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ON SOME BOUNDARY VALUE PROBLEMS FOR THE EQUATION $(\Delta-c^2)^p u=0$ IN A HALF-SPACE

Let

$$X = (x_1, x_2, \dots, x_n), \quad Y = (y_1, y_2, \dots, y_n),$$

$$r^2 = \sum_{i=1}^{n} (x_i - y_i)^2, \quad r_1^2 = \sum_{i=1}^{n-1} (x_i - y_i)^2 + (x_n + y_n)^2,$$

$$r_0^2 = \sum_{i=1}^{n-1} (x_i - y_i)^2 + x_n^2,$$

$$E_n^+ = \left\{ X:\ -\infty < x_i < \infty,\ i=1,2,\dots,n-1,\ x_n > 0 \right\}.$$

In the present paper we shall solve of the Riquier, Neumann and mixed problems for the equation

$$(1) \qquad (\Delta - c^2)^p u(X) = 0$$

in the domain E_n^+, where c is a positive constant, p and n is a positive integer and $n \geq 2$.

We need two lemmas:

L e m m a 1. (see [1]). The function $(cr)^s K_s(cr)$ is the fundamental solution of the equation (1), where $K_q(cr)$ is MacDonald function and $s = p - \frac{n}{2}$.

L e m m a 2. (see [1]). If q is an arbitrary real number, p is an arbitrary nonnegative integer and c is an arbitrary positive real, then

$$\Delta_Y^p((cr)^q K_q(cr)) = c^{2p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} a_k (cr)^{q-k} K_{q-k}(cr),$$

where $a_0 = 1$, $a_k = \prod_{j=1}^{k} (n+2q-2j)$, $k=1,2,\dots,p$.

By Lemmas 1, 2 we have

L e m m a 3. The function

$$G(X,Y) = (cr)^s K_s(cr) - (cr_1)^s K_s(cr_1),$$

where $s = p - \frac{n}{2}$ is Green function with a pole at the point X in the domain E_n^+ for the equation (1) satisfying the following boundary conditions (Riquier's conditions).

$$\Delta_Y^i G(X,Y)\Big|_{y_n=0} = 0 \quad \text{for} \quad i=0,1,\dots,p-1. \tag{2}$$

We shall prove

L e m m a 4. We have

$$\lim_{x_n \to 0} \int_{E_{n-1}} D_{y_n} \Delta_Y^i G(X,Y)\Big|_{y_n=0} dY' = \begin{cases} 0 & \text{for} \quad i=0,1,\dots,p-2 \\ \pi^{\frac{n}{2}}(-1)^{p-1} c^{2p-n} (p-1)!\, 2^{p+\frac{n}{2}-1} & \end{cases}$$

for $i = p-1$ and where $Y' = (y_1, y_2, \dots, y_{n-1})$.

P r o o f . By the formulas ([2] p.79, 417)

$$\frac{d}{dz}(z^q K_q(z)) = -z^q K_{q-1}(z), \quad K_q(z) = K_{-q}(z), \tag{3}$$

$$\int_0^\infty \frac{K_\mu(\alpha\sqrt{u^2+z^2})}{\left(\sqrt{u^2+z^2}\right)^\mu} u^{2q+1}\, du = \frac{2^q\, \Gamma(q+1)}{\alpha^{q+1} z^{\mu-q-1}} K_{\mu-q-1}(\alpha z), \tag{4}$$

where $\alpha > 0$, $q > -1$; and by Lemma 2 we have

(5)
$$D_{y_n} \Delta_Y^i G(X,Y)\Big|_{y_n=0} =$$
$$= 2c^{2i+2} x_n \sum_{k=0}^{i} (-1)^k \binom{i}{k} a_k (cr_o)^{s-k-1} K_{s-k-1}(cr_o).$$

