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## SOME SHEAVES OVER A DIFFERENTIAL SPACE

Introduction

In this paper we show that in the case of the locally free sheaves of modules over a differential space, one can use, investigating their local properties, the analogical methods of calculations as those used in the theory of the differential manifolds. In consequence, as one can expect, many theorems known about the differential manifolds may be transferred without difficulty to locally free sheaves of modules over a differential space, what we show in the next parts.

1. The sheaves of the  $\mathcal{F}_k$ -modules over a differential space

Let  $\mathcal{F}$  be a non empty set of real functions defined on the set  $M$ . The set  $M$  will be interpreted as a topological space with the weakest topology  $\tau_{\mathcal{F}}$  in which all functions of  $\mathcal{F}$  are continuous.

As it is known [5], the set  $\mathcal{F}$  is called a differential structure on  $M$  iff the set  $\mathcal{F}$  is closed with respect to localization and superposition with the smooth functions on  $\mathbb{R}^n$ ,  $n = 1, 2, \dots$ . Then the couple  $(M, \mathcal{F})$  is called a differential space.

Analogically as in the theory of differential manifolds we define a tangent vector to the differential space  $(M, \mathcal{F})$  at the point  $p \in M$  as well as the smooth tangent vector

fields on  $(M, \mathcal{F})$  [5]. The  $\mathcal{F}$ -module of all the smooth vector fields tangent to the differential space will be denoted by  $\mathfrak{X}$ .

Let  $(M, \mathcal{F})$  be a differential space. Then  $\mathcal{F}$  is a linear ring over  $R$  containing all constant functions on  $M$  and the topological space  $(M, \tau_{\mathcal{F}})$  is a  $\mathcal{F}$ -regular. Of course the ring  $\mathcal{F}_{\mathbb{C}} := \{f + ig : f, g \in \mathcal{F}\}$  of complex functions on  $M$  is in natural way connected with the differential structure on  $(M, \mathcal{F})$ . It is easy to see that the ring  $\mathcal{F}_{\mathbb{C}}$  has a structure of 2-dimensional differential  $\mathcal{F}$ -module as well.

It is evident also that the  $\mathcal{F}$ -modules  $\mathcal{F}$ ,  $\mathfrak{X}$  and  $\mathcal{F}_{\mathbb{C}}$  generate respectively in the canonical way, sheaves of  $\mathcal{F}$ -modules  $\hat{\mathcal{F}}$ ,  $\hat{\mathfrak{X}}$  and  $\hat{\mathcal{F}}_{\mathbb{C}}$  over the differential space  $(M, \mathcal{F})$ . A natural inclusion of the ring  $\mathcal{F}$  into ring  $\mathcal{F}_{\mathbb{C}}$  allows us to treat the sheaves of  $\mathcal{F}_{\mathbb{C}}$ -modules also as the sheaves of  $\mathcal{F}$ -modules over the differential space  $(M, \mathcal{F})$ .

From this moment on we will consider sheaves of  $\mathcal{F}$ -modules and  $\mathcal{F}_{\mathbb{C}}$ -modules over the differential space  $(M, \mathcal{F})$ .

Let  $\pi$  be an arbitrary sheaf of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$ , where  $k=R$  or  $k=C$  as well as  $\mathcal{F}_R = \mathcal{F}$ . It is easy to prove:

**Lemma 1.1.** If  $U \in \tau_{\mathcal{F}}$  and  $\varrho \in \pi(U)$  then for any point  $p \in U$  there exist an open neighbourhood  $V$  of  $p$  and  $\bar{\varrho} \in \pi(M)$  such that

$$\varrho|V = \bar{\varrho}|V.$$

Now we are introducing the following definition.

**Definition 1.1.** Let  $\pi_1, \dots, \pi_k, \pi_{k+1}$ ,  $k \in \mathbb{N}$ , be any sheaves of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$ . Then each map

$$f : \pi_1(U) \times \dots \times \pi_k(U) \longrightarrow \pi_{k+1}(U)$$

satisfying the condition

$$(LF) \quad \text{if } \varrho_1|V = \varrho'_1|V, \varrho_1, \varrho'_1 \in \pi_1(U) \text{ for } i=1, 2, \dots, k$$

and  $V, U \in \tau_{\mathcal{F}}$  as well as  $V \subset U$ , then

$$f(\varrho_1, \dots, \varrho_k)|V = f(\varrho'_1, \dots, \varrho'_k)|V,$$

will be called the LF-mapping of  $\mathcal{F}_k(U)$ -modules  $\pi_1(U), \dots, \pi_k(U)$  into  $\mathcal{F}_k(U)$ -module  $\pi_{k+1}(U)$ .

Similarly as in [5] one can prove

**L e m m a 1.2.** If  $f_i: \pi_1(U) \times \dots \times \pi_k(U) \rightarrow \pi_{k+1}(U)$ ,  $i=1,2$ , are LF-mappings of  $\mathcal{F}_k(U)$ -modules  $\pi_1(U), \dots, \pi_k(U)$  into the  $\mathcal{F}_k(U)$ -module  $\pi_{k+1}(U)$  satisfying the condition

$$f_1(\bar{\varrho}_1|U, \dots, \bar{\varrho}_k|U) = f_2(\bar{\varrho}_1|U, \dots, \bar{\varrho}_k|U)$$

for any  $(\bar{\varrho}_1, \dots, \bar{\varrho}_k) \in \pi_1(V) \times \dots \times \pi_k(V)$ ,  $U, V \in \tau_{\mathcal{F}}$  and  $U \subset V$  then  $f_1 = f_2$ .

**L e m m a 1.3.** For an arbitrary LF-map

$$f: \pi_1(U) \times \dots \times \pi_k(U) \rightarrow \pi_{k+1}(U)$$

of the  $\mathcal{F}_k(U)$ -modules  $\pi_1(U), \dots, \pi_k(U)$  into the  $\mathcal{F}_k(U)$ -module  $\pi_{k+1}(U)$  as well as for any open set  $V \subset U$  there exists unique LF-map

$$f_V: \pi_1(V) \times \dots \times \pi_k(V) \rightarrow \pi_{k+1}(V)$$

such that

$$f_V(\varrho_1|V, \dots, \varrho_k|V) = f(\varrho_1, \dots, \varrho_k)|V$$

for any  $(\varrho_1, \dots, \varrho_k) \in \pi_1(U) \times \dots \times \pi_k(U)$ .

Analogically as in [5] we prove as well

**L e m m a 1.4.** Let  $\mathcal{B}$  be an open covering of  $U$  and

$$\left\{ f^B: \pi_1(B) \times \dots \times \pi_k(B) \rightarrow \pi_{k+1}(B) \right\}_{B \in \mathcal{B}}$$

family of the LF-mappings such that  $f^B|B \cap B' = f^{B'}|B \cap B'$  for any  $B, B' \in \mathcal{B}$ , then there exists unique LF-map

$$f : \pi_1(U) \times \dots \times \pi_k(U) \longrightarrow \pi_{k+1}(U)$$

such that  $f|_B = f^B$  for an arbitrary  $B \in \mathcal{B}$ .

Now, let  $\pi_1, \dots, \pi_k, \pi_{k+1}$ ,  $k \in N$ , be still any sheaves of  $\mathcal{F}_k$ -modules over a differential space  $(M, \mathcal{F})$ . By  $LF(\pi_1, \dots, \pi_k; \pi_{k+1})$  will be denoted the category which the  $\mathcal{F}_k(U)$ -modules

$$(1.1) \quad LF(\pi_1(U), \dots, \pi_k(U); \pi_{k+1}(U))$$

for  $U \in \tau_{\mathcal{F}}$ , are objects. Of course  $\mathcal{F}_k(U)$ -modules (1.1), where  $U \in \tau_{\mathcal{F}}$ , contain all LF-mappings  $f_U$  defined in Lemma 1.3.

Let  $U$  and  $V$  be an arbitrary open sets in  $M$  and  $U \subset V$ . By  $\varphi_U^V$  will be denoted the morphism of restriction

$$\varphi_U^V : LF(\pi_1(V), \dots, \pi_k(V); \pi_{k+1}(V)) \longrightarrow LF(\pi_1(U), \dots, \pi_k(U); \pi_{k+1}(U))$$

defined by the formula

$$\varphi_U^V(f) = f|_U$$

for an arbitrary  $f \in LF(\pi_1(V), \dots, \pi_k(V); \pi_{k+1}(V))$ .

From our lemmas there follows immediately

**C o r o l l a r y 1.1.** The triple  $(LF(\pi_1, \dots, \pi_k; \pi_{k+1}), F, \tau_{\mathcal{F}})$ , where  $F$  is a contravariant functor from the category  $\tau_{\mathcal{F}}$  into the category  $LF(\pi_1, \dots, \pi_k; \pi_{k+1})$  of  $\mathcal{F}_k(U)$ -modules of LF-mappings such that  $F(U) = LF(\pi_1(U), \dots, \pi_k(U); \pi_{k+1}(U))$  and  $F(\iota) = \varphi_V^U$ , where  $\iota : V \hookrightarrow U$  an inclusion, is a sheaf of LF-mappings over the differential space  $(M, \mathcal{F})$ .

