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Jacek Dąbrowski

EXISTENCE OF OPTIMAL CONTROLS FOR PARTIALLY OBSERVED GENERAL SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

In this paper we are concerned with the existence of optimal controls for problems of the following kind.

Let there be given stochastic one dimensional differential equations

$$(1)\qquad \begin{cases} dX_t = f(t,X,\bar{u}(Y))dt + \sigma(t,X,\bar{u}(Y))dW_t \\ X_0 = x_0, \end{cases}$$

$$(2)\qquad \begin{cases} dY_t = h(t,X_t)dt + d\tilde{W}_t \\ Y_0 = 0 \qquad\qquad 0 \le t \le t, \end{cases}$$

where $\tilde{W}$ and W are independent Brownian motions, $h : \langle 0,1\rangle \times R \longrightarrow R$.

Let U be a compact metric space and $\mathcal{P}(U)$ denote the space of all probability measures defined on the σ-field of Borel subsets of U. Let $\mathcal{V}$ be the space of all functions $V : \langle 0,1\rangle \longrightarrow \mathcal{P}(U)$ such that for every Borel subset $A \subset U$ the function $V_t(A)$ is measurable with respect to $t \quad \langle 0,1\rangle$.

The real functions f, σ are defined on $\langle 0,1\rangle \times C(\langle 0,1\rangle) \times \mathcal{V}$ where $C(\langle 0,1\rangle)$ denotes the space of all real, continuous functions defined on $\langle 0,1\rangle$. A control function

$\bar{u} : C(\langle 0,1\rangle) \longrightarrow \vartheta$ is measurable. More precise definitions will be given in the sequel.

The functions $\bar{u}$ seem to be generalizations of usually applied control functions $u_0 : \langle 0,1\rangle \times C(\langle 0,1\rangle) \longrightarrow R$. Namely for every u_0 we can define $\bar{u}$ as follows

$$(\bar{u}(\varphi))_t = \delta_u \Longleftrightarrow u_0(t,\varphi) = u$$

where $\varphi \in C(\langle 0,1\rangle)$ and δ_u denotes Dirac measure concentrated at $u \in U$. One can identify u_0 with $\bar{u}$.

The problem is to minimize a criterion of the form

$$J[\bar{u}] = E\left\{\int_0^1 F(t,X,\bar{u}(Y))dt + G(X)\right\}$$

where $F : \langle 0,1\rangle \times C(\langle 0,1\rangle) \times \vartheta \longrightarrow R$ and $G : C(\langle 0,1\rangle) \longrightarrow R$.

In this paper we shall show that under certain assumptions there exists an admissible control $\bar{u}^*$ such that $J[\bar{u}^*] \leqslant \leqslant J[\bar{u}]$ for all admissible $\bar{u}$. We formulate first the control problem on the canonical sample space using the Fleming's and Pardoux's construction [1]. Next we shall show that the system (1), (2) has the unique weak solution X, Y. Therefore we shall be able to formulate the control problem on any probability space.

To make assumptions about the functions appearing in (1), (2) we must endow the space ϑ with a topology as follows. An element $V \in \vartheta$ we shall identify with the probability measure (defined on Borel subsets of $\langle 0,1\rangle \times U$) by the formula

$$V(dt,\ du) = V_t(du)dt.$$

We can endow ϑ with Prohorov metric (see [7]). The space ϑ becomes a separable complete space. Moreover ϑ is a compact space.

We make the following assumptions about the functions appearing in (1), (2).

(3) The functions $f, \sigma, F : \langle 0,1\rangle \times C\langle 0,1\rangle \times \mathcal{V} \longrightarrow R$ are bounded, continuous with $|\sigma(t,\varphi, v)| \geqslant \Delta > 0$, such that for every $t \in \langle 0,1\rangle$ we have $f(t,\varphi,v) = f(t,\varphi',v')$, $\sigma(t,\varphi, v) = \sigma(t,\varphi',v')$ and $F(t,\varphi,v) = F(t,\varphi',v')$ if $\varphi_s = \varphi'_s$ for all $s \leqslant t$ and $v_s = v'_s$ for almost all $s \leqslant t$; $\varphi, \varphi' \in C(\langle 0,1\rangle)$, $v, v' \in \mathcal{V}$.

(4) There exists a constant $L > 0$ such that we have

$$|f(t,\varphi,v) - f(t,\varphi',v)|^2 + |\sigma(t,\varphi,v) - \sigma(t,\varphi',v)|^2 \leqslant$$

$$\leqslant L\int_0^t |\varphi_s-\varphi'_s|^2 ds + L|\varphi_t-\varphi'_t|^2 \quad \text{for all} \quad v \in \mathcal{V}, \; t \in \langle 0,1\rangle.$$

(5) $h : \langle 0,1\rangle \times R \longrightarrow R$ is a bounded function whose partial derivatives of orders $\leqslant 2$ are bounded and continuous; $G : C(\langle 0,1\rangle) \longrightarrow R$ is bounded and continuous function.

D e f i n i t i o n 1. An admissible control $\bar{u}$ is a measurable function $\bar{u} : C(\langle 0,1\rangle) \longrightarrow \mathcal{V}$.

Let $\mathcal{U}$ denote the set of all admissible controls $\bar{u}$.

We shall prove now the following theorem.

T h e o r e m 1. Let $\bar{u} \in \mathcal{U}$ be fixed. Let (X,Y) and (X',Y') be solutions of the system (1), (2) (not necessary defined on the same probability space). Then under (3), (4), (5) the joint distributions of (X,Y) and (X',Y') are the same (uniqueness of the weak solution (X,Y) if it exists).

We shall consider the following system of stochastic equations:

$$(6) \qquad \begin{cases} dX_t = \sigma(t,X,\bar{u}(Y))dB_t \\ X_0 = x_0, \end{cases}$$

$$(7) \qquad \begin{cases} dY_t = d\tilde{B}_t \\ Y_0 = 0, \qquad \bar{u} \in \mathcal{U} \text{ is fixed,} \end{cases}$$

where B and $\tilde{B}$ are independent Brownian motions.

To prove Theorem 1 we shall need the following lemmas.

Lemma 1. Let (X,Y) and (X',Y') be solutions of (6), (7) defined on probability spaces $(\Omega,\mathcal{F},(\mathcal{F}_t),P,B,\tilde{B})$ and $(\Omega',\mathcal{F}',(\mathcal{F}'_t),P',B',\tilde{B}')$ respectively, where $(B,\tilde{B})$ and $(B',\tilde{B}')$ are systems of independent Brownian motions. We assume that σ satisfies (3), (4) with $f \equiv 0$. Then the joint distributions of $(X,Y,B,\tilde{B})$ and $(X',Y',B',\tilde{B}')$ are the same.

