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LOWER AND UPPER QUASI-CONTINUOUS FUNCTIONS

I. If A is a subset of a topological space X, then by $\overline{A}$, Int A, Fr A we will denote the closure, the interior and the boundary of A respectively.

A set A is said to be:
- semi-open, if there is an open set U such that $U \subset A \subset \overline{U}$,
- semi-closed, if its complementary $X \setminus A$ is semi-open.

Any semi-open set A such that $x \in A$ we will call a semi-neighbourhood of a point x (shortly we shall write s-neighbourhood). A intersection of all semi-closed sets containing a set A is said to be a semi-closure of A and we denote it by $\underline{A}$. [1].

Let X,Y be two topological spaces. A map $f : X \to Y$ is said to be [2,4,6]:
- quasi-continuous, if for any open set $V \subset Y$ a set $f^{-1}(V)$ is semi-open;
- semi-open (pre-semi-open), if for any open (semi-open) set $U \subset X$ a set $f(U)$ is semi-open.

Let R be the space of real numbers with the natural topology. For any $a,b \in R$, $a < b$, we denote $(a,b) = \{x \in R : a < x < b\}$.

D e f i n i t i o n 1.1. A function $f : X \to R$ is said to be lower-quasi-continuous (upper-quasi-continuous) at a point $x_0 \in X$, if for any number $a \in R$ such that $a < f(x_0)$ $(a > f(x_0))$ there exists s-neighbourhood U of the point x_0 such that $f(U) \subset (a,\infty)$; $(f(U) \subset (-\infty,a))$.

(Shortly we shall write l-quasi-continuous and u-quasi-continuous). A function f is said to be l-quasi-continuous (u-quasi-continuous) in X, if it is l-quasi-continuous (u-quasi-continuous) at every point.

From the definition there immediately follows:

Corollary 1.2. The following conditions are equivalent:

1) a function f is l-quasi-continuous (u-quasi-continuous) at a point x_0;

2) for any neighbourhood U of a point x_0 and a real number $a \in R$ such that $a < f(x_0)$ $(f(x_0) < a)$ holds $U \cap \operatorname{Int}\{x \in X: a < f(x)\} \neq \emptyset$ $(U \cap \operatorname{Int}\{x \in X: f(x) < a\} \neq \emptyset)$;

3) for any neighbourhood U of a point x_0 and a real number $a \in R$ such that $a < f(x_0)$ $(f(x_0) < a)$ there exists an open set U', $\emptyset \neq U' \subset U$ such that $f(U') \subset (a, \infty)$ $(f(U') \subset (-\infty, a))$.

Corollary 1.3. The following conditions are equivalent:

1) a function f is l-quasi-continuous (u-quasi-continuous) in X;

2) for any real number $a \in R$ the set $\{x \in X: a < f(x)\}$ $(\{x \in X: f(x) < a\})$ is semi-open;

3) for any real number $a \in R$ the set $\{x \in X: f(x) \leqslant a\}$ $(\{x \in X: a \leqslant f(x)\})$ is semi-closed.

II. Theorem 2.1. Let D be a dense subset of X. If a function $f: X \to R$ is l-quasi-continuous and u-quasi-continuous at every point of D simultaneously then f is l-quasi-continuous or u-quasi-continuous at every point of X.

Proof. Let $x_0 \in X \setminus D$. Assume that for every neighbourhood U of the point x_0 there is a point $x \in U \cap D$ such that $f(x_0) \leqslant f(x)$. Then for any $a \in R$ such that $a < f(x_0)$ there exists an open set U', $\emptyset \neq U' \subset U$ such that $a < f(x')$ for each $x' \in U'$. Hence f is l-quasi-continuous at x_0. In the other case, i.e. if there exists a neighbourhood U_0 of the point x_0 such that $f(x) < f(x_0)$ for any $x \in U_0 \cap D$, then every neighbourhood U of x_0 contains a point $x \in U \cap D$ such that $f(x) < f(x_0)$.