This sheaf will be denoted by  $LF(\pi_1, \dots, \pi_k; \pi_{k+1})$  and called the sheaf of LF-mappings over the differential space  $(M, \mathcal{F})$ .

Now, we will give some examples of LF-mappings on the differential space  $(M, \mathcal{F})$ .

1. Let  $\lambda \in \mathfrak{X}$  be an arbitrary smooth vector field on  $M$  and  $\varphi = f + ig \in \mathcal{F}_{\mathbb{C}}$  - any complex function on  $M$ . Let us define

$$(1.2) \quad \tilde{X}(\varphi) = X(f) + i X(g)$$

for any function  $\varphi = f + ig \in \mathcal{F}_C$ . By means of immediate calculations we verify that the map  $\tilde{X} : \mathcal{F}_C \rightarrow \mathcal{F}_C$  is a differential operator. It is easy to observe as well that  $\tilde{X}$  is a LF-mapping. The set of all mappings of the form (1.2) is denoted by  $\tilde{\mathcal{X}}$ . This set has the natural structure of the  $\mathcal{F}_C$ -module.

A sheaf of the derivations of the ring  $\mathcal{F}_C$  corresponding to  $\mathcal{F}$ -module  $\tilde{\mathcal{X}}$  will be denoted by  $\tilde{\mathcal{X}}$  as well.

2. Let  $\pi_1, \dots, \pi_k$  and  $\pi_{k+1}$ ,  $k \in N$ , be the sheaves of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$  as well as

$$(1.3) \quad \omega : \pi_1(U) \times \dots \times \pi_k(U) \rightarrow \pi_{k+1}(U)$$

$\mathcal{F}_k(U)$ -k-linear map, where  $U \in \tau_{\mathcal{F}}$ .

One can show that the map  $\omega$  is a LF-mapping. The set of all LF-mappings  $\omega$  of the form (1.3) we denote by  $L_{\mathcal{F}_k}(\pi_1(U), \dots, \pi_k(U); \pi_{k+1}(U))$  and endow it in a natural way with a structure of  $\mathcal{F}_k(U)$ -module.

At the same time we have

**C o r o l l a r y 1.2.** The triple  $(L_{\mathcal{F}_k}(\pi_1, \dots, \pi_k, \pi_{k+1}), F, \tau_{\mathcal{F}})$ , where  $F$  is a contravariant functor from the category  $\tau_{\mathcal{F}}$  into the category  $L_{\mathcal{F}_k}(\pi_1, \dots, \pi_k; \pi_{k+1})$  or  $\mathcal{F}_k(U)$ -k-linear mappings and  $F(\iota) = \varphi_V^U$  where  $\iota : V \hookrightarrow U$  an inclusion, is a sheaf over the differential space  $(M, \mathcal{F})$ .

This sheaf will be denoted by  $L_{\mathcal{F}_k}(\pi_1, \dots, \pi_k, \pi_{k+1})$  and called a sheaf of the tensor fields on  $M$  with the values in the sheaf  $\pi_{k+1}$  of  $\mathcal{F}_k(U)$ -modules over the differential space  $(M, \mathcal{F})$ .

Of course, in particular when  $\pi_1 = \pi_2 = \dots = \pi_k = \mathcal{X}$  and  $\pi_{k+1} = \mathcal{F}_k$  this sheaf will be denoted by  $L_{\mathcal{F}_k}^k(\mathcal{X}, \mathcal{F}_k)$  and its elements will be in this case the  $\mathcal{F}_k$ -k-tensor fields on  $(M, \mathcal{F})$ .

Moreover if these  $\mathcal{F}_k$ -k-linear mappings will be skew-symmetric, then we call them the exterior  $\mathcal{F}_k$ -k-forms on  $M$ .

The sheaf of exterior  $\mathcal{F}_k$ - $k$ -forms on the differential space  $(M, \mathcal{F})$  will be denoted by  $\wedge^k(\mathfrak{X}, \mathcal{F}_k)$ . Next, by  $\wedge^k(\mathfrak{X}, \mathfrak{N})$  we will denote the sheaf of the exterior  $k$ -forms on  $M$  with the values in the sheaf  $\mathfrak{N}$ .

Now, let  $\omega \in \wedge^k(\mathfrak{X}, \mathcal{F}_k)$ , where  $k = C$ , then  $\omega(X_1, \dots, X_k) \in \mathcal{F}_C$  for any  $X_1, \dots, X_k \in \mathfrak{X}$ . Consequently

$$\omega(X_1, \dots, X_k) = \omega_1(X_1, \dots, X_k) + i\omega_2(X_1, \dots, X_k),$$

where  $\omega_1, \omega_2 \in \wedge^k(\mathfrak{X}, \mathcal{F})$ .

It is not difficult to observe that the operator of exterior derivation  $d$  as well as the linear connections in the  $\mathcal{F}_k$ -module  $\mathfrak{N}$  are the LF-mappings.

In our work we accept the following definition of the exterior derivation of exterior  $k$ -forms:

1° if  $\alpha \in \wedge^0(\mathfrak{X}(U), \mathcal{F}(U))$  then  $(d\alpha)(X) = X(\alpha)$  for an arbitrary  $X \in \mathfrak{X}(U)$  and if  $\alpha \in \wedge^0(\mathfrak{X}(U), \mathcal{F}_C(U))$  then  $(d\alpha)(X) = \tilde{X}\alpha = X(\alpha_1) + iX(\alpha_2)$  for an arbitrary  $X \in \mathfrak{X}(U)$ , where  $\alpha = \alpha_1 + i\alpha_2$ .

2° if  $\omega \in \wedge^k(\mathfrak{X}(U), \mathcal{F}_k(U))$ ,  $k = C$  or  $k = R$ ,  $U \in \tau_{\mathcal{F}}$ , then

$$\begin{aligned} (d\omega)(X_1, \dots, X_{k+1}) &= \sum_{i=1}^k (-1)^{i+1} \tilde{X}_i(\omega(X_1, \dots, \check{X}_i, \dots, X_{k+1})) + \\ &+ \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \check{X}_i, \dots, \check{X}_j, \dots, X_{k+1}) \end{aligned}$$

for an arbitrary  $X_1, \dots, X_{k+1} \in \mathfrak{X}(U)$  and  $k \geq 1$ .

Of course

$$d : \wedge^k(\mathfrak{X}(U), \mathcal{F}_k(U)) \longrightarrow \wedge^{k+1}(\mathfrak{X}(U), \mathcal{F}_k(U))$$

for  $k = 0, 1, \dots$  and as it is easy to show [5], we have the identity  $d \circ d = 0$ . This fact gives us possibility to consider analogically as in the theory of differential manifolds, the chain complex

$$\mathcal{L}^0(\mathfrak{X}(U), \mathcal{F}_k(U)) \xrightarrow{d} \mathcal{L}^1(\mathfrak{X}(U), \mathcal{F}_k(U)) \xrightarrow{d} \mathcal{L}^2(\mathfrak{X}(U), \mathcal{F}_k(U)) \xrightarrow{d} \dots$$

and consequently we have possibility to consider de Rham cohomology groups of the differential space  $(M, \mathcal{F})$ .

Accordingly with the denotation generally accepted in the theory of cohomology we put

$$Z^k(U) := \ker(d: \mathcal{L}^k(\mathfrak{X}(U), \mathcal{F}_k(U)) \rightarrow \mathcal{L}^{k+1}(\mathfrak{X}(U), \mathcal{F}_k(U)))$$

and

$$B^k(U) := \text{Im}(d: \mathcal{L}^{k-1}(\mathfrak{X}(U), \mathcal{F}_k(U)) \rightarrow \mathcal{L}^k(\mathfrak{X}(U), \mathcal{F}_k(U)))$$

$$H^k(U) := Z^k(U)/B^k(U)$$

for  $k = 1, 2, \dots$  and  $U \in \tau_{\mathcal{F}}$ . Moreover we accept also  $H^0(U) = \mathcal{F}_k(U)$ .

The group  $H^k(U)$ ,  $k = 0, 1, 2, \dots$  is called the  $k$ -th group of de Rham cohomology of the differential space  $(U, \mathcal{F}|_U)$  for  $U \in \tau_{\mathcal{F}}$ .

Next the linear connection in the  $\mathcal{F}_k(U)$ -module  $\pi(U)$  we define as a  $k$ -linear mapping

$$D: \pi(U) \rightarrow \mathcal{L}^1(\mathfrak{X}(U), \pi(U))$$

satisfying the condition

$$D(\alpha \varrho) = (d\alpha) \varrho + \alpha D \varrho$$

for any  $\varrho \in \pi(U)$  and  $\alpha \in \mathcal{F}_k(U)$ , where  $(d\alpha \cdot \varrho)(X) := (d\alpha)(X) \varrho = \tilde{X}(\alpha) \varrho$  for  $X \in \mathfrak{X}(U)$ .

Now we prove

**L e m m a 1.5.** The operator  $d$  of exterior derivative as well as the linear connection  $D$  in the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ , for  $U \in \tau_{\mathcal{F}}$ , are the LF-mappings.