Lemma 2. Let $\bar{u} \in \mathcal{U}$ be fixed. We assume that (X,Y) and (X',Y') are solutions of (1), (2) defined on probability spaces $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le 1},P_{\bar{u}},W,\tilde{W})$ and $(\Omega',\mathcal{F},(\mathcal{F}'_t)_{0\le t\le 1},P'_{\bar{u}},W',\tilde{W}')$ respectively, where $(W,\tilde{W})$ and $(W',\tilde{W}')$ are systems of independent Brownian motions. Let P_0 and P'_0 be the probability measures on $(\Omega,\mathcal{F})$ and $(\Omega',\mathcal{F}')$ respectively defined by the formula

$$\frac{dP_0}{dP_{\bar{u}}} = \exp\left\{ -\int_0^1 \frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,u(Y))}\, dW_t - \int_0^1 h(t,X_t)\,d\tilde{W}_t - \right.$$

$$\left. - \frac{1}{2}\int_0^1 \left(\left[\frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\right]^2 + h(t,X_t)^2\right)dt \right\}$$ with P'_0 defined analogously.

Then under (3), (4), (5) there exists two couples $(B,\tilde{B})$ and $(B',\tilde{B}')$ of independent Brownian motions such that the two-dimensional processes (X,Y) and (X',Y') satisfy (6), (7) under P_0 and P'_0 respectively.

Proof of Lemma 1. Using (3) there exists the unique strong solution of (6), (7). The assertion of Lemma 1 is now evident. Q.E.D.

Proof of Lemma 2. The assertion of Lemma 2 is the consequence of Lemma 6.4.1, Theorem 4.5.1, of [2] and Lemma 1. Q.E.D.

Remark. More precisely to prove Lemma 2 we must apply the following generalization of Lemma 6.4.1 of [2].

L e m m a 3. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le 1},P,W)$ be any probability space, where W is d-dimensional Wiener process. Let $b:\langle 0,1\rangle\times\Omega\to R^d$, $c:\langle 0,1\rangle\times\Omega\to R^d$ and $a:\langle 0,1\rangle\times\Omega\to S_d^+$ (S_d^+ is the set of strictly positive definite $d\times d$-symmetric matrices) be progressively measurable functions such that a, b, c^*ac are bounded. Let ξ be an Ito process defined by the formula

$$\xi_t = x + \int_0^t b(s)ds + \int_0^t e(s)dW_s \quad \text{for} \quad x\in R^d,$$

where $e:\langle 0,1\rangle\times\Omega\to R^d\times R^d$ is progressively measurable function such that $e\cdot e^* = a$. Let us define a new probability measure Q on $(\Omega,\mathcal{F})$ using the following Cameron-Martin-Girsanov formula

$$\frac{dQ}{dP} = \exp\left[\int_0^1 c^*(t)e(t)dW_t - \frac{1}{2}\int_0^1 c^*(t)\cdot a(t)\cdot c(t)dt\right].$$

Then there exists a d-dimensional Q-Wiener process B such that

$$\xi_t = x + \int_0^t (b(s) + a(s)c(s))ds + \int_0^t e(s)d\,B_s.$$

P r o o f of Lemma 3. Using Theorems 4.3.7 and 4.5.1 of [2] the prooves of Lemma 3 and Lemma 6.4.1 of [2] are the same. Q.E.D.

We shall also need the following lemma.

L e m m a 4. Under the assumptions of Lemma 2 we have

$$\frac{dP_{\bar{u}}}{dP_0} = \exp\left\{\int_0^1 \frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\,dB_t + \int_0^1 h(t,X_t)d\tilde{B}_t - \frac{1}{2}\int_0^1\left[\left(\frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\right)^2 + (h(t,X_t))^2\right]dt\right\},$$

where B, $\tilde{B}$ are independent Wiener processes appearing in the assertion of Lemma 2.

P r o o f of Lemma 4. We have to show that

$$\int_0^1 \frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\,dB_t + \int_0^1 h(t,X_t)\,d\tilde{B}_t - \frac{1}{2}\int_0^1 \left[\left(\frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\right)^2 + (h(t,X_t))^2\right]dt =$$

$$= -\left\{-\int_0^1 \frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\,dW_t - \int_0^1 h(t,X_t)\,d\tilde{W}_t - \frac{1}{2}\int_0^1 \left[\left(\frac{f(t,X,\bar{u}(Y))}{\sigma(t,X,\bar{u}(Y))}\right)^2 + (h(t,X_t))^2\right]dt\right\}$$

P_0 a.s. (almost sure).

Using the elementary computation and the fact that

$$\begin{bmatrix} W_t \\ \tilde{W}_t \end{bmatrix} = \int_0^t \begin{bmatrix} \sigma^{-1}(s,X,\bar{u}(Y)), & 0 \\ 0 & , 1 \end{bmatrix} \cdot \begin{bmatrix} d\,X_s - \int_0^s f(\tau,X,\bar{u}(Y))\,d\tau \\ d\,Y_s - \int_0^s h(\tau,X_\tau)\,d\tau \end{bmatrix} \quad \text{under } P_u$$

and

$$\begin{bmatrix} B_t \\ \tilde{B}_t \end{bmatrix} = \int_0^t \begin{bmatrix} \sigma^{-1}(s,X,\bar{u}(Y)), & 0 \\ 0 & , 1 \end{bmatrix} \begin{bmatrix} dX_s \\ dY_s \end{bmatrix} \quad \text{under } P_0,$$

one can easily complete the proof. The technique of this computation is very similar to the one given in the proof of Theorem 6.4.2 of [2]. We omit the details. Q.E.D.

