DEMONSTRATIO MATHEMATICA
Vol. XVI No 1 1983

Paweł Olszewski

CAUCHY'S PROBLEM FOR A CERTAIN SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS OF ORDER 2p

1. The Problem

Let B be a non-empty set of real numbers and R^n_T the zone $R^n \times \langle 0,T\rangle$ $(n \geq 2,\ 0 < T < \infty)$. Let us denote by u_B a set of real functions u_β, with $\beta \in B$, and $D^{\alpha m}u_B$ the set of the derivatives $D^{\alpha m}u_\beta = \dfrac{\partial^{|\alpha|+m}u_\beta}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}\partial t^m}$ $(\beta \in B)$, where $|\alpha| = \sum_{i=1}^{n}\alpha_i$ and $m \geq 0$ is an integer. The domain of the parameters α and m will be given in the sequel.

In this paper we shall prove the existence of a solution of the following system of integro-differential equations

$$(1.1)\quad L_\beta^{(u_B)}\left[u_\beta(X,t)\right] \equiv$$

$$\equiv \sum_{j=0}^{p-1}(-1)^j\binom{p}{j}\sum_{i_1,\dots,i_k=1}^{n} A_\beta^{i_1,\dots,i_k}(X,t,D^{\alpha m}u_B(X,t))\,\frac{\partial^{k+j}u_\beta(X,t)}{\partial x_{i_1}\dots\partial x_{i_k}\partial t^j}+$$

$$+(-1)^p\frac{\partial^p u_\beta(X,t)}{\partial t^p} = F_\beta\left(X,t,D^{\nu k}u_B(X,t),\int_0^t\int_{R^n}H_B(X,t;Y,\tau,D^{\nu k}u_B(Y,\tau))\,dY\,d\tau\right)$$

$(\beta \in B,\ (X,t) \in R^n_T,\ k=2(p-j),\ 0 \leq |\alpha|+2m \leq 2p-1,\ 0 \leq |\nu|+2k \leq 2p,\ p \geq 1)$, where

$$A_\beta^{i_1 \dots i_k} = a_\beta^{i_1 i_2} \dots a_\beta^{i_{k-1} i_k},$$

with the initial conditions

$$(1.2) \qquad \lim_{t \to 0} \frac{\partial^j u_\beta(X,t)}{\partial t^j} = f_\beta^j(X) \quad (\beta \in B;\ j=0,1,\dots,p-1;\ X \in R^n).$$

Applying the potential method we shall reduce the above Cauchy problem to a system of integral equations and then prove the existence of a solution of this system basing on the Schauder-Tikhonov fixed-point theorem.

Let us note that a similar Cauchy problem for a linear differential equation of order 2p was solved by A.Borzymowski in [1] and that M. Tryjarska have examined in [8] a Cauchy problem with one homogeneous initial condition for quasi-linear parabolic system of order 2p. Systems of an arbitrary power have been already considered in connection with other kinds of problems (see [4] and [7]).

2. Assumptions

We make the following assumptions:

I. a_β^{ij} $(i,j=1,\dots,n;\ \beta \in B)$ with $a_\beta^{ij} = a_\beta^{ji}$ are bounded and continuous functions of $(X,t) \in \bar{R}_T^n$ and real $z_\gamma^{\alpha m}$ $(\gamma \in B;\ 0 \leq |\alpha| + 2m \leq 2p-1)$ such that $\sup_{\gamma \in B} |z_\gamma^{\alpha m}| < \infty$ and fulfil the Hölder inequalities*)

$$(2.1) \qquad \left|a_\beta^{ij}(X,t,z_B^{\alpha m}) - a_\beta^{ij}(\bar{X},\bar{t},\bar{z}_B^{\alpha m})\right| \leq \text{const}\left\{|X\bar{X}|^{h'} + |\bar{t}-t|^{h''} + \sup_{\gamma \in B} \sum_{|\alpha|+2m=0}^{2p-1} \left[\exp(-b'|OX|)\,\left|z_\gamma^{\alpha m} - \bar{z}_\gamma^{\alpha m}\right|\right]^{h^*}\right\}$$

where $|OX| \geq |O\bar{X}|$; h', h'', $h^* \in (0,1\rangle$ and $b' > 0$.

---

*) $z_B^{\alpha m}$ denotes the set of $z_\gamma^{\alpha m}$ with $\gamma \in B$ and $0 \leq |\alpha| + 2m \leq 2p-1$.

