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SUMS OF DOUBLE SYSTEMS OF PARTIALLY ORDERED SETS

Sums of double systems of abstract algebras and lattices were first introduced in [1] and [2] respectively. They are derived from Plonka systems and their sums as defined in [5], in fact, Plonka systems of abstract algebras are special double systems. In [3] the concept of a double system of lattices is reexamined and with a change in the original definition of such a system it becomes possible to represent any lattice in a very natural way as the sum of the double system of its congruence classes with respect to a given congruence relation (Theorem I and II of [3]). The approach in [3] is algebraic rather than order-theoretic. This paper focuses on the order-theoretic properties of double systems of lattices and extends the concept of a double system to arbitrary partially ordered sets. The main result here is similar to the one for lattices in [3]. It will be possible to represent a partially ordered set as the sum of the double system of its equivalence classes with respect to a given acyclic equivalence relation (Theorem 2.1 and Theorem 2.2). The results of [3] for lattices will be discussed as special cases of the results for partially ordered sets (section 3).

1. Acyclic equivalence relations on partially ordered sets

If L is a lattice and θ a congruence relation on L, then the partial order on the quotient lattice L/θ, whose elements are the congruence classes of L modulo θ, can be described as follows: For $a,b \in L$ the following conditions are equivalent.

(1.1) $\theta(a) \leq \theta(b)$

(1.2) $\theta(a) \vee \theta(b) = \theta(b)$

(1.3) $\theta(a) \wedge \theta(b) = \theta(a)$

(1.4) For each $x \in \theta(a)$ there exists a $y \in \theta(b)$ such that $x \leq y$ and for each $z \in \theta(b)$ there exists $u \in \theta(a)$ such that $u \leq z$.

(1.5) There exists $x \in \theta(a)$ and $y \in \theta(b)$ such that $x \leq y$.

If P is a partially ordered set and θ an equivalence relation then there is of course no algebraic quotient structure that induces an order relation on the equivalence classes as in (1.2) and (1.3) for lattices. But condition (1.5) can be used to define an order relation on the equivalence classes.

An equivalence relation θ on P is <u>acyclic</u> if and only if the following is true for all $a_1,\dots,a_n \in P$:

(1.6) If there are $x_1,\dots,x_n$, $y_2,\dots,y_{n+1} \in P$ such that $x_i \leq y_{i+1}$, $x_i, y_i \in \theta(a_i)$ and $y_{n+1} \in \theta(a_1)$, then $\theta(a_i) = \theta(a_1)$ for $1 \leq i \leq n$.

For $a,b \in P$ we now define a binary relation $\trianglelefteq$ on P/θ by

(1.7) $\theta(a) \trianglelefteq \theta(b)$ iff (1.5) is fulfilled.

The transitive closure R of this relation is a partial order on P/θ. It is obviously reflexive and transitive by definition. The antisymmetry is a consequence of the fact that θ is acyclic.

This relation R will be called the <u>θ-induced partial order</u> on P/θ. Note that if P is a lattice and θ a congruence relation, then $\trianglelefteq$ is already a partial order on P/θ, in fact, the usual order of the quotient lattice.

Let $L(P)$ be the lattice of lower ends of a partially ordered set P, $U(P)$ the lattice of upper ends, where $E \subset P$ is a lower (upper) end if for $x \in E$ and $y \le x$ $(x \le y)$ also $y \in E$. $(x]_P([x)_P)$ is the principal lower (upper) end in $L(P)$ $(U(P))$ generated by $x \in P$.