P r o o f of Theorem 1. Let (X,Y) and (X',Y') be solutions of (1), (2) defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le 1},P_{\bar{u}},W,\tilde{W})$ and $(\Omega',\mathcal{F}',(\mathcal{F}'_t)_{0\le t\le 1},P'_{\bar{u}},W',\tilde{W}')$ respectively. Let Φ: $C(\langle 0,1\rangle)\times C(\langle 0,1\rangle) \longrightarrow R$ be a bounded, measurable function. To complete the proof we have to show that

$$\int_\Omega \Phi(X,Y)\,dP_{\bar{u}} = \int_{\Omega'} \Phi(X',Y')\,dP'_{\bar{u}}.$$

Namely, we have

$$\int_{\Omega} \Phi(X,Y)\,dP_{\bar{u}} = \int_{\Omega} \Phi(X,Y)\,\frac{dP_{\bar{u}}}{dP_0}\,dP_0 .$$

From the very definition of stochastic integral and Lemma 1 we see that the joint distribution of $(X,Y,B,\hat{B},\int_0^1 \frac{f(t,X,\bar{u}(Y))}{(t,X,\bar{u}(Y))}\,dB_t,$ $\int_0^1 h(t,X_t)\,d\tilde{B}_t)$ (under P_0) is determined by $\bar{u} \in \mathcal{U}$. Therefore, using Lemma 4, there exists a measurable function $\psi : C(<0,1>) \times C(<0,1>) \times R \times R \to R$ such that

$$\frac{dP_{\bar{u}}}{dP_0} = \psi\left(X, Y, \int_0^1 \frac{f(t,X,\bar{u}(Y))}{(t,X,\bar{u}(Y))}\,dB_t, \int_0^1 h(t,X_t)\,d\hat{B}_t\right)$$

and analogously $\frac{dP'_{\bar{u}}}{dP'_0} = \psi(X',Y',\ldots)$.

Finally we have

$$\int_{\Omega} \Phi(X,Y)\,dP_{\bar{u}} = \int_{\Omega} \Phi(X,Y)\,\psi(X,Y,\ldots)\,dP_0 =$$

$$= \int_{\Omega'} \Phi(X',Y')\,\psi(X',Y',\ldots)\,dP'_0 = \int_{\Omega'} \Phi(X',Y')\,dP'_{\bar{u}}.$$

This ends the proof. Q.E.D.

We can formulate, now, the control problem on the canonical sample space

$$\Omega^0 = C(<0,1>) \times \vartheta \times C(<0,1>)$$

which will be endowed with an increasing family of σ-fields and probability measure. Because of Theorem 1, if we show that there exists an optimal control $\bar{u}^* \in \mathcal{U}$ for the control problem formulated on Ω^0, then we shall prove the existence of optimal control for the control problem formulated on any probability space. Our construction of the probability space is very similar to the one given in [1]. To define precisely the new probability space, let $\vartheta_{t_1}^{t_2}$ denote the space of all measures V defined on Borel subsets of $\langle t_1,t_2\rangle \times U$ such that $V(\langle s_1,s_2\rangle \times U) = s_2 - s_1$, for all $0 < t_1 \leq s_1 \leq s_2 \leq t_1 \leq 1$. Obviously we have $\vartheta_0^1 = \vartheta$. The space $\vartheta_{t_1}^{t_2}$ is also endowed with Prohorov metric. The elements $\omega = (\omega_0,\omega_1,\omega_2)$ of Ω^0 satisfy

$$\omega(t) = (\omega_0(t),\omega_1(t),\omega_2(t)) = (W_t(\omega),V_t(\omega),Y_t(\omega))$$

for all $0 \leq t \leq 1$. Let $\mathcal{F}_t(W) = \sigma(W_s\colon 0 \leq s \leq t)$ be the σ-field generated by the random variables $\{W_s\colon 0 \leq s \leq t\}$, with $\mathcal{F}_t(Y)$ defined similarly. We can regard $\mathcal{F}_t(W)$ and $\mathcal{F}_t(Y)$ as σ-fields of subsets of $C(\langle 0,1\rangle)$. Let $\mathcal{F}_t(V) = \mathcal{B}_{\vartheta_0^t} \times \vartheta_t^1$ where $\mathcal{B}_{\vartheta_0^t}$ denotes the Borel σ-field of ϑ_0^t. Let us define the family of σ-fields $(\mathcal{F}_t^0)$ as follows

$$\mathcal{F}_t^0 = \mathcal{F}_t(W) \times \mathcal{F}_t(V) \times \mathcal{F}_t(Y).$$

We put $\mathcal{F}^0 = \mathcal{F}_1^0$. Let $\mathcal{W}$ be the Wiener measure defined on the Borel subsets of $C(\langle 0,1\rangle)$. Let ω_1 be fixed. Under the conditions (3), (4) there exists a process $\bar{X}^{\omega_1}$ defined on the probability space $(C(\langle 0,1\rangle), \mathcal{F}_1(\bar{W}), \mathcal{F}_t(\bar{W})_{0\leq t\leq 1}, \mathcal{W})$ such that $\bar{X}^{\omega_1}$ is the unique strong solution of the following equation

$$d\bar{X}_t^{\omega_1} = f(t, \bar{X}^{\omega_1}, \omega_1)dt + \sigma(t, \bar{X}^{\omega_1}, \omega_1)d\bar{W}_t \tag{8}$$

where $\bar{W}_t(\omega_0) = \omega_0(t)$ is the canonical Wiener process defined on $C(\langle 0,1\rangle)$. Obviously we have $W_t(\omega_0,\omega_1,\omega_2) = \bar{W}_t(\omega_0)$ for all $(\omega_0,\omega_1,\omega_2) \in \Omega^0$.

Let $\bar{u} \in \mathcal{U}$ be fixed. We define the probability measure $\tilde{P}_{\bar{u}}$ on $\mathcal{F}^0$ as follows

$$\tilde{P}_{\bar{u}}(d\omega_0, d\omega_1, d\omega_2) = \mathcal{W}(d\omega_0)\, \delta_{\bar{u}(\omega_2)}(d\omega_1)\, \mathcal{W}(d\omega_2),$$

where $\delta_{\bar{u}(\omega_2)}$ denotes Dirac measure on ϑ concentrated at $\bar{u}(\omega_2)$. Let X^* be the function defined on $\langle 0,1\rangle \times \Omega^0$ as follows.

$$X_t^*(\omega) = X_t^*(\omega_0, \omega_1, \omega_2) = \bar{X}_t^{\omega_1}(\omega_0) \text{ for all } (\omega_0, \omega_1, \omega_2) \in \Omega^0.$$

The following lemma shows that X^* has measurable modification X under $\tilde{P}_{\bar{u}}$. In other words we shall show that X is measurable, $\mathcal{F}_t^0$ - adapted process defined on $(\Omega^0, \mathcal{F}^0, \tilde{P}_{\bar{u}})$. Besides, Lemma 5 will be necessary to prove the existence of an optimal control function.