II. The characteristic forms $\sum_{i,j=1}^{n} a_{\beta}^{ij}(X,t,z_B^{\alpha m})\lambda_i\lambda_j$ $(\beta \in B)$ are positive-definite and the inequality

$$(2.2) \qquad \sum_{i,j=1}^{n} a_{\beta}^{ij}(X,t,z_B^{\alpha m})\,\lambda_i\lambda_j \geq c_0 \sum_{k=1}^{n} \lambda_k^2 \quad (c_0 > 0)$$

holds true.

III. The functions $f_{\beta}^{j}(X)$ $(\beta \in B;\ j=0,\dots,p-1)$ belong to the class $C^{2p-2j-1}(R^n)$ and satisfy the conditions

$$(2.3) \qquad |D^{\alpha} f_{\beta}^{j}(X)| \leq M_f \exp(b_f |OX|),$$

$$(2.4) \qquad |D^{\varrho} f_{\beta}^{j}(X) - D^{\varrho} f_{\beta}^{j}(\bar{X})| \leq \bar{M}_f \exp(b_f |OX|)\,|X\bar{X}|^{h_f}$$

where $D^{\alpha} = D^{\alpha 0}$, $0 \leq |\alpha| \leq 2p-2j-1$, $|\varrho| = 2p-2j-1$; $|OX| \geq |O\bar{X}|$; $h_f \in (0,1\rangle$, $b_f \in \langle 0,b')$, $M_f > 0$, $\bar{M}_f > 0$.

IV. The functions $H_{\gamma}(X,t;Y,\tau,z_B^{\nu k})$ $(\gamma \in B)$ are continuous for (X,t), $(Y,\tau) \in R_T^n$ and real $z_{\delta}^{\nu k}$ $(\delta \in B;\ 0 \leq |\nu| + 2k \leq 2p)$, such that $\sup_{\delta \in B} |z_{\delta}^{\nu k}| < \infty$ $(0 \leq |\nu| + 2k \leq 2p)$, and satisfy the inequalities

$$(2.5) \quad |H_{\gamma}(X,t;Y,\tau,z_B^{\nu k})| \leq M_H \tau^{-\mu_H}\Big[\exp(b_H|OX|) + \sup_{\delta \in B} \sum_{|\alpha|+2m=0}^{2p-1} |z_{\delta}^{\alpha m}|^{r_H}\Big] + m_H \sup_{\delta \in B} \sum_{|\alpha|+2m=2p} |z_{\delta}^{\alpha m}|^{r_H},$$

$$(2.6) \quad |H_{\gamma}(X,t;Y,\tau,z_B^{\nu k}) - H_{\gamma}(\bar{X},\bar{t};Y,\tau,\bar{z}_B^{\nu k})| \leq$$
$$\leq \bar{M}_H \tau^{-\mu_H}\Big[\exp(b_H|OX|)\big(|X\bar{X}|^{h_H} + |\bar{t}-t|^{\frac{1}{2}h_H}\big) + \sup_{\delta \in B} \sum_{|\alpha|+2m=0}^{2p-1} |z_{\delta}^{\alpha m} - \bar{z}_{\delta}^{\alpha m}|^{h'_H}\Big] + \bar{m}_H \sup_{\delta \in B} \sum_{|\alpha|+2m=2p} |z_{\delta}^{\alpha m} - \bar{z}_{\delta}^{\alpha m}|^{h'_H},$$

where $|OX| \geq |O\bar{X}|$, $|XY| \leq |\bar{X}Y|$, $\mu_H \in \langle 0, \frac{1}{2})$; h_H, $r_H \in (0,1\rangle$, $b_H \in \langle 0, b')$ and M_H, $\bar{M}_H$, m_H, $\bar{m}_H$ are positive constants.

V. The functions $F_\beta(X,t,z_B^{\nu k},z_B^*)$ $(\beta \in B)$ are continuous for $(X,t) \in R_T^n$ and real $z_\gamma^{\nu k}$, z_γ^* $(\gamma \in B;\ 0 \leq |\nu|+2k \leq 2p)$, such that $\sup_{\gamma \in B} |z_\gamma^{\nu k}| < \infty$ $(0 \leq |\nu|+2k \leq 2p)$, $\sup_{\gamma \in B} |z_\gamma^*| < \infty$, and fulfil the inequalities