For $T = P/\theta$ with the θ-induced order and $t \in T$, let P_t denote the corresponding equivalence class as a partially ordered subset of P. If now $s \le t$ in T, then there are two natural maps:

$$\phi_s^t = L(P_t) \longrightarrow L(P_s)$$

defined as

$$\phi_s^t(E) = (E]_P \cap P_s \quad \text{for} \quad E \in L(P_t),$$

and

$$\psi_s^t : U(P_s) \longrightarrow U(P_t)$$

defined as

$$\psi_s^t(E) = [E)_P \cap P_t \quad \text{for} \quad E \in U(P_s).$$

The partial order on P can now be described in terms of these mappings. For $x,y \in P$, $x \in P_s$, $y \in P_t$ we have

$$(1.8) \qquad x \le y \text{ in } P \text{ iff } s \le t \text{ in } T \text{ and } x \in \phi_s^t(y]_{P_t}$$
$$\text{iff } s \le t \text{ in } T \text{ and } y \in \psi_s^t[x)_{P_s}.$$

Note that $\phi_s^t E = \bigcup \left\{ \phi_s^t(x] \mid x \in E \right\}$ and $\psi_s^t E = \bigcup \left\{ \psi_s^t[x) \mid x \in E \right\}$. If $s \le t \le r$ in T, then $\phi_s^r \supseteq \phi_s^t \phi_t^r$ and $\psi_s^r \supseteq \psi_t^r \psi_s^t$ and equality does, in general, not hold here. It does, if P is a lattice and θ is a congruence relation.

T h e o r e m 1.1. Let P be a lattice, θ an acyclic equivalence relation. The following are equivalent.

(1) θ is a congruence relation

(2) P_t is a sublattice of P for all $t \in T$, and for $s \le t \le r$ in T, $\phi_s^t \phi_t^r = \phi_s^r$ and $\psi_t^r \psi_s^t = \psi_s^r$.

P r o o f . (1) $\Rightarrow$ (2): The θ-induced order on T is the order of the quotient lattice $T = P/\theta$ and the congruence classes P_t are sublattices of P. To show $\phi_s^r \subseteq \phi_s^t \phi_t^r$. Let $x \in \phi_s^r E = (E]_p \cap P_s$. Then $x \leq z \in E \subset P_r$, but also by (1.4) $x \leq y \in P_t$. Now $z \wedge y \in P_t$ and $x \leq z \wedge y \in (E]_p \cap P_t$. Hence $x \in ((E]_p \cap P_t] \cap P_s = \phi_s^t \phi_t^r E$. Dually, $\psi_s^r \subseteq \psi_t^r \psi_s^t$.

(2) $\Rightarrow$ (1): Suppose $x \equiv y(\theta)$, $a \in P$, to show $x \vee a \equiv$ $\equiv y \vee a(\theta)$ and $x \wedge a \equiv y \vee a(\theta)$. Let $x, y \in P_s$, $x \vee a \in P_{t_1}$, $y \vee a \in P_{t_2}$, $x \vee y \vee a \in P_r$. Then $s \leq t_1 \leq r$ and $s \leq t_2 \leq r$ and by assumption $\phi_s^r (x \vee y \vee a]_{P_r} = \phi_s^{t_1} \phi_{t_1}^r (x \vee y \vee a]_{P_r}$ and also $\phi_s^r (x \vee y \vee a]_{P_r} = \phi_s^{t_2} \phi_{t_2}^r (x \vee y \vee a]_{P_r}$. This implies $(x \vee y \vee a]_p \cap P_s =$ $= ((x \vee y \vee a]_p \cap P_{t_1}]_p \cap P_s = ((x \vee y \vee a]_p \cap P_{t_2}] \cap P_s$. But then $y \leq u_1$ for some $u_1 \in P_{t_1}$, and $x \leq u_2$ for some $u_2 \in P_{t_2}$ and $x \vee y \vee a \leq x \vee u_1 \vee a \in P_{t_1}$ and $x \vee y \vee a \leq u_2 \vee y \vee a \in P_{t_2}$. This implies $r \leq t_1$, and $r \leq t_2$ hence $r = t_1 = t_2$ and $x \vee a \equiv y \vee a(\theta)$. Dually $x \wedge a \equiv y \wedge a(\theta)$.