**P r o o f .** It is known that for any two exterior forms  $\omega \in \mathcal{L}^k(U, \mathcal{F}_k(U))$  and  $\theta \in \mathcal{L}^1(\mathfrak{X}(U), \mathcal{F}_k(U))$  we have the identity

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta.$$

Now let  $\omega$  and  $\eta$  be any  $k$ -form from  $\wedge^k(\mathfrak{X}(U), \mathcal{F}_k(U))$  such that

$$\omega|_V = \eta|_V$$

for some open set  $V \subset U$ . We will show that

$$d\omega|_V = d\eta|_V.$$

Indeed, let  $p \in V$  be an arbitrary point and  $\varphi$  the function separated point  $p$  in the set  $V$ . Then we have

$$\varphi(\omega - \eta) = 0|_V.$$

Hence we have

$$0 = d(\varphi(\omega - \eta)) = d\varphi \wedge (\omega - \eta) + \varphi d(\omega - \eta)$$

as well as

$$d\varphi|_V(\omega - \eta)|_V + \varphi|_V d(\omega - \eta)|_V = 0.$$

The last identity implies

$$d\omega|_V = d\eta|_V.$$

Of course the remaining part of our proof runs analogically.

At the end let us observe also that for any vector fields  $X_1, \dots, X_s \in \mathfrak{X}(U)$  the composition  $X_1 \circ X_2 \circ \dots \circ X_s$ ,  $s \in \mathbb{N}$ , of those vector fields defined by the formula

$$(X_1 \circ X_2 \circ \dots \circ X_s)(\alpha) = X_1(X_2(\dots(X_s(\alpha))\dots))$$

for  $\alpha \in \mathcal{F}(U)$  is a LF-mapping as well.

## 2. The connection and curvature forms in the sheaves of $\mathcal{F}_k$ -modules over a differential space

Let  $\pi$  be a sheaf of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$ ,  $k = \mathbb{R}$  or  $k = \mathbb{C}$  and  $U \in \tau_{\mathcal{F}}$  any open set in  $M$ .



**D e f i n i t i o n 2.1.** The elements  $e_1, \dots, e_n \in \pi(V)$  are called a local  $\mathcal{F}_k(V)$ -base of  $\mathcal{F}_k(U)$ -module  $\pi(U)$ , where  $V, U \in \tau_{\mathcal{F}}$  and  $V \subset U$ , iff the  $e_1, \dots, e_n$  form an  $\mathcal{F}_k(V)$ -base for  $\pi(V)$ .

Of course if  $e_1, \dots, e_n \in \pi(V)$  for a local  $\mathcal{F}_k(V)$ -base of  $\mathcal{F}_k(U)$ -module  $\pi(U)$  then for any  $\eta \in \pi(U)$  we have the identity

$$\eta|_V = \alpha^1 e_1 + \alpha^2 e_2 + \dots + \alpha^n e_n,$$

where  $\alpha^1, \alpha^2, \dots, \alpha^n \in \mathcal{F}_k(V)$ . The collection  $(\alpha^1, \dots, \alpha^n)$  are called local coordinates of the element  $\eta \in \pi(U)$  with respect to the local  $\mathcal{F}_k(V)$ -base  $e_1, \dots, e_n$  in  $\pi(U)$ .

It is easy to prove

**L e m m a 2.1.** If the elements  $e_1, \dots, e_n \in \pi(V)$  form a local  $\mathcal{F}_k(V)$ -base of  $\mathcal{F}_k(U)$ -module  $\pi(U)$  then for any open set  $W \subset V$  the elements  $e_1|_W, \dots, e_n|_W \in \pi(W)$  form a local  $\mathcal{F}_k(W)$ -base of  $\mathcal{F}_k(U)$ -module  $\pi(U)$ .

As we know, if the  $\mathcal{F}_k(U)$ -module  $\pi(U)$  has a  $\mathcal{F}_k(U)$ -base then it has infinitely many  $\mathcal{F}_k(U)$ -bases and moreover every two bases of this  $\mathcal{F}_k(U)$ -module contain the same number of elements. Hence if  $e_1, \dots, e_n$  and  $e'_1, \dots, e'_n$  are two  $\mathcal{F}_k(U)$ -bases of  $\mathcal{F}_k(U)$ -module  $\pi(U)$  then there exists a matrix  $g \in GL(n, \mathcal{F}_k(U))$

$$g = \begin{pmatrix} g_1^1 & g_1^2 & \dots & g_1^n \\ g_2^1 & g_2^2 & \dots & g_2^n \\ \dots & \dots & \dots & \dots \\ g_n^1 & g_n^2 & \dots & g_n^n \end{pmatrix}$$

such that

$$e'_i = g_i^j e_j,$$

$i = 1, 2, \dots, n$ , called the matrix of change from one  $\mathcal{F}_k(U)$ -base to another  $\mathcal{F}_k(U)$ -base. Of course the element  $\det(g) \in \mathcal{F}_k(U)$  is reversible element of the ring  $\mathcal{F}_k(U)$ .

**Definition 2.2.** A sheaf  $\pi$  over the differential space  $(M, \mathcal{F})$  is called an  $n$ -dimensional locally free sheaf of  $\mathcal{F}_k$ -modules if for any point  $p \in M$  there exists an open set  $U \ni p$  as well as a  $\mathcal{F}_k(U)$ -local base of  $\mathcal{F}(U)$ -module  $\pi(U)$  containing  $n$  elements. The number  $n$  is then called the dimension of the sheaf  $\pi$ .

**Example 2.1.** Let  $\pi(M)$  be a  $n$ -dimensional differential module of the linear  $\phi$ -fields on the differential space  $(M, \mathcal{F})$  [5], where  $\phi$  is a function on  $M$  such that for every point  $p \in M$   $\phi(p)$  is a  $n$ -dimensional vector space over  $k$ . Of course each differential module  $\pi(M)$  on  $M$  is associated with the sheaf  $\pi$  of  $\mathcal{F}_k$ -modules over  $M$ . It is not difficult to observe that the sheaf  $\pi$  is a  $n$ -dimensional locally free sheaf over the differential space  $(M, \mathcal{F})$ .

**Example 2.2.** Let us take under consideration  $\mathcal{F}_k$ -module  $GL(n, \mathcal{F}_k)$  of the  $n \times n$ -matrices which elements are functions on  $M$  with values in  $k$ . A  $\mathcal{F}_k$ -module  $GL(n, \mathcal{F}_k)$  is an  $n$ -dimensional free module over  $M$  and the matrices  $a_{kl}^i = (a_{kl})$ ,  $i, j = 1, 2, \dots, n$ , satisfying the conditions

$$a_{kl}^i = \begin{cases} 1 & \text{if } k=i \text{ and } l=j \\ 0 & \text{if } k \neq i \text{ or } l \neq j \end{cases}$$

for  $i, j = 1, 2, \dots, n$  and for any point  $p \in M$ , form the  $\mathcal{F}_k(M)$ -base of  $GL(n, \mathcal{F}_k)$ .

Now, let  $\alpha_j^i \in \mathcal{F}_k$  for  $i, j = 1, 2, \dots, n$ , be fixed functions on  $M$ , among which at most one is equal to zero at a countable subset of  $M$ . Next, let us consider the  $\mathcal{F}_k$ -module generated by the vector of the form

$$(2.1) \quad \alpha_1^1 e_1^1 + \alpha_1^2 e_1^2 + \dots + \alpha_n^n e_n^n.$$

It is easy to observe that the  $\mathcal{F}_k$ -module constructed in such a manner is an  $n^2$ -dimensional free  $\mathcal{F}_k$ -module over the differential space  $(M, \mathcal{F})$ , where the vectors of the form (2.1) make a  $\mathcal{F}_k$ -base of this  $\mathcal{F}_k$ -module.

Of course the  $\mathcal{F}_K$ -module constructed above is a  $\mathcal{F}_K$ -submodule of the  $\mathcal{F}_K$ -module  $GL(n, \mathcal{F}_K)$  which is a differential module but the considered  $\mathcal{F}_K$ -submodule is not in general a differential module. However the sheaf associated with this  $\mathcal{F}_K$ -module will be a locally free sheaf of  $\mathcal{F}_K$ -modules over  $M$ , of dimension  $n^2$ .

Now, let  $\pi$  be an arbitrary  $n$ -dimensional locally free sheaf over the differential space  $(M, \mathcal{F})$ . Then by virtue of definition for any point  $p \in M$  there exists an open neighbourhood  $U$  of  $p$  in  $M$  as well as a  $\mathcal{F}_K(U)$ -base  $e_1, \dots, e_n$  of  $\mathcal{F}_K$ -module  $\pi(M)$  such that for any  $\varrho \in \pi(M)$  we have the identity

$$\varrho|U = \alpha^1 e_1 + \alpha^2 e_2 + \dots + \alpha^n e_n,$$

where  $\alpha^i \in \mathcal{F}_K(U)$  for  $i = 1, 2, \dots, n$ .

If  $\bar{e}_1, \dots, \bar{e}_n$  is another local  $\mathcal{F}_K(U)$ -base of the  $\mathcal{F}_K$ -module  $\pi(M)$  then we also have the identity

$$\varrho|U = \beta^1 e_1 + \beta^2 e_2 + \dots + \beta^n e_n,$$

where  $\beta^i \in \mathcal{F}_K(U)$  for  $i = 1, 2, \dots, n$ . Hence if  $g \in GL(n, \mathcal{F}_K(U))$  is a matrix of change from the  $\mathcal{F}_K(U)$ -base  $e_1, \dots, e_n$  to the  $\mathcal{F}_K(U)$ -base  $\bar{e}_1, \dots, \bar{e}_n$ , i.e.  $\bar{e}_i = g_i^j e_j$  for  $i = 1, 2, \dots, n$ , then as is easy to show we get the identity

$$(2.2) \quad \beta^i = g_i^j \alpha^j$$

for  $i = 1, 2, \dots, n$ .