From this there follows u-quasi-continuity of f at the point x_0.

T h e o r e m 2.2. If $f : X \to R$ is a l-quasi-continuous (u-quasi-continuous) function, then the set M of all points at which f is not lower semi-continuous (upper semi-continuous) is of the first category.

P r o o f . Let $x_0 \in M$. Then there exists a number $a \in R$ such that $a < f(x_0)$, the set $A = \{x: a < f(x)\}$ is semi-open and $x_0 \in \operatorname{Fr} A$.

Let $Q = \{q_n\}_{n=1}^{\infty}$ be the set of all rational numbers and $A_n = \{x: q_n < f(x)\}$. Then for any number q_n such that $a \leqslant q_n < f(x_0)$ we have $x_0 \in A_n \subset A$. Hence $x_0 \in \operatorname{Fr} A_n$ and $M \subset \bigcup_{n=1}^{\infty} \operatorname{Fr} A_n$.

Because the boundary of a semi-open set is nowhere dense, M is of the first category. For u-quasi-continuous function the proof is analogous.

R e m a r k 2.3. Let f be a l-quasi-continuous function. From the proof of Theorem 2.2 it easy follows that the set of all lower semi-continuity points of f is equal to $X \setminus \bigcup_{n=1}^{\infty} (A_n \cap \overline{X \setminus A_n})$, where $A_n = \{x: q_n < f(x)\}$. If f is u-quasi-continuous, then the set of all upper semi-continuity points is equal to $X \setminus \bigcup_{n=1}^{\infty} (B_n \cap \overline{X \setminus B_n})$, where $B_n = \{x: f(x) < q_n\}$.

III. T h e o r e m 3.1. A function $f : X \to R$ is l-quasi-continuous (u-quasi-continuous) if and only if for every open set $U \subset X$ and dense $D \subset X$ the condition $f(U) \cap (a, \infty) \neq \emptyset$ implies $f(U \cap D) \cap (a, \infty) \neq \emptyset$ (the condition $f(U) \cap (-\infty, a) \neq \emptyset$ implies $f(U \cap D) \cap (-\infty, a) \neq \emptyset$).

P r o o f . Assume that f is l-quasi-continuous. Let U be an open set, and let D be dense in X. Let us put $x_0 \in U$, and $a < f(x_0)$. There exists an s-neighbourhood V of x_0 such that $f(V) \subset (a, \infty)$. Because $V_1 = U \cap V$ is non-empty, semi-open set and D is dense, there exists a point $x_1 \in \operatorname{Int} V_1 \cap D$. Then $f(x_1) \in f(U \cap D)$ and $a < f(x_1)$.

Let us assume now that the condition $f(U) \cap (a, \infty) \neq \emptyset$ implies $f(U \cap D) \cap (a, \infty) \neq \emptyset$ for every open set U and dense D. On the contrary let f be not l-quasi-continuous at a point x_0. Then there exists a number $a < f(x_0)$ and a neighbourhood U of x_0 such that for every open set $U' \subset U$ we have $f(U') \not\subset (a, \infty)$. The set $D_1 = \{x \in U: f(x) \leq a\}$ is dense in U, so $D = (X \setminus U) \cup D_1$ is dense in X. But $f(U) \cap (a, \infty) \neq \emptyset$ and $f(U \cap D) \cap (a, \infty) = \emptyset$. For u-quasi--continuous function the proof is similar.

T h e o r e m 3.2. Let $f : X \to R$ be an arbitrary function. If there exists a dense set $D \subset X$ such that:

1) the function $f_{/D}$ is upper semi-continuous (lower semi-continuous),

2) for every open set $U \subset X$ the condition $f(U) \cap (-\infty, a) \neq \emptyset$ implies $f(U \cap D) \cap (-\infty, a) \neq \emptyset$,

3) for every open set $U \subset X$ the condition $f(U) \cap (a, \infty) \neq \emptyset$ implies $f(U \cap D) \cap (a, \infty) \neq \emptyset$; then f is u-quasi-continuous (l-quasi-continuous).