2. Double systems of partially ordered sets

Let T be a partially ordered set and let $\{P_t \mid t \in T\}$ be a set of pairwise disjoint partially ordered sets. For $s \leq t$ in T let

$$\phi_s^t : L(P_t) \longrightarrow L(P_s)$$

and

$$\psi_s^t : U(P_s) \longrightarrow U(P_t)$$

be mappings with the following properties:

(2.1) $$\phi_s^t E = \cup \left\{ \phi_s^t (x]_{P_t} \mid x \in E \right\}$$

and

$$\psi_s^t E = \bigcup \left\{ \psi_s^t [x)_{P_s} \mid x \in E \right\},$$

(2.2) ϕ_s^s and ψ_s^s are the identity maps on P_s,

(2.3) $(x]_{P_s} \subseteq \phi_s^t (y]_{P_t}$ iff $[y)_{P_t} \subseteq \psi_s^t [x)_{P_s}$,

(2.4) ϕ_s^t and ψ_s^t are order-preserving,

(2.5) for $s \leq t \leq r$ in T, $\phi_s^t \circ \phi_t^r \subseteq \phi_s^r$ and $\psi_t^r \circ \psi_s^t \subseteq \psi_s^r$,

(2.6) there is a chain $s_1, \ldots, s_n$ in T, $s \leq s_1 \leq \ldots \leq s_n \leq t$, such that either all of $\phi_s^{s_1}$, $\phi_{s_i}^{s_{i+1}}$, $\phi_{s_n}^t$ for $1 \leq i \leq n-1$, or all of $\psi_s^{s_1}$, $\psi_{s_i}^{s_{i+1}}$, $\psi_{s_n}^t$ for $1 \leq i \leq n-1$ are non--trivial.

A mapping ϕ_s^t is considered <u>non-trivial</u> if there is at least one non-empty lower end E in P_t such that $\phi_s^t E$ is non--empty.

Such a system $\mathcal{P}$ of partially ordered sets $\{P_t \mid t \in T\}$ with the mappings ϕ_s^t and ψ_s^t will be called a <u>double system of partially ordered sets</u>. For a double system $\mathcal{P}$ we define an order relation on $P = \bigcup \{P_t \mid t \in T\}$ as follows: For $x \in P_s$, $y \in P_t \in$ let

(2.7) $x \leq y$ in P iff $s \leq t$ in T and $x \in \phi_s^t (y]_{P_t}$.

By (2.5) this is equivalent to

(2.8) $x \leq y$ in P iff $s \leq t$ in T and $y \in \psi_s^t [x)_{P_s}$.

As a consequence of (2.3), $\leq$ is reflexive and antisymmetric. (2.4) and (2.5) imply the transitivity of $\leq$.

$(P, \leq)$, where $\leq$ is the order-relation of (2.7), is called the <u>sum of the double system</u> $\mathcal{P}$. Note that for $x, y \in P$, $x \leq y$ in P_t iff $x \leq y$ in P.

Let now θ be the natural equivalence relation on P defined as $x \equiv y(\theta)$ iff $x,y \in P_t$ for some $t \in T$. Then θ is acyclic and the θ-induced order $\underset{\theta}{\leq}$ of the equivalence classes P_t, $t \in T$, is order-isomorphic to the given order of T.

We must show:

$$(2.9) \qquad P_s \underset{\theta}{\leq} P_t \text{ iff } s \leq t \text{ in } T.$$