Let

$$C_k(X) = \int_{E_{n-1}} x_n \,(r_o)^{s-k-1}\, K_{s-k-1}(cr_o)\,dY'.$$

Using the polar coordinates and the formulas (3), (4) we have

$$C_k(X) = \omega_{n-1} \int_0^{\infty} x_n \left(\sqrt{\varrho^2+x_n^2}\right)^{s-k-1} K_{s-k-1}\left(c\sqrt{\varrho^2+x_n^2}\right) \varrho^{n-2}\, d\varrho =$$
$$= \omega_{n-1} x_n \int_0^{\infty} \frac{K_{k+1-s}\left(c\sqrt{\varrho^2+x_n^2}\right)}{\left(\sqrt{\varrho^2+x_n^2}\right)^{k+1-s}} \varrho^{2\left(\frac{n}{2}-\frac{3}{2}\right)+1}\, d\varrho =$$
$$= \omega_{n-1} x_n \frac{2^{\frac{n}{2}-\frac{3}{2}} \Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{c^{\frac{n}{2}-\frac{1}{2}}\, x_n^{k+1-s-\frac{n}{2}+\frac{1}{2}}} K_{k+1-s-\frac{n}{2}+\frac{1}{2}}(cx_n),$$

where ω_{n-1} denotes the surface area of the unit sphere in E_{n-1}.

Since $s = p - \frac{n}{2}$, then

$$C_k(X) = \omega_{n-1} x_n\, 2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right) c^{\frac{-n+1}{2}}\, x_n^{p-k-3/2}\, K_{p-k-3/2}(cx_n).$$

Hence, by formula $K_v(z) \underset{z\to 0,\ v>0}{\approx} z^{-v}\, 2^{v-1}\, \Gamma(v)$ we obtain

(6) $$\lim_{x_n\to 0} C_k(X) = 0 \quad \text{for} \quad k=0,1,2,\ldots,p-2.$$

If $k = i = p - 1$, then we have

(7) $$\lim_{x_n\to 0} C_{p-1}(X) = \omega_{n-1}\, 2^{\frac{n-4}{2}}\, \Gamma\left(\tfrac{n-1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) c^{-n/2}.$$

In view of (5), (6), (7) we can write

$$\lim_{x_n\to 0} \int_{E_{n-1}} D_{y_n} \Delta_Y^i G(X,Y)\Big|_{y_n=0} dY' =$$

$$= \begin{cases} 0 & \text{for } i=0,1,\ldots,p-2, \\ 2c^{2p}(-1)^{p-1}\, a_{p-1}\, c^{-\frac{n}{2}} \lim\limits_{x_n\to 0} C_{p-1}(X) & \text{for } i=p-1. \end{cases}$$

Since $\omega_{n-1} = 2\pi^{\frac{n-1}{2}}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, then

$$\lim_{x_n\to 0} \int_{E_{n-1}} D_{y_n} \Delta_Y^i G(X,Y)\Big|_{y_n=0} dY' =$$

$$= \begin{cases} 0 & \text{for } i=0,1,\ldots,p-2, \\ \left(\frac{\pi}{2}\right)^{n/2}(-1)^{p-1}c^{2p-n}(p-1)!\, 2^{p+\frac{n}{2}-1} & \text{for } i=p-1 \end{cases}$$

which completes the proof of Lemma 4.

Let

$$a(n,p) = \left(\frac{\pi}{2}\right)^{\frac{n}{2}} (-1)^{p-1}\, c^{2p-n}(p-1)!\, 2^{p+\frac{n}{2}-1}.$$

In virtue of Lemma 4 we obtain

Lemma 5. Let $f(Y') \in L_1(E_{n-1})$. If the function $f(Y')$ is continuous at the point $X_0 \in E_{n-1}$, then

$$\lim_{X\to(X_0,0)} (a(n,p))^{-1} \int_{E_{n-1}} f(Y')D_{y_n}\Delta_Y^i G(X,Y)\Big|_{y_n=0} dY' =$$

$$= \begin{cases} 0 & \text{for} \quad i=0,1,\dots,p-2, \\ f(X_0) & \text{for} \quad i=p-1. \end{cases}$$