P r o o f . Let U be an open set, $x_0 \in U$ and $a > f(x_0)$. Let us choose a number b such that $f(x_0) < b < a$; then we have $f(U \cap D) \cap (-\infty, b) \neq \emptyset$.

Let $x_1 \in U \cap D$ and $f(x_1) < b$. There exists an open set G such that $x_1 \in G \subset U$ and $f(x') < b$ for every $x' \in G \cap D$. We shall show that $f(x) < a$ for every $x \in G$. On the contrary let there be a point x_2 such that $f(x_2) \geq a$ and $x_2 \in G$. Then $f(x_2) > b$ and $f(G \cap D) \cap (b, \infty) \neq \emptyset$ in view of the assumption 3) of the theorem. This means that there exists a point $x' \in G \cap D$ such that $f(x') > b$; it is the contradiction. Proof of the l-quasi-continuity is similar.

IV. From the definition of l-quasi-continuous and u-quasi-continuous functions there immediately follows:

R e m a r k 4.1.

1) Every lower (upper) semi-continuous function is l-quasi-continuous (u-quasi-continuous).
2) Any quasi-continuous function is l-quasi-continuous and u-quasi-continuous simultaneously.
3) Any l-quasi-continuous and upper semi-continuous (u-quasi-continuous and lower semi-continuous) function is quasi-continuous.

Simple examples show that a function which is l-quasi-continuous and u-quasi-continuous simultaneously need not be either quasi-continuous lower semi-continuous or upper semi-continuous.

T h e o r e m 4.2. Any pre-semi-open and u-quasi-continuous (l-quasi-continuous) function $f: X \to R$ is lower semi-continuous (upper semi-continuous).

P r o o f . On the contrary let us assume that f is not lower semi-continuous. Then for some number a the set $A = \{x \in X: a < f(x)\}$ is not open, thus $A \neq X$. There exists a point $x_0 \in A$ such that $x_0 \notin \operatorname{Int} A$. Let $\varepsilon > 0$ be a number such that $a < f(x_0) - \varepsilon$ and let us put $B = [f(x_0)-\varepsilon, \infty)$. The sets B and $f^{-1}(B)$ are semi-closed. Because $x_0 \in f^{-1}(B) \subset A$ we have $x_0 \in \operatorname{Fr}[f^{-1}(B)]$. The set $X \setminus f^{-1}(B)$ is semi-open, $x_0 \in \operatorname{Fr}[X \setminus f^{-1}(B)]$ so $U = [X \setminus f^{-1}(B)] \cup \{x_0\}$ is semi-open. Therefore $f(U)$ and $f(U) \cap (f(x_0)-\varepsilon, \infty)$ are semi-open sets. On the other hand $f(U) \cap (f(x_0)-\varepsilon, \infty) = [(R \setminus B) \cup \{f(x_0)\}] \cap$ $\cap (f(x_0)-\varepsilon, \infty) = \{f(x_0)\}$ and the set $\{f(x_0)\}$ has the empty interior; this contradiction finishes the proof.

From the above there follows:

C o r o l l a r y 4.3. Any pre-semi-open and u-quasi-continuous (l-quasi-continuous) function is quasi-continuous.

C o r o l l a r y 4.4. Any pre-semi-open, u-quasi-continuous and l-quasi-continuous function is continuous.

R e m a r k 4.5. The following example shows that the assumption of pre-semi-openess is essential.

The function $f : R \to R$ given by:

$$f(x) = \begin{cases} x & \text{for } x \leqslant 1 \\ x+1 & \text{for } 1 < x < 2 \\ x+2 & \text{for } 2 \leqslant x \end{cases}$$

is l-quasi-continuous, u-quasi-continuous and is not pre-semi-open. This function is neither lower semi-continuous nor upper semi-continuous.

V. Let $\{f_n: n=1,2...\}$ be a sequence of l-quasi-continuous (u-quasi-continuous) functions defined on the space X. In general the function f given by $f(x) = \lim_{n\to\infty} f_n(x)$ need not be l-quasi-continuous (u-quasi-continuous) as shows the following example.