L e m m a 5. Let $(\omega_1^n, \omega_2^n) \in \vartheta \times C(\langle 0,1\rangle)$ and $(\omega_1^n, \omega_2^n) \xrightarrow{n\to\infty} (\omega_1^0, \omega_2^0) \in \vartheta \times C(\langle 0,1\rangle)$, $n = 1,2,\ldots$. Then we have

$$\lim_{n\to\infty} \int_{C(\langle 0,1\rangle)} \sup_{0\le t\le 1} \left| X_t^*(\omega_0, \omega_1^n, \omega_2^n) - X_t^*(\omega_0, \omega_1^0, \omega_2^0) \right|^2 \mathcal{W}(d\omega_0) = 0.$$

P r o o f . We have to show that

$$\lim_{n\to\infty} \int_{C(\langle 0,1\rangle)} \sup_{0\le t\le 1} \left| \bar{X}^{\omega_1^n}(\omega_0) - \bar{X}^{\omega_1^0}(\omega_0) \right|^2 \mathcal{W}d(\omega_0) = 0.$$

The proof is standard. Namely using (3), (4), (8) and martingale's inequality we have

$$\int\limits_{C(\langle 0,1\rangle)} \sup_{0\leq s\leq t} \left| \overline{X}_s^{\omega_1^n}(\omega_o) - \overline{X}_s^{\omega_1^o}(\omega_o)\right|^2 \mathcal{W}(d\omega_o) \leq$$

$$\leq 2 \int\limits_{C(\langle 0,1\rangle)} \left(\sup_{0\leq s\leq t} \int_0^s \left|f(\tau,\overline{X}^{\omega_1^n}(\omega_o),\omega_1^n) - f(\tau,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^o)\right|^2 d\tau\right)\mathcal{W}(d\omega_o) +$$

$$+ 2 \int\limits_{C(\langle 0,1\rangle)} \sup_{0\leq s\leq t} \left|\int_0^s \left(\sigma(\tau,\overline{X}^{\omega_1^n}(\omega_o),\omega_1^n) - \sigma(\tau,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^o)\right) d\overline{W}_\tau\right|^2 \mathcal{W}d(\omega_o) \leq$$

$$\leq 2 \int\limits_{C(\langle 0,1\rangle)} \int_0^t \left|f\left(s,\overline{X}^{\omega_1^n}(\omega_o),\omega_1^n\right) - f\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^o\right)\right|^2 ds\,\mathcal{W}(d\omega_o) +$$

$$+ 4 \int\limits_{C(\langle 0,1\rangle)} \int_0^t \left|\sigma\left(s,\overline{X}^{\omega_1^n}(\omega_o),\omega_1^n\right) - \sigma\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^o\right)\right|^2 ds\,\mathcal{W}(d\omega_o) \leq$$

$$+ 4 \int\limits_{C(\langle 0,1\rangle)} \int_0^t \left|f\left(s,\overline{X}^{\omega_1^n}(\omega_o),\omega_1^n\right) - f\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^n\right)\right|^2 ds\,\mathcal{W}(d\omega_o) +$$

$$+\int_0^t 4 \int\limits_{C(\langle 0,1\rangle)} \left| f\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^n\right) - f\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^o\right)\right|^2 ds\,\mathcal{W}(d\omega_o) +$$

$$+ \int\limits_{C(\langle 0,1\rangle)} 8\int_0^t \left|\sigma\left(s,\overline{X}^{\omega_1^n}(\omega_o),\omega_1^n\right) - \sigma\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^n\right)\right|^2 ds\,\mathcal{W}(d\omega_o) +$$

$$+ \int\limits_{C(\langle 0,1\rangle)} 8\int_0^t \left|\sigma\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^n\right) - \sigma\left(s,\overline{X}^{\omega_1^o}(\omega_o),\omega_1^o\right)\right|^2 ds\,\mathcal{W}(d\omega_o) \leq$$

$$\leq 8L\int_0^t \int\limits_{C(\langle 0,1\rangle)} \left| \overline{X}_s^{\omega_1^n}(\omega_o)-\overline{X}_s^{\omega_1^o}(\omega_o)\right|^2 \mathcal{W}(d\omega_o)ds+\beta_n(t) \leq 8L\int_0^t \alpha_n(s)ds+\beta_n(t),$$

where

$$\alpha_n(t) = \int_{C(\langle 0,1\rangle)} \sup_{0\le s\le t} \left| \bar{X}_s^{\omega_1^n}(\omega_0) - \bar{X}_s^{\omega_1^0}(\omega_0)\right|^2 \mathcal{W}(d\omega_0)\cdot$$

and

$$\beta_n(t) = 4 \int_{C(\langle 0,1\rangle)} \int_0^t \left| f\left(s,\bar{X}^{\omega_1^0}(\omega_0),\omega_1^n\right) - f\left(s,\bar{X}^{\omega_1^0}(\omega_0)\omega_1^0\right)\right|^2 ds\,\mathcal{W}(d\omega_0) +$$

$$+ 8 \int_{C(\langle 0,1\rangle)} \int_0^t \left|\sigma\left(s,\bar{X}^{\omega_1^0}(\omega_0),\omega_1^n\right) - \sigma\left(s,\bar{X}^{\omega_1^0}(\omega_0),\omega_1^0\right)\right|^2 ds\,\mathcal{W}(d\omega_0).$$

Therefore we have

$$\alpha_n(t) \le 8L \int_0^t \alpha_n(s)ds + \beta_n(t),$$

where $\beta_n(t) \xrightarrow{n\to\infty} 0$ uniformly with respect to $t \in \langle 0,1\rangle$. Very easy computation shows that.

$$\alpha_n(t) \le 8L \int_0^t \beta_n(s) \exp\left[8L(t-s)\right] ds + \beta_n(t) \quad \text{(see [4] p.368).}$$

Finally we obtain $\alpha_n(1) \xrightarrow{n\to\infty} 0.$ Q.E.D.

C o r o l l a r y 1. There exists $\mathcal{F}_t^0$-adapted measurable process X defined on $(\Omega^0, \mathcal{F}^0, \tilde{P}_{\bar{u}})$ such that for every ω_1, ω_2 we have

$$X_t(\omega_0,\omega_1,\omega_2) = X_t^*(\omega_0,\omega_1,\omega_2)$$

for $\omega_0 \in A^{\omega_1\omega_2} \subset C(\langle 0,1\rangle)$ with $\mathcal{W}(A^{\omega_1\omega_2}) = 1$ and for all $t \in \langle 0,1\rangle$.