We shall prove

Theorem 1. Let $f_i(Y') \in L_1(E_{n-1})$, $i=0,1,\dots,p-1$. If the functions $f_i(Y')$ are continuous at the point $X_0 \in E_{n-1}$, then the function

$$U(X) = (a(n,p))^{-1} \sum_{i=0}^{p-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i}^{p-1} e_k^p D_{y_n}\Delta_Y^{k-i} G(X,Y)\Big|_{y_n=0} dY',$$

where $e_k^p = \binom{p}{k+1}(-c^2)^{p-1-k}$ is a solution of the problem (Riquier's problem)

1° $(\Delta - c^2)^p U(X) = 0$ for $X \in E_n^+$,

2° $\lim\limits_{X\to(X_0,0)} \Delta^i U(X) = f_i(X_0)$ for $i=0,1,\dots,p-1$.

Proof. Observe, that the function $U(X)$ belongs to the class C^∞ in E_n^+ and

$$D_X^\alpha U(X) =$$

$$= (a(n,p))^{-1} \sum_{i=0}^{p-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i}^{p-1} e_k^p D_X^\alpha D_{y_n}\Delta_Y^{k-i} G(X,Y)\Big|_{y_n=0} dY'$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the multiindex. Hence by Lemma 3 the function $U(X)$ satisfies 1^o.

We shall prove 2^o. Let

$$P_i(X) = (a(n,p))^{-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i}^{p-1} e_k^p D_{y_n} \Delta_Y^{k-i} G(X,Y)\Big|_{y_n=0} dY'.$$

We remark that

$$\Delta_Y G(X,Y) = \Delta_X G(X,Y) \quad \text{for} \quad X \neq Y. \tag{8}$$

Hence by Lemma 5 we get

$$\begin{cases} \Delta_X^i P_i(X) \longrightarrow f_i(X_o) & \text{as} \quad X \longrightarrow (X_o, 0) \\ \Delta_X^j P_i(X) \longrightarrow 0 & \text{as} \quad X \longrightarrow (X_o, 0) \quad \text{for} \quad 0 \le j < i. \end{cases} \tag{9}$$

In sequel we prove that

$$\Delta_X^j P_i(X) \longrightarrow 0 \quad \text{as} \quad X \longrightarrow (X_o, 0) \quad \text{for} \quad j > i.$$

From (8) it follows that

$$\Delta_X^{i+1} P_i(X) = \tag{10}$$

$$= (a(n,p))^{-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i}^{p-1} e_k^p D_{y_n} \Delta_Y^{k+1} G(X,Y)\Big|_{y_n=0} dY' =$$

$$= (a(n,p))^{-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i+1}^{p-1} e_{k-1}^p D_{y_n} \Delta_Y^{k} G(X,Y)\Big|_{y_n=0} dY' +$$

$$+ (a(n,p))^{-1} \int_{E_{n-1}} f_i(Y') D_{y_n} \Delta_Y^{p} G(X,Y)\Big|_{y_n=0} dY'.$$

Since

$$(\Delta_Y - c^2)^p G(X,Y) = 0 \quad \text{for} \quad X \neq Y,$$

then

$$\Delta_Y^p G(X,Y) = -\sum_{k=0}^{p-1} \binom{p}{k} (-c^2)^{p-k} \Delta_Y^k G(X,Y).$$

Thus, by (10) we have

$$\Delta_X^{i+1} P_i(X) =$$

$$= -(a(n,p))^{-1} \int_{E_{n-1}} f_i(Y') \sum_{k=0}^{i} \binom{p}{k} (-c^2)^{p-k} D_{y_n} \Delta_Y^k G(X,Y)\Big|_{y_n=0} dY'.$$