Let $P_s \underset{\theta}{\leq} P_t$, then there are $s_1,\ldots,s_n \in T$ such that $P_s \trianglelefteq P_{s_1} \trianglelefteq \ldots \trianglelefteq P_{s_n} \trianglelefteq P_t$. But then there are $x \in P_s$, x_i, $y_i \in P_{s_i}$, $y \in P_t$ such that $x \leq y_1$, $x_i \leq y_{i+1}$, $x_n \leq y$, therefore $s \leq s_1 \leq \ldots \leq s_n \leq t$ in T. Let $s \leq t$ in T. By (2.6) there is a chain $s_1,\ldots,s_n$ in T, $s \leq s_1 \leq \ldots$ $\ldots \leq s_n \leq t$, where either all the ϕ's or all the ψ's are non-trivial. Suppose the ϕ's are non-trivial. There is then a lower end E in P_t with $\phi^t_{s_n} E \neq \emptyset$, i.e. there is $x_n \in P_{s_n}$ such that $x_n \in \phi^t_{s_n}(E)$ and by (2.1) this means $x_n \in \phi^t_{s_n}(y]$ for some $y \in E$. This implies $x_n \leq y$ in P. Similarly we can find $x_{n-1} \in P_{s_{n-1}}$ and $y_n \in P_{s_n}$ where $x_{n-1} \leq y_n$ in P. We continue this for decreasing indices and get that $P_s \trianglelefteq P_{s_1} \trianglelefteq \ldots \trianglelefteq P_{s_n} \trianglelefteq P_t$, hence $P_s \underset{\theta}{\leq} P_t$. Should the ψ's be non-trivial then (2.1), (2.5), (2.6) and (2.8) allow a similar argument.

Nex, we claim that the ϕ^t_s and ψ^t_s of the double system $\mathcal{P}$ are the concrete ones discussed in section 1:

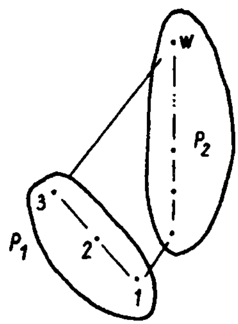
$$(2.10) \qquad \phi^t_s E = (E]_P \cap P_s \text{ and } \psi^t_s E = [E)_P \cap P_t.$$

Let $x \in (E]_P \cap P_s$, then $x \in P_s$ and $x \leq y \in E$, therefore $x \in \phi^t_s(y]_{P_t} \subset \phi^t_s E$ by (2.2). Let $x \in \phi^t_s E$, then by (2.1) $x \in \phi^t_s(y]$ for some $y \in E$. This implies $x \leq y$ in P, hence $x \in (E]_P$ and $x \in (E]_P \cap P_s$.

Note that condition (2.1) for the mappings is not needed for the representation of P as sum of the double system $\mathscr{P}$. It is also not essential to establish the order-isomorphism between the order of T and the Θ-induced order of the equivalence classes, since we could simply restate (2.6) for principal ends rather than arbitrary ends and would not have to use (2.1). However the equations (2.10) will no longer hold, if we drop condition (2.1):

E x a m p l e : Let $T = \underline{2}$, $P_1 = \underline{3}$, $P_2 = N \cup \{w\}$. We define $\phi_1^2 : L(P_2) \to L(P_1)$ by $\phi_1^2 \emptyset = \emptyset$, $\phi_1^2(n] = (1]$ for all $n \in N$, $\phi_1^2 N = (2]$, $\phi_1^2 P_2 = P_1$, and $\psi_1^2 : U(P_1) \to U(P_2)$ by $\psi_1^2 \emptyset = \psi_1^2 [3) = \psi_1^2 [2) = [w)$, $\psi_1^2 P_1 = P_2$. If we let ϕ_i^i, ψ_i^i, $i = 1,2$, be the identity maps, then the ϕ's and ψ's fulfill (2.2)-(2.6) but violate (2.1) since $\phi_1^2 N \neq \bigcup \{\phi_1^2(n] \mid n \in N\}$.

The sum of this double system is



and $\phi_1^2 N = (2]_{P_1} \neq (1] = (N]_P \cap P_1$.

We collect our results in the following theorem.

T h e o r e m 2.1. The sum P of a double system $\mathscr{P}$ of partially ordered sets $\{P_t \mid t \in T\}$ is a partially ordered set where the order on the parts P_t is preserved. The equivalence relation Θ on P, where $x \equiv y(\Theta)$ iff $x, y \in P_t$ for some $t \in T$, is acyclic and the Θ-induced order on the equivalence classes is order-isomorphic to the index-set T.

Moreover, the mappings ϕ_s^t and ψ_s^t are the natural ones: $\phi_s^t E = (E]_p \cap P_s$ and $\psi_s^t E = [E)_p \cap P_t$.