P r o o f . First we shall show that there exists a measurable process X such that for every t, ω_1, ω_2 we have

$$X_t(\omega_0,\omega_1,\omega_2) = X^*_t(\omega_0,\omega_1,\omega_2)$$

for all $\omega_0 \in A^{\omega_1\omega_2,t}$ with $\mathcal{W}(A^{\omega_1,\omega_2,t}) = 1$.

It suffices to show (see [4]) that for every $\varepsilon > 0$ we have

$$\lim_{(t_n,\omega_1^n)\to(t_0,\omega_1^0)} \mathcal{W}\left\{\omega_0 : \left|\bar{X}^{\omega_1^n}_{t_n}(\omega_0) - \bar{X}^{\omega_1^0}_{t_0}(\omega_0)\right| > \varepsilon\right\} = 0.$$

The processes $\bar{X}^{\omega_1^n}$, $\bar{X}^{\omega_1^0}$ have continuous trajectories and the family $\left\{\left|\bar{X}^{\omega_1^n}_t - \bar{X}^{\omega_1^0}_t\right|^2 : t \in \langle 0,1\rangle\right\}$ is uniformly integrable with respect to $\mathcal{W}$. (see for example [5]). Therefore using Lemma 5 we have

$$\mathcal{U}\left\{\omega_0 : \left|\bar{X}^{\omega_1^n}_{t_n} - \bar{X}^{\omega_1^0}_{t_0}\right| > \varepsilon\right\} \leqslant \frac{1}{\varepsilon^2} \int_{C(\langle 0,1\rangle)} \left|\bar{X}^{\omega_1^n}_{t_n} - \bar{X}^{\omega_1^0}_{t_0}\right|^2 d\mathcal{W} \leqslant$$

$$\leqslant \frac{2}{\varepsilon^2} \int_{C(\langle 0,1\rangle)} \left|\bar{X}^{\omega_1^n}_{t_n} - \bar{X}^{\omega_1^0}_{t_0}\right|^2 d\mathcal{W} + \frac{2}{\varepsilon^2} \int_{C(\langle 0,1\rangle)} \left|\bar{X}^{\omega_1^0}_{t_n} - \bar{X}^{\omega_1^0}_{t_0}\right|^2 d\mathcal{W} \leqslant$$

$$\leqslant \frac{2}{\varepsilon^2}\alpha_n(1) + \frac{2}{\varepsilon^2} \int_{C(\langle 0,1\rangle)} \left|\bar{X}^{\omega_1^0}_{t_n} - \bar{X}^{\omega_1^0}_{t_0}\right|^2 d\mathcal{W} \xrightarrow{n\to\infty} 0.$$

Finally we put

$$A^{\omega_1,\omega_2} = \bigcap_{t\,\text{rational numbers}\,\cap\,\langle 0,1\rangle} A^{\omega_1,\omega_2,t}$$

and we obtain the assertion of Corollary 1 because of continuity of $\bar{X}^{\omega_1}$. Q.E.D.

We shall need the following lemma.

Lemma 6. Let $\bar{u} \in \mathcal{U}$ be fixed. The processes W, Y are $\mathcal{F}_t^0$-adapted independent Wiener processes and the process X satisfies (1) under $\tilde{P}_{\bar{u}}$.

Proof. From the definition of $\tilde{P}_{\bar{u}}$ we obtain that W, Y are independent Wiener processes. To prove the second part of the assertion of Lemma 6 let us note that, using Lemma 5, for every fixed $(\omega_1, \omega_2) \in \mathcal{V} \times C(<0,1>)$ we can find a version $*$ of stochastic integral with respect to $\bar{W}$ such that we have

$$(9) \qquad X_t(\cdot, \omega_1, \omega_2) = X_0 + \int_0^t f(s, X(\cdot, \omega_1, \omega_2), \omega_1) ds +$$

$$+ \int_0^t \sigma(s, X(\cdot, \omega_1, \omega_2), \omega_1) * d\bar{W}_s \quad \text{for all } \omega_0 \in C(<0,1>), t \in <0,1>.$$

What we need is to show that

$$(10) \qquad \int_0^t \sigma(s, X(\omega_0, \omega_1, \omega_2), \omega_1) * d\bar{W}_s(\omega_0) =$$

$$= \int_0^t \sigma(s, X(\omega_0, \omega_1, \omega_2), \omega_1) dW_s(\omega_0, \omega_1, \omega_2) \quad \text{a.s. under } \tilde{P}_{\bar{u}}.$$

Let $H_s(\omega_0, \omega_1, \omega_2) = \sigma(s, X(\omega_0, \omega_1, \omega_2), \omega_1)$. We have to show that

$$\int_0^t H_s * d\bar{W}_s = \int_0^t H_s \, dW_s \quad \text{a.s. under } \tilde{P}_{\bar{u}} \text{ for all } t \in <0,1>.$$

Let $\{H^n\}_{n=1}$ be a sequence of step processes such that

$$(11) \qquad \lim_{n \to \infty} \int_{\Omega^0} \int_0^1 |H_t^n - H_t|^2 \, dt \, d\tilde{P}_{\bar{u}} = 0$$

From the definition of stochastic integral we have

$$\int_0^t H_s^n d\overline{W}_s = \int_0^t H_t^n dW_s \quad \text{for all} \quad \omega \in \Omega^0, \quad t \in \langle 0,1\rangle.$$

Using (11) we can find a subsequence $\{H^{n'}\} \subset \{H^n\}$ such that

$$\lim_{n'\to\infty} \int_{C(\langle 0,1\rangle)} \left(\int_0^1 \left| H_t^{n'}(\cdot,\omega_1,\omega_2) - H_t(\cdot,\omega_1,\omega_2)\right|^2 dt \right) d\mathcal{W} = 0$$

for almost all $(\omega_1,\omega_2) \in \vartheta \times C(\langle 0,1\rangle)$ with respect to $\delta_{\overline{u}(\omega_2)}(d\omega_1)\mathcal{W}(d\omega_2)$. Therefore we have

$$(12) \quad \varphi_{n'}(t,\omega_1,\omega_2) = \int_{C(\langle 0,1\rangle)} \left| \int_0^t H_s^{n'}(\omega_0,\omega_1,\omega_2) d\overline{W}_s(\omega_0) - \int_0^t H_s(\omega_0,\omega_1,\omega_2) * d\overline{W}_s(\omega_0)\right|^2 \mathcal{W}(d\omega_0) \longrightarrow 0$$

for $\delta_{\overline{u}(\omega_2)}(d\omega_1)\mathcal{W}(d\omega_2)$ - almost all (ω_1,ω_2).