Since $i < p-1$, then by Lemma 5

$$\Delta_X^{i+1} P_i(X) \longrightarrow 0 \quad \text{as} \quad X \longrightarrow (X_0,0).$$

Similarly we have

$$\Delta_X^{i+1+l} P_i(X) =$$

$$= -(a(n,p))^{-1} \int_{E_{n-1}} f_i(Y') \sum_{k=0}^{i} \binom{p}{k} (-c^2)^{p-k} D_{y_n} \Delta_Y^{k+l} G(X,Y)\Big|_{y_n=0} dY'$$

If $j \leq p-1$, then $i+1+l \leq p-1$ and $l \leq p-2-i$.
Hence, by Lemma 5 we have

$$(11) \quad \Delta_X^{i+1+l} P_i(X) \longrightarrow 0 \quad \text{as} \quad X \longrightarrow (X_0,0) \quad \text{for} \quad l \leq p-2-i.$$

Finally in view of (9), (10), (11) we have

$$\Delta_X^j \sum_{k=0}^{p-1} P_k(X) = \Delta_X^j U(X) \longrightarrow f_j(X_0) \quad \text{as} \quad X \longrightarrow (X_0,0)$$

for $j=0,1,\ldots,p-1$, which completes the proof of Theorem 1.

Let

$G_1(X,Y) = (cr)^s K_s(cr) + (cr_1)^s K_s(cr_1)$, where $s = p - \frac{n}{2}$.

Observe that

$$D_{x_n} G_1(X,Y) = -D_{y_n} G(X,Y).$$

Hence by Theorem 1 we obtain the following

Theorem 2. Let $f_i(Y') \in L_1(E_{n-1})$, $i=0,1,\dots,p-1$. If the functions $f_i(Y')$ are continuous at the point $X_0 \in E_{n-1}$, then the function

$$U(X) = -(a(n,p))^{-1} \sum_{i=0}^{p-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i}^{p-1} e_k^p \Delta_Y^{k-i} G_1(X,Y)\Big|_{y_n=0} dY'$$

is a solution of the problem (Neumann problem)

1° $(\Delta - c^2)^p U(X) = 0$ for $X \in E_n^+$,

2° $D_{x_n} \Delta_Y^i U(X) \longrightarrow f_i(X_0)$ as $X \longrightarrow (X_0, 0)$ for $i=0,1,\dots,p-1$.

Let

$$r_2^2 = \sum_{i=1}^{n-1} (x_i - y_i)^2 + (x_n + y_n + v)^2.$$

and let

$$G_2(X,Y) = h^{-1}\Big(G_1(X,Y) + 2\int_0^\infty e^{hs}(cr_2)^s K_s(cr_2)dv\Big),$$

where $s = p - \frac{n}{2}$, $h < 0$ and v is non-negative real number.

Observe that

$$(12) \quad D_{x_n} G_2(X,Y) + h\, G_2(X,Y)\Big|_{y_n=0} = -h^{-1} D_{y_n} G(X,Y)\Big|_{y_n=0}.$$

From (12) and Theorem 1 it follows

Theorem 3. Let $f_i(Y') \in L_1(E_{n-1})$, $i=0,1,\dots,p-1$. If the functions $f_i(Y')$ are continuous at the point $X_0 \in E_{n-1}$, then the function

$$U(X) = -(a(n,p))^{-1} h \sum_{i=0}^{p-1} \int_{E_{n-1}} f_i(Y') \sum_{k=i}^{p-1} e_k^p \Delta_Y^{k-i} G_2(X,Y)\Big|_{y_n=0} dY'$$

is a solution of the problem (mixed problem)

1^o $(\Delta - c^2)^p U(X) = 0$ for $X \in E_n^+$

2^o $D_{x_n} \Delta_X^i U(X) + h \Delta_X^i U(X) \to f_i(X_0)$ as $X \to (X_0,0)$

for $i=0,1,\dots,p-1$.

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