The formula (2.2) gives us a transformation law of local coordinates of an element  $\varrho \in \pi(M)$  on changing of one  $\mathcal{F}_K(U)$ -base to another.

Now let  $\mathcal{B}$  be an open covering of  $M$  such that for each  $U \in \mathcal{B}$  there exists an  $\mathcal{F}_K(U)$ -base  $e_1^U, \dots, e_n^U$  of the  $\mathcal{F}_K(U)$ -module  $\pi(U)$ . Then for any  $\varrho \in \pi(M)$  as well as for each  $U \in \mathcal{B}$  we have

$$\varrho|U = \alpha_U^i e_i^U.$$

Of course, if  $U, V \in \mathcal{B}$  and  $U \cap V \neq \emptyset$  then

$$(\varrho|U)|U \cap V = (\varrho|V)|U \cap V$$

and consequently, if

$$\varrho|U = \alpha_i^U e_i^U \quad \text{as well as} \quad \varrho|V = \alpha_i^V e_i^V$$

then

$$\alpha_i^U |U \cap V \cdot e_i^U|_{U \cap V} = \alpha_i^V |U \cap V \cdot e_i^V|_{U \cap V}$$

or equivalently

$$\alpha_i^U |_{U \cap V} = g_j^i \alpha_j^V |_{U \cap V}$$

for  $i = 1, 2, \dots, n$ , where  $g = (g_j^i) \in GL(n, \mathcal{F}_k(U \cap V))$  is the matrix of change from one  $\mathcal{F}_k(U \cap V)$ -base to another.

Inversely, if for each  $U \in \mathcal{B}$  and for any local  $\mathcal{F}_k(U)$ -base  $e_1^U, \dots, e_n^U$  of the  $\mathcal{F}_k(U)$ -module  $\pi(M)$  are determined collections  $\left( \alpha_1^U, \dots, \alpha_n^U \right)$  such that if  $U, V \in \mathcal{B}$  and  $U \cap V \neq \emptyset$  then

$$\alpha_i^U |_{U \cap V} = g_j^i \alpha_j^V |_{U \cap V}$$

for  $i = 1, 2, \dots, n$ , where  $g \in GL(n, \mathcal{F}_k(U \cap V))$ , then there exists exactly one  $\varrho \in \pi(M)$  such that

$$\varrho|U = \alpha_i^U e_i^U$$

for any  $U \in \mathcal{B}$ .

Now, let us notice that the way of thinking and proving shown above, is well known from the classical theory of the differential manifolds. It results from this that in case of the  $n$ -dimensional locally free sheaves of  $\mathcal{F}_k$ -modules over a differential space one can use, investigating their local

properties, the analogical methods of calculations as those used in the theory of manifolds. In consequence, as one can expect, many theorems known on differential manifolds may be transferred without difficulty on  $n$ -dimensional locally free sheaves of  $\mathcal{F}_k$ -modules over the differential spaces  $(M, \mathcal{F})$  which we would like to show in the next parts of the paper.

To this effect let us consider an arbitrary  $k$ -form  $\omega \in \wedge^k(\mathcal{X}(U), \mathcal{N}(U))$  with the values in the  $\mathcal{F}_k(U)$ -module  $\mathcal{N}(U)$ . Then for arbitrary vector fields  $X_1, \dots, X_k \in \mathcal{X}(U)$   $\omega(X_1, \dots, X_k) \in \mathcal{N}(U)$ . Hence if the  $\mathcal{F}_k(U)$ -module  $\mathcal{N}(U)$  has an  $\mathcal{F}_k(U)$ -base  $e_1, \dots, e_n$  then

$$(2.3) \quad \omega(X_1, \dots, X_k) = \omega^i(X_1, \dots, X_k) e_i$$

where  $\omega^i(X_1, \dots, X_k) \in \mathcal{F}_k(U)$  for  $i = 1, 2, \dots, n$  as well as  $X_1, \dots, X_k \in \mathcal{X}(U)$ .

It results from this that  $\omega^i \in \wedge^k(\mathcal{X}(U), \mathcal{F}_k(U))$  is an exterior  $k$ -form on  $U$  for  $i = 1, \dots, n$ . In consequence: each  $k$ -form  $\omega \in \wedge^k(\mathcal{X}(U), \mathcal{N}(U))$  with values in the  $\mathcal{F}_k(U)$ -module  $\mathcal{N}(U)$  with a  $\mathcal{F}_k(U)$ -base determines unique  $n$  exterior  $k$ -forms  $\omega^1, \dots, \omega^n$  on  $U$ .

Of course from univocal decomposition (2.3) there results a univocal decomposition

$$(2.4) \quad \omega = \omega^i e_i$$

of the  $k$ -form  $\omega$  in the given  $\mathcal{F}_k(U)$ -base.

Next, let  $e'_1, \dots, e'_n$  be another  $\mathcal{F}_k(U)$ -base of the  $\mathcal{F}_k(U)$ -module  $\mathcal{N}(U)$  and let

$$(2.5) \quad e_i = g_i^j e'_j$$

for  $i = 1, 2, \dots, n$  where  $(g_j^i) \in GL(n, \mathcal{F}_k(U))$  is a matrix of change from one  $\mathcal{F}_k(U)$ -base to another. Now in the  $\mathcal{F}_k(U)$ -base  $e$  we have the decomposition

$$\omega = \omega'^i e'_i.$$

Hence as well as from (2.4) and (2.5) we obtain

$$(2.6) \quad \omega^i = -\frac{1}{g_j} \omega^j$$

for  $i = 1, 2, \dots, n$ . Of course the formula (2.6) gives us the transformation law of coordinates of the  $k$ -form  $\omega \in \wedge^k(\mathfrak{X}(U), \mathfrak{F}_k(U))$  on changing of one  $\mathfrak{F}_k(U)$ -base in  $\pi(U)$  to another.

Now, let  $\mathcal{B}$  be an open covering of  $M$  such that for each  $U \in \mathcal{B}$  there exists an  $\mathfrak{F}_k(U)$ -base  $e_1^U, \dots, e_n^U$  of the  $\mathfrak{F}_k(M)$ -module  $\pi(M)$ . Then for any  $\omega \in \wedge^k(\mathfrak{X}(M), \pi(M))$  as well as for any  $U \in \mathcal{B}$  we have the decomposition

$$\omega|_U = \omega^i e_i^U,$$

where  $\omega^i \in \wedge^k(\mathfrak{X}(U), \mathfrak{F}_k(U))$  for  $i = 1, 2, \dots, n$ . Moreover if  $U, V \in \mathcal{B}$  and  $U \cap V \neq \emptyset$  then

$$(\omega|_U)|_{U \cap V} = (\omega|_V)|_{U \cap V}$$

and in consequence if

$$\omega|_U = \omega^i e_i^U \quad \text{as well as} \quad \omega|_V = \omega^i e_i^V$$

then

$$\omega^i|_{U \cap V} e_i^U|_{U \cap V} = \omega^i|_{U \cap V} e_i^V|_{U \cap V}.$$

Hence, if  $g \in GL(n, \mathfrak{F}_k(U \cap V))$  is a matrix of change from one  $\mathfrak{F}_k(U \cap V)$ -base of  $\pi(U \cap V)$  to another, then by virtue of (2.6) we obtain

$$\omega^i|_{U \cap V} = -\frac{1}{g_j^i} \omega^j|_{U \cap V}$$

for  $i = 1, 2, \dots, n$ .

Conversely, if for each  $U \in \mathcal{B}$  and for any local  $\mathfrak{F}_k(U)$ -base  $e_1^U, \dots, e_n^U$  of the  $\mathfrak{F}_k(M)$ -module  $\pi(M)$  is a given collec-

tion  $(\omega^1_U, \dots, \omega^n_U)$  of exterior  $k$ -forms on  $U$  such that if  $U, V$  and  $U \cap V \neq \emptyset$  then

$$\omega^i|_{U \cap V} = g^i_j \omega^j|_{U \cap V}$$

for  $i = 1, 2, \dots, n$ , where  $g \in GL(n, \mathcal{F}_k(U \cap V))$ , then there exists exactly one  $k$ -form  $\omega \in \wedge^k(\mathfrak{X}(M), \mathcal{F}_k(M))$  such that

$$\omega|_U = \omega^i_U e^U_i$$

for each  $U \in \mathcal{B}$ .