From the definition of the version $*$ the process

$$Z_t = \int_0^t H_s * d\overline{W}_s$$

is measurable as a process defined on $(\Omega^0, \mathcal{F}^0, \tilde{P}_{\overline{u}})$. It is evident that $\{\varphi_{n'}\}$ are uniformly bounded functions ($H_s^{n'}$, H_s are bounded) and integrating φ_n we obtain

$$\int_{\Omega^0} \left| \int_0^t H_s^{n'} d\overline{W}_s - \int_0^t H_s * d\overline{W}_s\right|^2 d\tilde{P}_{\overline{u}} \xrightarrow{n'\to\infty} 0.$$

On the other hand we have

$$\int_{\Omega^0} \left| \int_0^t H_s^n dW_s - \int_0^t H_s dW_s\right|^2 d\tilde{P}_{\overline{u}} \xrightarrow{n\to\infty} 0.$$

Finally

$$\int_0^t H_s dW_s = \int_0^t H_s d\overline{W}_s \quad \tilde{P}_{\bar{u}} - \text{a.s.}$$

This ends the proof of (10). But the equation (9) is equivalent to (1) for we have

$$\tilde{P}_{\bar{u}}\left\{(\omega_0,\omega_1,\omega_2) \in \Omega^0 : \bar{u}(\omega_2) = \omega_1\right\} = 1. \qquad \text{Q.E.D.}$$

Now, following the Fleming and Pardoux paper [1] we define the new probability measure $P_{\bar{u}}$ on $(\Omega^0, \mathcal{F}^0)$ by the formulas

$$\frac{dP_u}{d\tilde{P}_{\bar{u}}} = \exp\left\{\int_0^1 h(t,X_t)dY_t - \frac{1}{2}\int_0^1 \left[h(t,X_t)\right]^2 dt\right\}.$$

It is well known that in this case $\tilde{W}$ and W are independent Brownian motions defined on $(\Omega^0, \mathcal{F}^0, P_{\bar{u}})$ (see [1]).

We shall show the following lemma.

L e m m a 7. The processes X, Y satisfy (1), (2) under $P_{\bar{u}}$.

P r o o f . From the Martin Girsanov formula we see that $\tilde{W}_t = Y_t + \int_0^t h(s,X_s)ds$ and (2) is evident. To prove (1) it suffice to show that

$$\int_0^t \sigma(s,X,\bar{u}(Y))dW_s = \int_0^{t *} \sigma(s,X,\bar{u}(Y))dW_s,$$

where $\int$, $\int^*$ denote the stochastic integrals under $P_{\bar{u}}$ and $\tilde{P}_{\bar{u}}$ respectively. We put $H_s = \sigma(s,X,\bar{u}(Y))$. There exists bounded sequence $\left\{H_s^n\right\}_{n=1}^{\infty}$ of step processes such that we have

$$\int_{\Omega^0}\left(\int_0^1 |H_s^n - H_s|^2 ds\right) d\tilde{P}_{\bar{u}} \xrightarrow{n\to\infty} 0.$$

It is evident that $\int_0^t H_s^n dW_s = \int_0^{t*} H_s^n dW_s$. One can choose a subsequence $\{H_s^{n'}\}_{n=1}^{\infty}$ of $\{H_s^n\}_{n=1}^{\infty}$ such that we have

$$\int_0^1 |H_s^{n'} - H_s|^2 ds \xrightarrow{n' \to \infty} 0 \quad \tilde{P}_{\bar{u}} \text{ - a.s.}$$

The $\tilde{P}_{\bar{u}}$ - null sets and $P_{\bar{u}}$ - null sets are the same so we have

$$\int_0^1 |H_s^{n'} - H_s|^2 ds \xrightarrow{n' \to \infty} 0 \quad P_{\bar{u}} \text{ - a.s.}$$

Therefore, using the Lebesque Theorem $\int_{\Omega^0} \left(\int_0^1 |H_s^{n'} - H_s|^2 ds \right) dP_{\bar{u}} \xrightarrow{n' \to \infty} 0$ and consequently we obtain

$$\int_{\Omega^0} \left| \int_0^t H_s^{n'} dW_s - \int_0^{t*} H_s dW_s \right|^2 d\tilde{P}_{\bar{u}} \xrightarrow{n' \to \infty} 0$$

and

$$\int_{\Omega^0} \left| \int_0^t H_s^{n'} dW_s - \int_0^t H_s dW_s \right|^2 dP_{\bar{u}} \xrightarrow{n' \to \infty} 0.$$

We can choose a further subsequence $\{H^{n''}\}$ of $\{H^{n'}\}$ such that

$$\int_0^t H_s^{n''} dW_s \xrightarrow{n'' \to \infty} \int_0^{t*} H_s dW_s \quad \text{and} \quad \int_0^t H_s^{n''} dW_s \xrightarrow{n'' \to \infty} \int_0^t H_s dW_s$$

$\tilde{P}_{\bar{u}}$, $P_{\bar{u}}$ - a.s. Q.E.D.

We can now formulate the main result of this paper.

T h e o r e m 2. Under (3), (4), (5) for every $\bar{u} \in \mathcal{U}$ there exists the unique weak solution X, Y (it may be defined on any probability space $(\Omega, \mathcal{F}, P)$) of (1), (2). Moroever there exists $\bar{u}^* \in \mathcal{U}$ such that we have

$$J[\bar{u}^*] \leq J[\bar{u}] \quad \text{for all} \quad \bar{u} \in \mathcal{U}.$$

The first part of the assertion of Theorem 2 is easy consequence of Theorem 1 and Lemma 7. To prove existence of optimal strategy $\bar{u}^* \in \mathcal{U}$ we shall need the following three lemmas.

Lemma 8. Let $\Gamma \subset \vartheta \times C(\langle 0,1\rangle)$ be a Borel such that we have $\mathrm{pr}_{C(\langle 0,1\rangle)} \Gamma := \{\omega_2 \in C(\langle 0,1\rangle) : \exists \ \omega_1 \in \vartheta$ such that $(\omega_1, \omega_2) \in \Gamma\} = C(\langle 0,1\rangle)$. Then there exists $\bar{u} \in \mathcal{U}$ such that $(\bar{u}(\omega_2), \omega_2) \in \Gamma\}$ for $\mathcal{W}$-almost all $\omega_2 \in C(\langle 0,1\rangle)$ where $\mathcal{W}$ is Wiener measure defined on Borel subsets of $C(\langle 0,1\rangle)$.

