contained in K and each of them is hyperbolic.

P r o o f . Because $H_0^r(M)$ is a dense and open subset of $H^r(M)$ (see [6]), there exists an open set $U_1 \subset U \cap H_0^r(M)$. Moreover Y has finitely many restpoints contained in K. Thus there exist $k \in \{0,1,\dots\}$, $x_1,..,x_k \in K$, open sets $B(x_i,\varepsilon) \subset K$, $i = 1,\dots k$ such that $B(x_i,\varepsilon) \cap B(x_j,\varepsilon) = \emptyset$ and only x_i are the restpoints of Y in K. By Lemma 3 and 4 there exist a neighbourhood $V_1 \subset U_1$ of Y and sets $B(x_i,\eta) \subset B(x_i,\varepsilon)$ such that if $Z \in V_1$ then Z has at $B(x_i,\eta)$ exactly one restpoint x_i^Z, $i=1,\dots,k$ which is hyperbolic. Because $D = \mathrm{cl}K - \bigcup_{i=1}^{k} B(x_i,\eta)$ is compact and $Y(x) \neq 0$ for $x \in D$, there exists a neighbourhood V_2 of Y, $V_2 \subset V_1$ such that $Z(x) \neq 0$ for $x \in D$ and $Z \in V_2$. Hence V_2 and k satisfy the thesis of Proposition 3.

T h e o r e m 3. Let $K \subset M$ be a compact region and $P \in E$. Then there exists an open and dense set $G_P^r(K) \subset H_P^r(K)$ such that, if $p \in K$ is a fixed saddle of $Y \in G_P^r(K)$ and $q \in k$ then neither branch γ_Y of $W_Y^-(p)$ (resp. $W_Y^+(p)$) can form a saddle at infinity with $O_Y^-(q)$ (resp. $O_Y^+(q)$) if $\omega^*(\gamma_Y) = \{P\}$ (resp. $\alpha^*(\gamma_Y) = \{P\}$).

P r o o f . First we will give a sketch of the proof. We argue by contradiction. If Theorem 3 were false, then some open set $U \subset H_P^r(K)$ would contain a dense subset V such that each $Z \in V$ had a fixed saddle $p^Z \in K$ and a branch γ_Z of $W_Z^-(p^Z)$ which forms a saddle at infinity with $O_Z^-(q^Z)$ for some $q^Z \in K$ and $\omega^*(\gamma_Z) = \{P\}$. By Proposition 3 there would exist an open set $U_0 \subset U$ and k such that if $Z \in U_0$ then Z has exactly k restpoints contained in K and each of them is hyperbolic. So any $Z \in U_0$ would have at most k fixed saddles contained in K. Moreover $V_0 = V \cap U_0$ is dense in U_0. Using Lemma 4 we can assume that p^Z is defined to vary continuously with $Z \in U_0$ and that we can pick out one branch γ_Z of $W_Z^-(p^Z)$, whose initial compact segments vary continuously with $Z \in U_0$, such that when $Z \in V_0$ then γ_Z forms a saddle at infinity with $q^Z \in K$ and $\omega^*(\gamma_Z) = \{P\}$. Note that we assume no continuous dependence of q^Z on Z. Assuming these choices of p^Z, γ_Z for $Z \in U_0$ and q^Z for $Z \in V_0$ we would produce a vector field $X \in U_0$ for which γ_X oscillates, $\omega(\gamma_X) \cap \mathrm{cl}K \neq \emptyset$ and $P \in \omega^*(\gamma_X)$. By Remark 1 there would exist a cycle $C_X(q)$ with $q \in \omega(\gamma_X) \cap \mathrm{cl}K$ such that $C_X(q) = \omega^*(\gamma_X)$, contrary to $X \in U_0 \subset H_P^r(K)$.

Now we present the proof with details. Pick $Y_1 \in V_0$, so that γ_{Y_1} and $O_{Y_1}^-(q^{Y_1})$ form a saddle at infinity for some $q^{Y_1} \in K$ and $\omega^*(\gamma_{Y_1}) = \{P\}$. Let J_n, n=1,2... denote the base of neighbourhoods of P such that $\mathrm{cl}J_{n+1} \subset \mathrm{int}J_n$. Set $q_1 = q^{Y_1} \in K$ and pick $A_{1Y_1} \in J_1 \cap \gamma_{Y_1}$. By Lemma 4 we can pick $A_{1Z} \in \gamma_Z$ varying continuously with Z near Y_1 such that $A_{1Z} \in J_1$. Using Proposition 2, there exists X_1 near Y_1 such that q_1 belongs to $O_{X_1}^+(A_{1X_1})$. Again using Lemma 4, we can pick q_1^Z varying continuously with Z near X_1 such that $q_1^Z \in O_Z^+(A_{1Z})$. Now, pick Y_2 near X_1 such that γ_{Y_2} forms a saddle at infinity with some $q^{Y_2} \in K$ and $\omega^*(\gamma_{Y_2}) = \{P\}$. Set $q_2 = q^{Y_2}$ and pick $A_2 \in O_{Y_2}^+(q_1^{Y_2}) \subset \gamma_{Y_2}$ such that $A_2 \in J_2$. Again by Lemma 4 we can choose