Really, for each collection of  $k$ -forms  $\omega^i_U \in \wedge^k(\mathfrak{X}(U), \mathcal{F}_k(U))$  where  $i = 1, 2, \dots, n$  and the set of  $\mathcal{F}_k(U)$ -bases  $e^U_1, \dots, e^U_n$  of the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ , there exists exactly one  $k$ -form  $\omega^U \in \wedge^k(\mathfrak{X}(U), \pi(U))$  such that

$$(2.7) \quad \omega^U = \omega^i_U e^U_i.$$

Now, let  $(\omega^U)_{U \in \mathcal{B}}$  be a family of  $k$ -forms from  $\mathcal{F}_k(U)$ -modules  $\wedge^k(\mathfrak{X}(U), \mathcal{F}_k(U))$ ,  $U \in \mathcal{B}$  such that for any two elements  $U, V \in \mathcal{B}$  with  $U \cap V \neq \emptyset$ , the following condition is fulfilled

$$\omega^i|_{U \cap V} = g^i_j \omega^j|_{U \cap V}$$

for  $i = 1, 2, \dots, n$ , where  $g \in GL(n, \mathcal{F}_k(U \cap V))$  when

$$e^V_i|_{U \cap V} = g^j_i e^U_j|_{U \cap V}.$$

Then of course,

$$\omega^V|_{U \cap V} = \omega^U|_{U \cap V}$$

for any  $U, V \in \mathcal{B}$  such that  $U \cap V \neq \emptyset$ . Therefore, there exists, by virtue of definition of a sheaf, a  $k$ -form  $\omega \in \wedge^k(\mathfrak{X}(M), \pi(M))$  such that

$$\omega|_U = \omega^U$$

for each  $U \in \mathcal{B}$ . From (2.7) it results moreover that

$$\omega|_U = \omega^i_{\phantom{i}i} e_i^U.$$

Now, let us take under consideration a  $k$ -form

$$\omega \in \mathcal{L}^k(\mathfrak{X}(U), \mathcal{A}(\pi(U), \pi(U)))$$

i.e. a  $k$ -form with the values in the  $\mathcal{F}_k(U)$ -module of endomorphisms of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ . Let us assume also that the  $\mathcal{F}_k(U)$ -module  $\pi(U)$  has an  $\mathcal{F}_k(U)$ -base  $e_1, \dots, e_n$ .

Analogically as above we find that each  $k$ -form  $\omega \in \mathcal{L}^k(\mathfrak{X}(U), \mathcal{A}(\pi(U), \pi(U)))$  determines the  $n^2$  exterior  $k$ -forms  $\omega^i_j \in \mathcal{L}^k(\mathfrak{X}(U), \mathcal{F}_k(U))$ , where  $i, j = 1, 2, \dots, n$ , such that

$$\omega(X_1, \dots, X_k)(e_i) = \omega^j_i(e)(X_1, \dots, X_k)e_j$$

for an arbitrary  $X_1, \dots, X_k \in \mathfrak{X}(U)$ .

Of course, the  $k$ -forms  $\omega^i_j(e) \in \mathcal{L}^k(\mathfrak{X}(U), \mathcal{F}_k(U))$  where  $e = (e_1, \dots, e_n)$ ,  $i, j = 1, 2, \dots, n$ , form the matrix

$$\omega(e) = \begin{pmatrix} \omega^1_1(e) & \dots & \omega^1_n(e) \\ \dots & \dots & \dots \\ \omega^n_1(e) & \dots & \omega^n_n(e) \end{pmatrix}$$

called the matrix of  $k$ -form  $\omega \in \mathcal{L}^k(\mathfrak{X}(U), \mathcal{A}(\pi(U), \pi(U)))$  with respect to the base  $e$  of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ .

Now, let  $e' = (e'_1, \dots, e'_n)$  be another  $\mathcal{F}_k(U)$ -basis of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$  and  $g \in GL(n, \mathcal{F}_k(U))$  the matrix of change from base  $e'$  to  $e$ , i.e.  $e' = g e = (g^i_j e_i)$ . Then by direct calculations we obtain the following transformation law

$$\omega^i_j(eg) = g^k_j \omega^l_k(e) g^i_l$$

where  $i, j = 1, 2, \dots, n$ , for  $k$ -forms  $\omega^i_j(e) \in \mathcal{L}^k(\mathfrak{X}(U), \mathcal{F}_k(U))$ , or equivalently in the matrix form



$$\omega(eg) = g^{-1} \omega(e)g.$$

Similarly as above it is easy to prove

**Theorem 2.1.** Let  $\pi$  be a  $n$ -dimensional locally free sheaf of the  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$  and  $\mathcal{B}$  such an open covering of  $M$  that for each  $B \in \mathcal{B}$  there exists an  $\mathcal{F}_k(B)$ -base  $e^B = (e_1^B, \dots, e_n^B)$  on  $B$  of the  $\mathcal{F}_k$ -module  $\pi(M)$ . Then, if for each local  $\mathcal{F}_k(B)$ -base of  $\pi(M)$ , there assigned is a matrix  $\omega(e^B)$  of  $k$ -form  $\omega \in \wedge^k(\mathfrak{X}(B), \mathcal{A}(\pi(B), \pi(B)))$  in such a way that for any  $B, B' \in \mathcal{B}$ ,  $B \cap B' \neq \emptyset$ , the following condition is fulfilled

$$\omega(e^{B'})|_{B \cap B'} = g^{-1}(\omega(e^B)|_{B \cap B'})g,$$

where  $g \in GL(n, \mathcal{F}_k(B \cap B'))$  is a matrix of change from base  $e$  to  $e'$ , then there exists exactly one  $k$ -form  $\omega \in \wedge^k(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  and its local decomposition with respect to the  $\mathcal{F}_k(B)$ -base  $e^B$  has the following form

$$(\omega|_B)(X_1, \dots, X_k)(e_i^B) = \omega_i^j(e^B)(X_1, \dots, X_k)e_j^B$$

for any  $X_1, \dots, X_k \in \mathfrak{X}(B)$  and  $i = 1, 2, \dots, n$ .

Next, let  $\pi$  be an  $n$ -dimensional locally free sheaf of  $\mathcal{F}_k$ -modules over  $M$  and  $D: \pi(M) \rightarrow \wedge^1(\mathfrak{X}(M), \pi(M))$  a linear connection in the  $\mathcal{F}_k$ -module  $\pi(M)$ . Decomposing 1-forms  $De_i \in \wedge^1(\mathfrak{X}(U), \mathcal{F}_k(U))$  for  $i = 1, 2, \dots, n$  with respect to the  $\mathcal{F}_k(U)$ -base  $e$  of the  $\mathcal{F}_k$ -module  $\pi(U)$  we get

$$(2.8) \quad De_i = \theta_i^j(D, e)e_j.$$

The decompositions (2.8), similarly as above, determines unequivocally the family of the exterior 1-forms

$$\theta_j^i(D, e) \in \wedge^1(\mathfrak{X}(U), \mathcal{F}_k(U)),$$

where  $i, j = 1, 2, \dots, n$ .

As it is known, the matrix

$$\theta(D, e) = \begin{pmatrix} \theta_1^1(D, e) & \dots & \theta_n^1(D, e) \\ \dots & \dots & \dots \\ \theta_1^n(D, e) & \dots & \theta_n^n(D, e) \end{pmatrix}$$

of the 1-forms, where  $\theta_j^i(D, e)$  for  $i, j = 1, 2, \dots, n$  are determined by formula (2.8), is called the matrix of the connection form  $D$  with respect to the  $\mathcal{F}_K(U)$ -base  $e$  of the  $\mathcal{F}_K(U)$ -module  $\pi(U)$ .

Next, putting

$$(2.9) \quad \theta_j^i(D, e) := d \theta_j^i(D, e) + \theta_k^i(D, e) \wedge \theta_j^k(D, e)$$

for  $i, j = 1, 2, \dots, n$  we obtain the matrix of 2-forms

$$\theta(D, e) = \begin{pmatrix} \theta_1^1(D, e) & \dots & \theta_n^1(D, e) \\ \dots & \dots & \dots \\ \theta_1^n(D, e) & \dots & \theta_n^n(D, e) \end{pmatrix}$$

called the matrix of curvature of the connection  $D$  with respect to the  $\mathcal{F}_K(U)$ -base  $e$  of the  $\mathcal{F}_K(U)$ -module  $\pi(U)$ .

Definition (2.9) is written briefly in the form

$$\theta(D, e) = d \theta(D, e) + \theta(D, e) \wedge \theta(D, e).$$

Henceforth, when the connection  $D$  in the  $\mathcal{F}_K(U)$ -module  $\pi(U)$  will be fixed, then the matrixes  $\theta(D, e)$  as well as  $\theta(D, e)$  are denoted by  $\theta(e)$  and  $\Theta(e)$ , respectively.

Let  $e = (e_1, \dots, e_n)$  be the  $\mathcal{F}_K(U)$ -base of the  $\mathcal{F}_K(U)$ -module  $\pi(U)$  and  $g \in GL(n, \mathcal{F}_K(U))$  the matrix of change from  $\mathcal{F}_K(U)$ -base  $e$  to  $\mathcal{F}_K(U)$ -base  $e' = eg$ . Analogically as in the theory of differential manifolds we prove

**L e m m a 2.1.** If  $\theta(e)$  and  $\theta(eg)$  as well as  $\theta(e)$  and  $\theta(eg)$  are respectively the matrices of the connection  $D$  and of the curvature of this connection  $D$  with respect to  $\mathcal{F}_k(U)$ -base  $e$  and  $eg$ , then the following identities are fulfilled

$$(a) \quad dg + \theta(e)g = g \theta(eg)$$

$$(b) \quad \theta(eg) = g^{-1} \theta(e)g.$$

Of course, the identities (a) and (b) give us the transformation laws for the matrices of the connection and of curvature connection respectively, on changing of one  $\mathcal{F}_k(U)$ -base to another.