Lemma 9. Let $\Phi : \vartheta \times C(\langle 0,1\rangle) \longrightarrow R$ be a real nonnegative continuous function. Then there exists $\bar{u}^* \in \mathcal{U}$ such that we have

$$\Phi(\bar{u}^*(\omega_2), \omega_2) \leq \Phi(\omega_1, \omega_2) \text{ for all } \omega_1 \in \vartheta \text{ and } \omega_2 \in \mathcal{A} \subset C(\langle 0,1\rangle) \text{ with } \mathcal{W}(\mathcal{A})=1.$$

Lemma 10. Under the conditions (3), (4), (5) there exists measurable function $I: C(\langle 0,1\rangle) \times \vartheta \times C(\langle 0,1\rangle) \longrightarrow R$ such that the following conditions are satisfied.

(13) For every sequence $(\omega_1^n, \omega_2^n) \longrightarrow (\omega_1^0, \omega_2^0) \in \vartheta \times C(\langle 0,1\rangle)$ the family of random variables $\{\exp I(\cdot, \omega_1^n, \omega_2^n) : (\omega_1^n, \omega_2^n) \in \vartheta \times C(\langle 0,1\rangle)\}$ defined on $(C(\langle 0,1\rangle), \mathcal{F}_1(W))$ is uniformly integrable under $\mathcal{W}$.

(14) For every $\bar{u} \in \mathcal{U}$ there exists a version of the stochastic integral $\int_0^1 h(t, X_t) dY_t$ under $\tilde{P}_{\bar{u}}$ such that we have
$$I(\omega_0, \omega_1, \omega_2) = \int_0^1 h(t, X_t(\omega_0, \omega_1, \omega_2)) dY_t(\omega_0, \omega_1, \omega_2)$$
for all $\omega_0 \in C(\langle 0,1\rangle)$ and for $(\omega_1, \omega_2) \in \mathcal{T}_{\bar{u}}$ with $\tilde{P}_{\bar{u}}(C(\langle 0,1\rangle) \times \mathcal{T}_{\bar{u}}) = 1$.

(15) If $(\omega_1^n,\omega_2^n) \xrightarrow{n\to\infty} (\omega_1^0,\omega_2^0) \in \vartheta \times C(<0,1>)$ then there exists a subsequence $\left\{(\omega_1^{n'},\omega_2^{n'})\right\}$ such that $I(\cdot,\omega_1^{n'},\omega_2^{n'}) \xrightarrow{n'\to\infty}$ $\to I(\cdot,\omega_1^0,\omega_2^0)$ $\mathcal{W}$- a.s.

The proof of Lemma 8 is given in the appendix of [6].

P r o o f . of Lemma 9. Let $\Gamma = \left\{(\omega_1,\omega_2)\in \vartheta\times C(<0,1>) : \phi(\omega_1,\omega_2) \le \phi(\omega_1',\omega_2) \text{ for all } \omega_1'\in\vartheta\right\}$. The space ϑ is compact and consequently $pr_{C(<0,1>)}\Gamma = C(<0,1>)$. It is evident that Γ is closed subset of $\vartheta\times C(<0,1>)$. Using Lemma 8 we obtain the assertion of Lemma 9.

P r o o f of Lemma 10. Since h satisfies (5) we can integrate $\int_0^1 h(t,X_t)dY_t$ by parts:

$$(16)\quad \int_0^1 h(t,X_t)dY_t = Y_1h(1,X_1) - \int_0^1 Y_t \frac{\partial h(t,X_t)}{\partial t}dt +$$

$$-\int_0^1 Y_t \frac{\partial h(t,X_t)}{\partial x} f(t,X,V)dt - \int_0^1 Y_t \frac{\partial h(t,X_t)}{\partial x} \sigma(t,X,V)dW_t +$$

$$-\frac{1}{2}\int_0^1 Y_t \frac{\partial^2 h(t,X_t)}{\partial^2 x}\left[\sigma(t,X,V)\right]^2 dt \quad \text{a.s. under } \tilde{P}_{\bar{u}}, \text{ for } \bar{u}\in\mathcal{U}.$$

Let us define

$$I^*(\omega_0,\omega_1,\omega_2) = \omega_2(1)h(1,X_1(\omega_0,\omega_1,\omega_2)) - \int_0^1 \omega_2(t)\frac{\partial h(t,X_t)}{\partial t}dt +$$

$$-\int_0^1 \omega_2(t)\frac{\partial h(t,X_t(\omega_0,\omega_1,\omega_2))}{\partial x} f(t,X(\omega_0,\omega_1,\omega_2),\omega_1)dt +$$

$$-\int_0^1 \omega_2(t)\frac{\partial h(t,X_t(\omega_0,\omega_1,\omega_2))}{\partial x}\sigma(t,X(\omega_0,\omega_1,\omega_2),\omega_1)d\overline{W}_t(\omega_0) +$$

$$-\frac{1}{2}\int_0^1 \omega_2(t)\frac{\partial^2 h(t,X_t(\omega_0,\omega_1,\omega_2))}{\partial^2 x}\left[\sigma(t,X(\omega_0,\omega_1,\omega_2)\omega_1)\right]^2 dt.$$

Using Lemma 5, and the assumption (3) we see that (15) is satisfied for I^*. The above construction of the process X shows that we can find a measurable version I of I^* such that $I(\cdot,\omega_1,\omega_2) = I^*(\cdot,\omega_1,\omega_2)$ $\mathcal{W}(d\omega_0)$ - a.s. for all $(\omega_1,\omega_2) \in \vartheta \times C(<0,1>)$ and I satisfies (15). Let $\bar{u}$ be any admissible control function. We can also find a measurable version of the stochastic integral

$$\int_0^1 Y_t \frac{\partial h(t,X_t)}{\partial x} \sigma(t,X,V)dW_t \quad \text{under} \quad \tilde{P}_{\bar{u}} \quad \text{such that we have}$$

$$\int_0^1 Y_t \frac{\partial h(t,X_t)}{\partial x} \sigma(t,X,V)dW_t = \int_0^1 Y_t \frac{\partial h(t,X_t)}{\partial x} \sigma(t,X,V)d\bar{W}_t$$

for all $\omega_0 \in C(<0,1>)$ and for $\delta_{\bar{u}(\omega_2)}(d\omega_1)\,\mathcal{W}(d\omega_2)$ - almost all (ω_1,ω_2). This ends the proof of (14).