E x a m p l e 5.1. Let $f_n : [0,1] \to R$, $f_n(x) = x^{2n-1}$ for $n=1,2,...$; all of f_n are continuous. The function f given by:

$$f(x) = \begin{cases} -1 & \text{for } x = -1 \\ 0 & \text{for } -1 < x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

is a limit of the sequence $\{f_n: n=1,2,...\}$, but it is neither l-quasi-continuous nor u-quasi-continuous.

T h e o r e m 5.2. Let $\{f : n=1,2,...\}$ be a sequence of l-quasi-continuous (u-quasi-continuous) functions. If this sequence converges to the function f uniformly, then f is l-quasi-continuous (u-quasi-continuous).

L e m m a 5.3. If $\{f_\alpha : \alpha \in \mathfrak{a}\}$ is a family of l-quasi-continuous (u-quasi-continuous) functions defined on a space X, then the function $g(x) = \sup_{\alpha\in\mathfrak{a}} f_\alpha(x)$ ($g(x) = \inf_{\alpha\in\mathfrak{a}} f_\alpha(x)$) is l-quasi-continuous (u-quasi-continuous).

C o r o l l a r y 5.4. If $\{f_n: n=1,2,...\}$ is an increasing (decreasing) sequence of l-quasi-continuous (u-quasi-continuous) functions, then the function $f(x) = \lim_{n\to\infty} f_n(x)$ is l-quasi-continuous (u-quasi-continuous).

Let Ω be the first uncountable ordinal number. The transfinite sequence $\{a_\xi : \xi < \Omega\}$ of real numbers is said to be convergent to a number $a \in R$ if for every $\varepsilon > 0$ there is an ordinal number $\mu < \Omega$ such that for every ξ, $\mu \leq \xi < \Omega$ the inequality $|a_\xi - a| < \varepsilon$ holds.

A transfinite sequence $\{f_\xi : \xi < \Omega\}$ of functions defined on the space X is called convergent to a function f, if $\{f_\xi(x) : \xi < \Omega\}$ is convergent to $f(x)$ for every $x \in X$.

Theorem 5.5. Let $\{f_\xi : \xi < \Omega\}$ be a transfinite sequence of functions defined on a separable space X, convergent to a function f. If every function f_ξ is l-quasi-continuous (u-quasi-continuous), then f is l-quasi-continuous (u-quasi-continuous).

The proof of the above facts is omitted here because it is analogous to the proofs for quasi-continuous, lower and upper semi-continuous functions, respectively (see [3,4,5]).

VI. Let X,Y be topological spaces. If f is a function defined on the product space $X \times Y$, we shall call an x-section for a given $x \in X$ the function f_x defined on Y such that $f_x(y) = f(x,y)$. The y-section f_y for a given $y \in Y$ is defined by the same way.

Theorem 6.1. Let X be a Baire space, Y a second countable space and let $f : X \times Y \to R$ be any function. If all of the sections f_x, f_y are l-quasi-continuous and f_y is u-quasi-continuous for every $y \in Y$, then f is l-quasi-continuous.

Proof: We shall prove that for every number $a \in R$ such that $\{(x,y) \in X \times Y : a < f(x,y)\} \neq \emptyset$ we have $\mathrm{Int}\{(x,y) : a < f(x,y)\} \neq \emptyset$. On the contrary, we assume that there exists a point $(x_0,y_0) \in X \times Y$ and a number $a \in R$ such that $a < f(x_0,y_0)$ and $\mathrm{Int}\{(x,y) : a < f(x,y)\} = \emptyset$. Let $a < a_1 < f(x_0,y_0)$. Since f_{y_0} is l-quasi-continuous, so $U = \mathrm{Int}\{x \in X : a_1 < f_{y_0}(x)\} \neq \emptyset$. By l-quasi-continuity of f_x and $f_x(y_0) = f(x,y_0)$ for every $x \in U$, we obtain