$A_{2Z} \in O_Z^+(q_1^Z) \subset \gamma_Z$ varying continuously with Z near Y_2 so that $A_{2Z} \in J_2$. Proceeding inductively, we find a vector field X_n such that for Z near X_n there exist points q_i^Z and p_i^Z varying continuously with Z near X_n such that

(i) $q_i^Z \in O_Z^+(A_{iZ}) \subset \gamma_Z$ for $i=1,\dots,n$;

(ii) $A_{iZ} \in O_Z^+(q_{i-1}^Z)$ for $i=2,\dots,n$;

(iii) $A_{iZ} \in J_i$ for $i=1,\dots,n$.

Then we find $Y_{n+1} \in V_o$ near X_n so that (i-iii) hold, $\gamma_{Y_{n+1}}$ forms a saddle at infinity with some $q_{n+1} \in K$ and $\omega^*(\gamma_{Y_{n+1}}) = \{P\}$. Pick A_{n+1Z} varying continuously with Z near Y_{n+1} and satisfying (ii) and (iii) for $i = n+1$; using Proposition 2 we find X_{n+1} with $q_{n+1} \in O^+_{X_{n+1}}(A_{n+1X_{n+1}})$, and pick q_{n+1}^Z varying continuously with Z near X_{n+1} such that (i) holds. This is the inductive step. It is clear that, since "near" can be defined arbitrarily at each stage, the sequence X_n can be made to converge C^r-uniformly on compact to some $X \in U_o$ by using a known method (see [4]). For this vector field above (i-iii) hold for all n, so we have alternating points q_n^X, $A_{nX} \in \gamma_X$ with $q_n^X \in K$ and $A_{nX} \in J_n$. Because clK is compact, then there exists an accumulation point q of the sequence q_n^X which belongs to clK. Moreover A_{nX} tend to P, so $P \in \omega^*(\gamma_X)$. This shows that γ_X oscillates, contrary to $X \in U_o \subset H_P^r(K)$ and Theorem 3 is proved.

T h e o r e m 4. There exists a residual set $G^r(M) \subset H^r(M)$ such that, if $Y \in G^r(M)$ then there is no fixed saddle of Y for which some separatrix forms a saddle at infinity with another semi-orbit.

P r o o f . Let K_i, $i=1,2\dots$ denote the compact regions such that $\bigcup_{i=1}^{\infty} K_i = M$ and $clK_i \subset intK_{i+1}$. Pick $P \in E$. Theorem 3 implies that for any K_i there exists an open and dense set $G_P^r(K_i) \subset H_P^r(K_i)$ with the property: if $p \in K_i$ is a fixed saddle of $Y \in G_P^r(K_i)$ and $q \in K_i$ then no branch γ_Y of $W_Y^-(p)$ (resp. $W_Y^+(p)$)

such that $\omega^*(\gamma_Y) = \{P\}$ (resp. $\alpha^*(\gamma_Y) = \{P\}$) can form a saddle at infinity with $O_Y^-(q)$ (resp. $O_Y^+(q)$). Because $H_P^r(K_i)$ is open and dense in $H^r(M)$ for any K_i by Lemma 2 then $G_P^r(K_i)$ is also open and dense in $H^r(M)$. Therefore $G_P^r(M) = \bigcap_{i=1}^{\infty} G_P^r(K_i)$ is residual in $H^r(M)$. Moreover, if $O_Y^+(p)$ and $O_Y^-(q)$ form a saddle at infinity of $Y \in G_P^r(M)$ such that either $\omega^*(O_Y^+(p)) = \{P\}$ or $\alpha^*(O_Y^-(q)) = \{P\}$, then neither $\alpha(O_Y^-(p))$ nor $\omega(O_Y^+(q)$ is a fixed saddle. Let $G^r(M) = \bigcap_{P \in E} G_P^r(M)$. Because E is countable, $G^r(M)$ is residual in $H^r(M)$. It is not difficult to see that if $O_Y^+(p)$ and $O_Y^-(q)$ form a saddle at infinity of $Y \in G^r(M)$, then neither $\alpha(O_Y^-(p))$ nor $\omega(O_Y^+(q))$ is a fixed saddle and $G^r(M)$ satisfies the thesis of Theorem 4.

C o r o l l a r y . If $Y \in H^r(M)$ is a globally structurally stable, then there is no fixed saddle of Y for which some separatrix forms a saddle at infinity with another semi--orbit.

Q u e s t i o n . Does a globally structurally stable vector field Y on M have the property that the stable separatrices of saddles at infinity are not also the unstable separatrices of saddles at infinity?

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