To prove (13) note that for every $\varepsilon > 0$ we have

$$\mathcal{W}\left\{\left|I(\cdot,\omega_1^n,\omega_2^n)\right| > \varepsilon\right\} \leq$$

$$\leq e^{-2\varepsilon} \int_{C(<0,1>)} \exp\left[2I(\cdot,\omega_1^n,\omega_2^n)\right] \, d\,\mathcal{W} \leq L_0 \, e^{-2\varepsilon}$$

because of the assumptions (3), (5). The constant L_0 does not depend on n. Therefore we have

$$\int_{\left\{\omega_0 : \exp I(\omega_0,\omega_1^n,\omega_2^n) \geq \Delta\right\}} \exp I(\omega_0,\omega_1^n,\omega_2^n)\, \mathcal{W}(d\omega_0) \leq$$

$$\leq \int_{\left\{\omega_0 : I(\omega_0,\omega_1^n,\omega_2^n) \geq \ln\Delta\right\}} \exp I(\omega_0,\omega_1^n,\omega_2^n)\, \mathcal{W}(d\omega_0) \leq$$

$$\leq \sup_n \sum_{k > \ln\Delta} \mathcal{W}\left\{\omega_0 : k-1 < I(\omega_0,\omega_1^n,\omega_2^n) \leq k\right\} \cdot e^k \leq$$

$$\leq L_0 \sum_{k > \ln\Delta} e^{-2(k-1)} \, e^k \xrightarrow{\Delta \to \infty} 0.$$

This ends the proof of Lemma 10.

Remark. Note that I does not depend on $\bar{u} \in \mathcal{U}$.

Now we can prove Theorem 2.

Proof of Theorem 2. It remains to prove that there exists an optimal control $\bar{u}^* \in \mathcal{U}$. Using the construction of $(\Omega^0, \mathcal{F}^0, P_{\bar{u}})$ the cost function can be represented as follows

$$J[\bar{u}] = \int_{\Omega^0} \left(\int_0^1 F(t,X,\bar{u}(Y))dt\right) dP_{\bar{u}} + \int_{\Omega^0} G(X)dP_{\bar{u}} =$$

$$= \int_{\Omega^0} \left[\int_0^1 F(t,X,\bar{u}(Y))dt + G(X)\right] \exp\left[\int_0^1 h(t,X_t)dY_t - \frac{1}{2}\int_0^1 h(t,X_t)^2 dt\right] d\tilde{P}_{\bar{u}} =$$

$$= \int_{C(\langle 0,1\rangle)} \int_{\vartheta} \int_{C(\langle 0,1\rangle)} \left(\int_0^1 F(t,X,\bar{u}(Y)dt + G(X)\right) \exp\left[I - \frac{1}{2}\int_0^1 h(t,X_t)^2 dt\right] \cdot$$

$$\cdot \mathcal{W}(d\omega_0)\, \delta_{\bar{u}(\omega_2)}(d\omega_1)\, \mathcal{W}(d\omega_2) = \int_{C(\langle 0,1\rangle)} \int_{\vartheta} \Phi(\omega_1,\omega_2)\, \delta_{\bar{u}(\omega_2)}(d\omega_1)\, \mathcal{W}(d\omega_2) =$$

$$= \int_{C(\langle 0,1\rangle)} \Phi(\bar{u}(\omega_2),\omega_2)\, \mathcal{W}(d\omega_2),$$

where

$$\Phi(\omega_1,\omega_2) = \int_{C(\langle 0,1\rangle)} \left\{\left[\int_0^1 F(t,X(\omega_0,\omega_1,\omega_2),\omega_1)dt + G(X(\omega_0,\omega_1,\omega_2))\right] \cdot\right.$$

$$\left. \cdot \exp\left[I(\omega_0,\omega_1,\omega_2) - \frac{1}{2}\int_0^1 h(t,X_t(\omega_0,\omega_1,\omega_2))^2 dt\right]\right\} \mathcal{W}(d\omega_0).$$

From Lemma 5 and Lemma 10 we have that for every sequence $(\omega_1^n,\omega_2^n) \longrightarrow (\omega_1^0,\omega_2^0)$ there exists subsequence $\left\{(\omega_1^{n'},\omega_2^{n'})\right\}$ of $\left\{(\omega_1^n,\omega_2^n)\right\}$ such that

$$\left[\int_0^1 F(t,X(\omega_0,\omega_1^{n'},\omega_2^{n'}),\omega_1^{n'})dt + G(X(\omega_0,\omega_1^{n'},\omega_2^{n'}))\right] \cdot \exp\left[I(\omega_0,\omega_1^{n'},\omega_2^{n'}) +\right.$$

$$\left. - \frac{1}{2}\int_0^1 h(t,X_t(\omega_0,\omega_1^{n'},\omega_2^{n'}))dt\right] \xrightarrow{n'\to\infty} \left[\int_0^1 F(t,X(\omega_0,\omega_1^{o},\omega_2^{o}),\omega_1^{o})dt + \right.$$

$$\left. + G(X(\omega_0,\omega_1^{o},\omega_2^{o}))\right] \cdot \exp\left[I(\omega_0,\omega_1^{o},\omega_2^{o}) - \frac{1}{2}\int_0^1 h(t,X_t(\omega_0,\omega_1^{o},\omega_2^{o}))^2 dt\right], \mathcal{W}\text{-a.s.}$$

The functions F, G are bounded and, according to Lemma 10, the family $\left\{\exp I(\cdot,\omega_1^{n},\omega_2^{n})\ n=1,2,\ldots\right\}$ is uniformly $\mathcal{W}$-integrable. Therefore Φ satisfies the assumptions of Lemma 9 and consequently there exists $\bar{u}^* \in \mathcal{U}$ such that we have

$$J[\bar{u}^*] = \int_{C(\langle 0,1\rangle)} \Phi(\bar{u}^*(\omega_2),\omega_2))\mathcal{W}(d\omega_2) = \int_{\alpha} \Phi(\bar{u}^*(\omega_2),\omega_2))\mathcal{W}(d\omega_2) \leq$$

$$\leq \int_{\alpha} \Phi(\bar{u}(\omega_2),\omega_2))\mathcal{W}(d\omega_2) = \int_{C(\langle 0,1\rangle)} \Phi(\bar{u}(\omega_2),\omega_2)\mathcal{W}(d\omega_2) = J[\bar{u}]$$

for all $\bar{u} \in \mathcal{U}$. Q.E.D.

Random controls are considered also in [8] and [9].

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
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