$G_x = \mathrm{Int}\{y \in Y : a_1 < f_x(y)\} \neq \emptyset$ for $x \in U$. Let $\{V_n : n=1,2,\ldots\}$ be a countable basis of the space Y. We denote $A_n = \{x \in U : V_n \subset G_x\}$; evidently $\bigcup_{n=1}^{\infty} A_n = U$. Let U_1 be any open set, $\emptyset \neq U_1 \subset U$. Because $\mathrm{Int}\{(x,y) \in X \times Y : a < f(x,y)\} = \emptyset$ for every n there exists a point $(x_n,y_n) \in U_1 \times V_n$ such that $f(x_n,y_n) \leq a$ and $f(x_n,y_n) < a_1$. By the u-quasi-continuity of f_{y_n} at the point x_n there exists a non-empty open set $U_n \subset U_1$ such that $f(x,y_n) < a_1$ for any $x \in U_n$. Thus $y_n \in V_n$ and $y_n \notin \{y \in Y : a_1 < f_x(y)\}$. This implies $V_n \not\subset G_x$. Hence $x \notin A_n$ for any $x \in U_n$, so $U_n \cap A_n = \emptyset$. This means that A_n is nowhere dense for $n=1,2,\ldots$, and $U = \bigcup_{n=1}^{\infty} A_n$ is the first category. It is a contradiction finishing the first part of the proof.

Now let $B \subset X$ be any open set. For every $x \in B$ we have $(f_{/B\times V_n})_x = (f_x)_{/V_n}$ where $f_{/B\times V_n}$ denotes the partial function; $f_{/B\times V_n} : B \times V_n \to R$. It is easy to see that the restriction of a l-quasi-continuous (u-quasi-continuous) function to an open subspace is l-quasi-continuous (u-quasi-continuous). Thus functions $(f_{/B\times V_n})_x$ are l-quasi-continuous for every $x \in B$ and $(f_{/B\times V_n})_y = (f_y)_{/B}$ are l-quasi-continuous and u-quasi-continuous for every $y \in V_n$. Therefore by the first part of the proof for every $a \in R$ such that $\{(x,y) \in B \times V_n : f_{/B\times V_n}(x,y) > a\} \neq \emptyset$ we have $\mathrm{Int}\{(x,y) \in B\times V_n : a < f_{/B\times V_n}(x,y)\} \neq \emptyset$. Let $p \in X \times Y$ and $a < f(p)$. For any open neighbourhood U of a point p there exists an open set $B \subset X$ such that $p \in B \times V_n \subset U$. Then we have $U' = \mathrm{Int}\{w \in B\times V_n : a < f(w)\} \neq \emptyset$ and $U' \subset U$, so f is l-quasi-continuous at the point p.

T h e o r e m 6.2. Let X be a Baire space, Y a second countable space and let $f: X\times Y \to R$ be any function. If all of the sections f_x, f_y are u-quasi-continuous and

f_y is l-quasi-continuous for every $y \in Y$, then f is u-quasi-continuous.

Proof of this theorem is similar as in Theorem 6.1.

REFERENCES

[1] S.G. Crossley, S.K. Hildebrand: Semi-closure, Tex. J. Sci. 22 (1971) 99-112.

[2] J.S. Lipiński, T. Šalát: On the points of a quasi-continuity and cliquishness of functions, Czech. Math. J. 21 (1971) 484-489.

[3] S. Łojasiewicz: Wstęp do teorii funkcji rzeczywistych, Warszawa 1976.

[4] A. Neubrunnová: On quasi-continuous and cliquish functions, Časopis Pest. Mat. 99 (1974) 109-114.

[5] A. Neubrunnová: On transfinite sequences of certain types of functions, Acta Fac. Rerum Natur. Univ. Comenian. Math. 30 (1975) 121-125.

[6] Z. Piotrowski: On semihomeomorphisms, Bolletino Un. Mat. Ital. 16-A (1979) 501-509.

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