The elements of the curvature matrix of a connection in the  $\mathcal{F}_k(U)$ -module  $\pi(U)$  are, by virtue of the definition, the exterior 2-forms on  $U$  i.e.  $\theta_j^i(e) \in \Lambda^2(\mathfrak{X}(U), \mathcal{F}_k(U))$  for  $i, j = 1, 2, \dots, n$  and for an arbitrary  $\mathcal{F}_k(U)$ -base  $e$  on  $U$  of the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ .

Hence and from Theorem 2.1 there results

**C o r o l l a r y 2.1.** Let  $\pi$  be an  $n$ -dimensional locally free sheaf of the  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$  and

$$D: \pi(M) \longrightarrow \Lambda^1(\mathfrak{X}(M), \pi(M))$$

a linear connection in the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ . Then there exists exactly one 2-form  $\theta \in \Lambda^2(\mathfrak{X}(M), \mathcal{L}(\pi(M), \pi(M)))$  called the curvature 2-form of the connection  $D$ , which has a local decomposition of the form

$$(\theta|U)(X_1, X_2)(e_1) = \theta_1^j(e)(X_1, X_2)e_j$$

for  $i = 1, 2, \dots, n$  and  $X_1, X_2 \in \mathfrak{X}(U)$  with respect to any  $\mathcal{F}_k(U)$ -base  $e$  on  $U$  of the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ .

Now, let us return for a moment to the general case.

Let  $\pi$  be an  $n$ -dimensional locally free sheaf of  $\mathcal{F}_k(U)$ -modules over a differential space  $(M, \mathcal{F})$  and let  $\omega$  be a  $k$ -form on  $M$  with values in the  $\mathcal{F}_k(M)$ -module  $\mathcal{L}(\pi(M), \pi(M))$ , which is a locally free  $\mathcal{F}_k(M)$ -module as well.

For an arbitrary local  $\mathcal{F}_k(U)$ -base  $e$  on  $U$  of the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ , by  $\omega(e)$  we denote the following matrix

$$\omega(e) = \begin{pmatrix} \omega_1^1(e) & \dots & \omega_n^1(e) \\ \dots & \dots & \dots \\ \omega_1^n(e) & \dots & \omega_n^n(e) \end{pmatrix},$$

where  $\omega_j^i(e)$  for  $i, j = 1, 2, \dots, n$  are defined by the formula

$$(\omega|U)(X_1, \dots, X_k)(e_i) = \omega_i^j(e)(X_1, \dots, X_k)e_j$$

called the matrix of a  $k$ -form  $\omega$  with respect to the local  $\mathcal{F}_k(U)$ -base  $e$  on  $U$  of the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ .

As we know for the matrix  $\omega(e)$  we have the following transformation law of changing one local  $\mathcal{F}_k(U)$ -base to another

$$(2.10) \quad \omega(eg) = g^{-1}\omega(e)g.$$

Let  $D$  be a linear connection in the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ . Now, we extend the action of the operator  $D$  on the  $\mathcal{F}_k(M)$ -module  $\wedge^k(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  putting

$$D\omega_j^i(e) := d\omega_j^i(e) + \theta_k^i(e) \wedge \omega_j^k(e) - (-1)^k \omega_k^i(e) \wedge \theta_j^k(e)$$

for  $i, j = 1, 2, \dots, n$ , or equivalently

$$(2.11) \quad D\omega(e) = d\omega(e) + \theta(e) \wedge \omega(e) - (-1)^k \omega(e) \wedge \theta(e).$$

Analogically as in the theory of manifolds we prove

**L e m m a 2.2.** If  $\omega \in \wedge^k(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  and  $e$  as well as  $eg$ , where  $g \in GL(n, \mathcal{F}_k(U))$ , are two local  $\mathcal{F}_k(U)$ -bases on  $U$  of the  $\mathcal{F}_k(M)$ -module  $\pi(M)$ , then

$$D\omega(eg) = g^{-1}D\omega(e)g,$$

where  $\omega(e)$  and  $\omega(eg)$  are the matrices of the  $k$ -form  $\omega$  with respect to  $\mathcal{F}_k(U)$ -bases  $e$  and  $eg$  respectively.

From the above lemma and from Theorem 2.1 there results

**C o r o l l a r y 2.2.** For an arbitrary  $k$ -form  $\omega \in \Lambda^k(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  there exists exactly one  $k+1$ -form  $D\omega \in \Lambda^{k+1}(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  such that its local decomposition with respect to the  $\mathcal{F}_k(U)$ -base  $e$  on  $U$  has the form

$$\begin{pmatrix} D\omega_1^1(e) \dots D\omega_n^1(e) \\ \dots\dots\dots \\ D\omega_1^n(e) \dots D\omega_n^n(e) \end{pmatrix},$$

where  $D\omega_j^i(e)$ , for  $i, j = 1, 2, \dots, n$ , are defined by formula (2.11).

The  $(k+1)$ -form  $D\omega$  is called a covariant derivative of the  $k$ -form  $\omega$ .

Let  $\pi(M)$  be an  $n$ -dimensional locally free  $\mathcal{F}_k(M)$ -module. We will prove

**L e m m a 2.3.** If  $\chi \in \Lambda^k(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  and  $\psi \in \Lambda^1(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  are any  $k$ -form and  $1$ -form respectively with values in the locally free  $\mathcal{F}_k(M)$ -module  $\mathcal{A}(\pi(M), \pi(M))$  while  $\chi(e)$  and  $\psi(e)$  are the matrices of these forms with respect to an arbitrary  $\mathcal{F}_k(U)$ -base  $e$  on  $U$ , then there exists exactly one  $k+1$ -form  $\chi \wedge \psi \in \Lambda^{k+1}(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  such that

$$(\chi \wedge \psi)(e) = \chi(e) \wedge \psi(e)$$

for any  $\mathcal{F}_k(U)$ -base  $e$  on  $U$  of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ .

**P r o o f .** Let us assume that fulfilled the assumptions of Lemma 2.3 and let  $\mathcal{B}$  be such an open covering of  $M$  that for each  $U \in \mathcal{B}$  there exists a  $\mathcal{F}_k(U)$ -base  $e^U$  on  $U$  of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ .

As a result for each  $\mathcal{F}_k(U)$ -base  $e^U$  on  $U$  given are the matrices  $\chi(e^U)$  and  $\psi(e^U)$  of  $k$ -form  $\chi$  and  $l$ -form  $\psi$  respectively as well as the matrix  $\chi(e^U) \wedge \psi(e^U)$ . Now, we will show that the family  $(\chi(e^U) \wedge \psi(e^U))_{U \in \mathcal{B}}$  of exterior  $k+l$ -forms with values in the  $\mathcal{F}_k(M)$ -module  $\mathcal{A}(\pi(M), \pi(M))$ , determine exactly one  $k+l$ -form  $\chi \wedge \psi \in \mathcal{A}^{k+l}(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$ .

In order to end, let us take two sets  $U, U' \in \mathcal{B}$  such that  $U \cap U' \neq \emptyset$  and let  $g \in GL(n, \mathcal{F}_k(U \cap U'))$  be a matrix of change from  $\mathcal{F}_k(U \cap U')$ -basis  $e^U|_{U \cap U'}$  to  $\mathcal{F}_k(U \cap U')$ -basis  $e^{U'}|_{U \cap U'}$ . By virtue of (2.10) we have

$$\chi(e^{U'})|_{U \cap U'} = g^{-1} \chi(e^U)|_{U \cap U'} g$$

$$\psi(e^U)|_{U \cap U'} = g^{-1} \psi(e^U)|_{U \cap U'} g.$$

Hence

$$\begin{aligned} (2.12) \quad \chi(e^{U'})|_{U \cap U'} \wedge \psi(e^U)|_{U \cap U'} &= (\chi(e^{U'}) \wedge \psi(e^{U'}))|_{U \cap U'} = \\ &= g^{-1} \chi(e^U)|_{U \cap U'} g \wedge g^{-1} \psi(e^U)|_{U \cap U'} g = \\ &= g^{-1} (\chi(e^U) \wedge \psi(e^U))|_{U \cap U'} g. \end{aligned}$$

From (1.2) and from Theorem 2.1 it results the existence of exactly one  $k+l$ -form  $\chi \wedge \psi \in \mathcal{A}^{k+l}(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$ , such that its matrix  $(\chi \wedge \psi)(e^U)$  with respect to  $\mathcal{F}_k(U)$ -base  $e^U$ , where  $U \in \mathcal{B}$ , has the form

$$(\chi \wedge \psi)(e^U) = \chi(e^U) \wedge \psi(e^U).$$

Defined in Lemma 2.3  $k+l$ -form  $\chi \wedge \psi$  is called an exterior product of the  $k$ -form  $\chi$  and  $l$ -form  $\psi$ .