On the other hand, if we start out with a partially ordered set P and an acyclic equivalence relation θ on P, we have shown that we can order the equivalence classes with the θ-induced order. The natural mappings ϕ_s^t and ψ_s^t as defined in section 1 fulfill (2.1)-(2.6), so the equivalence classes, together with these mappings, form a double system of partially ordered sets and by (1.8) the order of P is precisely the order of the sum of this double system.

T h e o r e m 2.2. Let P be a partially ordered set and let θ be an acyclic equivalence relation on P. Then P is the sum of the double system $\mathcal{P}$ of equivalence classes for θ, where the index-set T is the set of equivalence classes with the θ-induced order and where the mappings ϕ_s^t, ψ_s^t for $s,t \in T$ are defined as $\phi_s^t E = (E]_p \cap P_s$ and $\psi_s^t E = [E)_p \cap P_t$.

3. Double systems of lattices

Let $\mathcal{P}$ be a double system of lattices $\{P_t | t \in T\}$, where T is also a lattice and the ϕ's and ψ's fulfill (2.1)-(2.6). For the sum P of this system to be a lattice, we must impose some additional requirements on the mappings. (2.4) and (2.5) will have to be strengthened to:

(3.1) ϕ_s^t and ψ_s^t are meet-homomorphisms,

(3.2) for $s \le t \le r$ in T, $\phi_s^t \circ \phi_t^r = \phi_s^r$ and $\psi_t^r \circ \psi_s^t = \psi_s^r$.

Instead of (2.6) we require

(3.3) For $s,t \in T$, $x \in P_s$, $y \in P_t$ $\phi_{s\wedge t}^s(x] \cap \phi_{s\wedge t}^t(y]$ is principal, and $\psi_s^{s\vee t}[x) \cap \psi_t^{s\vee t}[y)$ is principal.

Note, that (3.3) implies (2.6) since principal lower ends are non-empty. Now suppose $\mathcal{P}$ is a double system of lattices,

T is a lattice, the ϕ's and ψ's fulfill (2.1)-(2.3) and (3.1)-(3.3). Let P be the sum of this system.

T h e o r e m 3.1. P is a lattice.

P r o o f . For $x \in P_s$, $y \in P_t$, we have to show that $x \vee y$ and $x \wedge y$ exist in P. By (3.3) we know $\phi^s_{s\wedge t}(x]_{P_s} \cap \cap \phi^t_{s\wedge t}(y]_{P_t} = (z]_{P_{s\wedge t}}$. Obviously z is a lower bound of x and y. Suppose a is a lower bound of x and y and $a \in P_r$. Then $r \le s \wedge t$ and $a \in \phi^s_r(x]_{P_s}$ and $a \in \phi^t_r(y]_{P_t}$. By (3.2), $a \in \phi^{s\wedge t}_r \phi^s_{s\wedge t}(x]_{P_s} \cap \phi^{s\wedge t}_r \phi^t_{s\wedge t}(y]_{P_t}$ which equals $\phi^{s\wedge t}_r\left[\phi^s_{s\wedge t}(x]_{P_s} \cap \phi^t_{s\wedge t}(y]_{P_t}\right]$ by (3.1).

Hence $a \in \phi^{s\wedge t}_r(z]_{P_{s\wedge t}}$, i.e. $a \le z$ in P and $z = z \wedge y$ in P.

Similarly we use (3.1)-(3.3) for ψ^t_s to show that $x \vee y = u$ where $\psi^{s\vee t}_s[x)_{P_s} \cap \psi^{s\vee t}_t[y)_{P_t} = [u)_{P_{s\vee t}}$.

This lattice P is the sum of the double system of lattices as defined in [3]. In fact, the two main theorems of [3] are now immediate consequences of Theorem 2.1 and Theorem 2.2. Note, however, that we are using a slightly and insignificantly different definition of a double system since the ϕ's and ψ's are defined on lower and upper ends whereas in [3] they are defined on ideals and dual ideals respectively.

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