Similarly we prove

**L e m m a 2.4.** For any  $k$ -form  $\chi \in \mathcal{A}^k(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  and any  $l$ -form  $\psi \in \mathcal{A}^l(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  there exists exactly one  $k+l$ -form  $[\chi, \psi] \in \mathcal{A}^{k+l}(\mathcal{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  such that its matrix with respect to an arbitrary  $\mathcal{F}_k(U)$ -base  $e^U$  on  $U$  is defined by the formula

$$(2.13) \quad [\chi, \psi](e^U) := [\chi(e^U), \psi(e^U)] = \\ = \chi(e^U) \wedge \psi(e^U) - (-1)^{kl} \chi(e^U) \wedge \psi(e^U).$$

The  $k+1$ -form defined above  $[\chi, \psi]$  is called the Lie bracket of the given forms. From (2.13) it results as well as that

$$[\chi, \psi] = \chi \wedge \psi - (-1)^{kl} \psi \wedge \chi.$$

Analogically as in the theory of manifolds one can prove

**L e m m a 2.4.** If  $D$  is a linear connection in the  $\mathcal{F}_k(M)$ -module  $\pi(M)$  then for any forms  $\chi \in \wedge^k(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  and  $\psi \in \wedge^l(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$ , there is fulfilled the identity

$$(2.14) \quad D(\chi \wedge \psi) = D\chi \wedge \psi + (-1)^k \chi \wedge D\psi$$

as well as

**L e m m a 2.6.** If  $\Theta \in \wedge^2(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  is the curvature 2-form of the connection  $D$  in a  $n$ -dimensional locally free  $\mathcal{F}_k(M)$ -module  $\pi(M)$ , then we have the identity

$$d\Theta(e) = [\Theta(e), \Theta(e)],$$

where  $\Theta(e)$  is the curvature matrix of the connection  $D$  and  $\Theta(e)$  is the matrix of this connection with respect to an arbitrary  $\mathcal{F}_k(U)$ -base on  $U$ .

From Lemma 2.6 there follows immediately

**C o r o l l a r y 2.3.** (Bianchi identity)

$$D\Theta = 0.$$

It is easy to prove

**L e m m a 2.7.** Let  $\psi \in \wedge^k(\mathfrak{X}(M), \mathcal{A}(\pi(M), \pi(M)))$  be an arbitrary  $k$ -form and  $D$  a linear connection in a  $n$ -dimensional locally free  $\mathcal{F}_k(M)$ -module  $\pi(M)$ . Then we have

$$(2.15) \quad D^2\psi := D(D\psi) = [\Theta, \psi].$$

**R e m a r k.** The proofs of all the propositions of this chapter from which the greater part was omitted may be proved

locally. Of course, this is a consequence of the assumption that the considered  $\mathcal{F}_k(M)$ -modules are finite dimensional locally free modules.

### 3. The locally free sheaves of $\mathcal{F}_k$ -modules and the differential $\mathcal{F}_k$ -modules over a differential space

Let  $\pi$  be an arbitrary sheaf over the differential Hausdorff space  $(M, \mathcal{F})$ . As it is known, the elements  $\varrho, \varrho' \in \pi(U)$  are said to be equivalent at the point  $p \in U$  if  $\varrho|_V = \varrho'|_V$  for some open set  $V \subset U$  such that  $p \in V$  ([2]).

The equivalence classes of this relation in  $\pi(U)$  are called the germs of the element  $\varrho \in \pi(U)$  at the point  $p \in U$ . The set of all germs at the point  $p$  of the sheaf  $\pi$  is denoted by  $\pi_p$  and called the stalk at  $p$  for the sheaf  $\pi$ .

Of course  $\mathcal{F} = \mathcal{F}_R$  and  $\mathcal{F}_C$  are the sheaves of smooth real and complex functions on  $M$  respectively. In those cases  $\mathcal{F}_p$  and  $\mathcal{F}_{Cp}$  are the stalks at the point  $p \in M$  of the germs of smooth real and complex functions on  $M$  respectively. Moreover as it is easy to observe the stalks  $\mathcal{F}_p$  and  $\mathcal{F}_{Cp}$  have the natural structure of a ring.

Now, let us denote by  $\mathcal{M}_{Rp}$  and  $\mathcal{M}_{Cp}$  the subsets of  $\mathcal{F}_p$  and  $\mathcal{F}_{Cp}$  respectively determined by the formulas

$$(3.1) \quad \begin{cases} \mathcal{M}_{Rp} := \{ [f]_p \in \mathcal{F}_p : f(p) = 0 \} \\ \mathcal{M}_{Cp} := \{ [f]_p \in \mathcal{F}_{Cp} : f(p) = 0 \}, \end{cases}$$

where  $[f]_p$  stand for the germs of smooth functions on  $M$  at the point  $p$ .

One can show that  $\mathcal{M}_{Rp}$  and  $\mathcal{M}_{Cp}$  are the maximal ideals of the rings  $\mathcal{F}_p$  and  $\mathcal{F}_{Cp}$  respectively and therefore the ring  $\mathcal{F}_{kp}$  is a local ring over  $k$ , where  $k=R$  or  $k=C$ . Hence  $\mathcal{F}_{kp}/\mathcal{M}_{kp}$  is a number field which is isomorphic to  $k$ .

Let us observe that if  $\pi$  is sheaf of  $\mathcal{F}_k$ -modules over  $M$ , then the stalk  $\pi_p$  at  $p$  of the germs of the sheaf  $\pi$  has a structure of the  $\mathcal{F}_{kp}$ -module.



Now let  $\pi$  and  $\pi'$  be the sheaves of  $\mathcal{F}_k$ -modules over a differential space  $(M, \mathcal{F})$ . It is known that the family

$$\left\{ h_V, h_V \in \text{Hom}_{\mathcal{F}_k(V)}(\pi(V), \pi'(V)), V \in \tau_{\mathcal{F}} \right\}$$

is called a homomorphisms of the sheaf  $\pi'$  into the sheaf  $\pi$  iff

$$\varphi_V^U \circ h_V = h_U \circ \varphi_V^U$$

for any  $V, U \in \tau_{\mathcal{F}}$ , where  $V \subset U$  and  $\varphi_V^U$  is the restriction map.

Now, let us observe that instead of Definition 2.2 we can equivalently give the following

**D e f i n i t i o n 3.1.** A sheaf  $\pi$  of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$  is said to be locally free of the rank  $n$  if for any point  $p \in M$  there exists a neighbourhood  $U$  of  $p$  such that  $\pi$  restricted to  $U$  is isomorphic to the sheaf  $(\mathcal{F}_k(U))^n$ .

It is easy to prove ([3])

**T h e o r e m 3.1.** A sheaf  $\pi$  of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$  is locally free of rank  $n$  iff for every point  $p \in M$  there exists a neighbourhood  $U$  of  $p$  and elements  $\varrho_1, \dots, \varrho_n \in \pi(U)$  such that for any open set  $V \subset U$  the elements  $\varphi_V^U(\varrho_1), \dots, \varphi_V^U(\varrho_n) \in \pi(V)$  form a base of  $\pi(V)$  over  $\mathcal{F}_k(V)$ .

**P r o o f .** In consequence of Definition 3.1 and Lemma 2.1, it is easy to see that a locally free sheaf has the property shown in Theorem 3.1. Conversely, if elements  $\varrho_1, \dots, \varrho_n$  over  $U$  exist, then for every open set  $V \subset U$  the map  $h_V$  of  $(\mathcal{F}_k(V))^n$  into  $\pi(V)$  defined by  $h_V(f_1, \dots, f_n) = f_1 \varrho_1 + \dots + f_n \varrho_n$  is bijective and the maps  $h_V$  define isomorphism of the sheaf  $\mathcal{F}_k^n$  of smooth maps  $U \rightarrow k^n$  onto the sheaf  $\pi$  restricted to  $U$ .

Now, let  $\pi$  be an  $n$ -dimensional locally free sheaf of  $\mathcal{F}_k$ -modules over  $(M, \mathcal{F})$ . Similarly as in the theory of differential manifolds one can show that the stalk  $\pi_p$  over  $p$

of germs of the elements belonging to  $\pi$  have the structure of a free  $\mathcal{F}_{kp}$ -module of rank  $n$ .

Of course  $\mathcal{U}_{kp} \cdot \pi_p$  is an  $\mathcal{F}_{kp}$ -submodule of  $\mathcal{F}_{kp}$ -module  $\pi_p$ , where  $\mathcal{U}_{kp}$  is an ideal of the ring  $\mathcal{F}_{kp}$  defined in (3.1), and  $\mathcal{U}_{kp} \cdot \pi_p$  denotes the set of all algebraic sums of the form  $r\alpha$ , where  $r \in \mathcal{U}_{kp}$  and  $\alpha \in \pi_p$ .

Analogically as in the theory of a differential manifolds we prove ([3]) the following theorem

**Theorem 3.2.** If  $\pi$  is an  $n$ -dimensional locally free sheaf of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$ , then for any point  $p \in M$ ,  $\pi_p / \mathcal{U}_{kp} \pi_p$  is an  $n$ -dimensional vector space over  $k$ .

From this Theorem there results

**Corollary 3.1.** Any locally free sheaf  $\pi$  of  $\mathcal{F}_k$ -modules of the rank  $n$  over  $(M, \mathcal{F})$  in a natural way determine a function  $\phi$  on  $M$  which assigns to each point  $p \in M$  an  $n$ -dimensional vector space  $E_p := \pi_p / \mathcal{U}_{kp} \pi_p$  over  $k$ .

Moreover, as it is not difficult to prove, if  $\pi(U)$  is a  $n$ -dimensional  $\mathcal{F}_k(U)$ -module and  $\varrho_1, \dots, \varrho_n \in \pi(U)$  form a  $\mathcal{F}_k(U)$ -base of  $\pi(U)$ , then for every point  $p \in U$  the elements  $[\varrho_1]_p, \dots, [\varrho_n]_p \in \pi_p$  form the  $\mathcal{F}_{kp}$ -base of the  $\mathcal{F}_{kp}$ -module  $\pi_p$ .

Similarly, if  $\pi(U)$  is an  $n$ -dimensional  $\mathcal{F}_k(U)$ -module and  $\varrho_1, \dots, \varrho_n \in \pi(U)$  form the  $\mathcal{F}_k(U)$ -base of  $\pi(U)$ , then for every point  $p \in U$  the elements  $\varrho_1^*(p), \dots, \varrho_n^*(p)$

$\in \pi_p / \mathcal{U}_{kp} \cdot \pi_p$  form a base of the vector space  $\pi_p / \mathcal{U}_{kp} \cdot \pi_p$  over  $k$ .

Let  $\pi$  be a locally free sheaf of  $\mathcal{F}_k$ -modules of the rank  $n$  over  $(M, \mathcal{F})$ . Then, according to Corollary 3.1, the sheaf  $\pi$  determines the function  $\phi$  on  $M$  which assigns to each point  $p \in M$  an  $n$ -dimensional vector space  $\pi_p / \mathcal{U}_{kp} \cdot \pi_p$  over  $k$ . Now, let us denote by  $\hat{\pi}(M)$  the  $\mathcal{F}_k$ -module of all linear  $\phi$ -fields on  $M$  in the following manner:  $\hat{\varrho} \in \hat{\pi}(M)$  if and only if for an arbitrary  $\mathcal{F}_k(U)$ -base  $\varrho_1, \dots, \varrho_n$  of  $\pi(U)$

$$\hat{\varrho}|_U = \alpha^1 \varrho_1^* + \dots + \alpha^n \varrho_n^*,$$

where  $\alpha^1, \dots, \alpha^n \in \mathcal{F}_k(U)$  and  $\varrho_1^*(p), \dots, \varrho_n^*(p)$  form a base of the vector space  $\pi_p / \mu_{kp} \cdot \pi_p$  over  $k$ , for any point  $p \in U$ , determined by the  $\mathcal{F}_k(U)$ -base  $\varrho_1, \dots, \varrho_n$  of  $\pi(U)$ .

One can show that the  $\mathcal{F}_k$ -module defined in such a way  $\hat{\pi}(M)$  is closed with respect to localisation. Hence the  $\mathcal{F}_k$ -module  $\hat{\pi}(M)$  is a differential module over a differential space  $(M, \mathcal{F})$  of linear  $\phi$ -fields.

It is easy to prove

**Theorem 3.3.** A linear  $\phi$ -field  $\hat{\varrho}$  on  $M$  is an element of the  $\mathcal{F}_k(M)$ -module  $\hat{\pi}(M)$  iff for an arbitrary family  $\{\varrho_1^B, \dots, \varrho_n^B\}_{B \in \mathcal{B}}$   $\mathcal{F}_k(B)$ -base of  $\mathcal{F}_k(B)$ -modules  $\mathcal{F}_k(B)$ , where  $\mathcal{B}$  is an open covering of  $M$ , there exists a decomposition

$$\hat{\varrho}|_B = \alpha_B^1 \varrho_1^{*B} + \dots + \alpha_B^n \varrho_n^{*B},$$

where  $\alpha_B^i \in \mathcal{F}_k(B)$  for  $i = 1, 2, \dots, n$  and  $B \in \mathcal{B}$ .

An  $\mathcal{F}_k(M)$ -module  $\hat{\pi}(M)$  will be called a differential  $\mathcal{F}_k(M)$ -module of linear  $\phi$ -fields associated with the  $\mathcal{F}_k$ -module  $\pi(M)$  or associated with the  $n$ -dimensional locally free sheaf of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$ .

Hence we have

**Corollary 3.2.** Any  $n$ -dimensional locally free sheaf  $\pi$  of  $\mathcal{F}_k$ -modules over  $(M, \mathcal{F})$  determine, in the canonical way, an  $n$ -dimensional differential  $\mathcal{F}_k$ -module  $\hat{\pi}(M)$  of linear  $\phi$ -fields on  $M$ .

Of course, with the  $\mathcal{F}_k$ -module  $\hat{\pi}(M)$ , there is associated, in the canonical way, a sheaf  $\hat{\pi}$  of differential  $\mathcal{F}_k$ -modules of the rank  $n$  over  $(M, \mathcal{F})$ .

Let  $\pi$  be a locally free sheaf of rank  $n$  over the differential space  $(M, \mathcal{F})$  and  $\hat{\pi}$  a sheaf associated with  $\pi$ . It is not difficult to observe if  $\varrho_1, \dots, \varrho_n$  is a local  $\mathcal{F}(U)$ -base on  $U$  of the sheaf  $\pi$ , or equivalently a local  $\mathcal{F}_k(U)$ -base of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ , then  $\varrho_1^*, \dots, \varrho_n^*$  is a local  $\mathcal{F}_k(U)$ -base on  $U$  of the sheaf  $\hat{\pi}$  of differential  $\mathcal{F}_k(U)$ -modules of linear  $\phi$ -fields on  $M$ .

We will prove

**T h e o r e m 3.4.** Any  $n$ -dimensional locally free sheaf  $\pi$  of  $\mathcal{F}_k$ -modules over the differential space  $(M, \mathcal{F})$  is canonically isomorphic to the associated sheaf  $\hat{\pi}$  of differential  $\mathcal{F}_k$ -modules of linear  $\phi$ -fields on  $M$ .

**P r o o f .** Let  $\varrho_1, \dots, \varrho_n$  be the  $\mathcal{F}_k(U)$ -base of the  $\mathcal{F}_k(U)$ -module  $\pi(U)$ . Let us consider the map  $h_U: \pi(U) \rightarrow \hat{\pi}(U)$  defined by the formula

$$h_U(\varrho_i) = \varrho_i^*$$

for  $i = 1, 2, \dots, n$ , where  $\varrho_1^*, \dots, \varrho_n^*$  is the  $\mathcal{F}_k(U)$ -base of the  $\mathcal{F}_k(U)$ -module  $\hat{\pi}(U)$  associated with  $\varrho_1, \dots, \varrho_n$ .

Now, we will show that the map  $h_U$  does not depend on the choice of a local  $\mathcal{F}_k(U)$ -base. Indeed, let  $\bar{\varrho}_1, \dots, \bar{\varrho}_n$  be another local  $\mathcal{F}_k(U)$ -base of  $\pi(U)$ . Then we have

$$\bar{\varrho}_i = a_i^j \varrho_j,$$

for  $i, j = 1, 2, \dots, n$ , where  $(a_i^j)$  is a matrix of change from one  $\mathcal{F}_k(U)$ -base to another, and for the associated  $\mathcal{F}_k(U)$ -base

$$\bar{\varrho}_i^* = a_i^j \varrho_j^*,$$

$i, j = 1, 2, \dots, n$ .

Consequently

$$h_U(\bar{\varrho}_i) = h_U(a_i^j \varrho_j) = a_i^j h_U(\varrho_j) = a_i^j \varrho_j^* = \bar{\varrho}_i^*.$$

Next, let  $V$  be an arbitrary open set of  $M$  and  $\mathcal{B}$  an open covering of  $V$  such that for any  $B \in \mathcal{B}$  there exists a  $\mathcal{F}_k(B)$ -base of the  $\mathcal{F}_k(B)$ -module  $\pi(B)$ . Of course, a covering  $\mathcal{B}$  of  $V$  determines a family  $\{h_B\}_{B \in \mathcal{B}}$  of isomorphisms. Moreover, by the immediate calculus one can verify that for any  $h_B, h_{B'} \in \{h_B\}_{B \in \mathcal{B}}$  we have

$$h_B|_{B \cap B'} = h_{B'}|_{B \cap B'}$$

Hence, by virtue of sheaf properties, there exists exactly one isomorphism  $h_V : \pi(V) \rightarrow \hat{\pi}(V)$  such that

$$h_V|_B = h_B$$

for any  $B \in \mathcal{B}$ .

Consequently, this family  $\{h_V\}_{V \in \tau_x}$ , as it is easy to verify is the canonical isomorphism of the sheaves  $\pi$  and  $\hat{\pi}$ .

This canonical isomorphism of sheaves  $\pi$  and  $\hat{\pi}$  allows us to define some geometrical structures on the sheaf  $\pi$  through pulling back correspondent structures from the associated sheaf  $\hat{\pi}$  of differential  $\mathcal{F}_k$ -modules, for example

**D e f i n i t i o n 3.2.** A function  $g^*$  is said to be a hermitian metric on a locally free sheaf  $\pi$  if  $g^* = h^*g$ , where  $g$  is a hermitian metric on the sheaf  $\hat{\pi}$  associated with  $\pi$  and  $h$  is the canonical isomorphism of those sheaves.

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