$$(2.7)\quad \left|F_\beta(X,t,z_B^{\nu k},z_B^*)\right| \leq M_F t^{-\mu_F} \Big\{ \exp(b_F|OX|) + \sup_{\gamma \in B} \sum_{|\alpha|+2m=0}^{2p-1} \left|z_\gamma^{\alpha m}\right|^{r_F} + \sup_{\gamma \in B} \left|z_\gamma^*\right|^{r_F} \Big\} + m_F \sup_{\gamma \in B} \sum_{|\alpha|+2m=2p} \left|z_\gamma^{\alpha m}\right|,$$

$$(2.8)\quad \left|F_\beta(X,t,z_B^{\nu k},z_B^*) - F_\beta(\bar{X},\bar{t},\bar{z}_B^{\nu k},\bar{z}_B^*)\right| \leq \bar{M}_F t^{-\mu_F} \Big\{ \exp(b_F|OX|)\left(|X\bar{X}|^{h_F} + |\bar{t}-t|^{\frac{1}{2}h_F}\right) + \sup_{\gamma \in B} \sum_{|\alpha|+2m=0}^{2p-1} \left|z_\gamma^{\alpha m} - \bar{z}_\gamma^{\alpha m}\right|^{h_F} + \sup_{\gamma \in B} \left|z_\gamma^* - \bar{z}_\gamma^*\right|^{h'_F} \Big\} + \bar{m}_F \sup_{\gamma \in B} \sum_{|\alpha|+2m=2p} \left|z_\gamma^{\alpha m} - \bar{z}_\gamma^{\alpha m}\right|,$$

where $|OX| \geq |O\bar{X}|$, $t \leq \bar{t}$, $\mu_f \in \langle 0, \frac{1}{2})$; r_F, h_F, $h'_F \in (0,1\rangle$, $b_F \in \langle 0, b')$ and M_F, $\bar{M}_F$, m_F, $\bar{m}_F$ are positive constants.

R e m a r k . If $b_f = b_H = b_F = 0$ then we can set $b' = 0$ in (2.1).

3. Solution of the problem

We assume the existence of the derivatives $D^{\alpha m} u_\beta(X,t)$ $(\beta \in B,\ 0 \leq |\alpha|+2m \leq 2p-1;\ (X,t) \in R_T^n)$ satisfying the following conditions (comp. (2.1) and (2.2) in [5])

(3.1) $$\left|D^{\alpha m}u_\beta(X,t)\right| \leq R\ \exp(b|OX|),$$

(3.2) $$\left|D^{\alpha m}u_\beta(X,t) - D^{\alpha m}u_\beta(\bar{X},\bar{t})\right| \leq R\ \exp(b|OX|)(|X\bar{X}|^h + |\bar{t}-t|^{\frac{1}{2}h})$$

where $|OX| \geq |O\bar{X}|$, $h \in (0,1\rangle$, $R > 0$, $b = \max(b_f, b_H, b_F)$.

In order to prove the existence of the solutions of the problem (1.1), (1.2) let us consider the relations

(3.3) $$u_\beta(X,t) = \int_0^t \int_{R^n} \omega_\beta^{(u_B),Y,\tau}(X,t;Y,\tau)\varrho_\beta(Y,\tau)dYd\tau +$$

$$+ \sum_{i=1}^{p} \int_{R^n} v_i(X,t;Y)\xi_\beta^i(Y)dY \equiv$$

$$\equiv V_\beta^{(u_B)}(X,t) + \sum_{i=1}^{p} \varkappa_\beta^i(X,t) \qquad (\beta \in B),$$

where

1^0. $\varrho_\beta(X,t)$ are the solution of the integral equations

(3.4) $$\varrho_\beta(X,t) = G_\beta^{(u_B)}(X,t) + \int_0^t \int_{R^n} N_\beta^{(u_B)}(X,t;Y,\tau)\varrho_\beta(Y,\tau)dYd\tau$$

with

(3.5) $$N_\beta^{(u_B)}(X,t;Y,\tau) =$$

$$= \frac{1}{(2\sqrt{\pi})^n(p-1)!}\left[\det\left|a_{ij}^\beta(X,t,D^{\alpha m}u_B(X,t))\right|\right]^{\frac{1}{2}} L_\beta^{(u_B)}\left[\omega_\beta^{(u_B),Y,\tau}(X,t;Y,\tau)\right]$$

and

(3.6) $$G_\beta^{(u_B)}(X,t) =$$

$$= \frac{1}{(2\sqrt{\pi})^n(p-1)!}\left[\det\left|a_{ij}^\beta(X,t,D^{\alpha m}u_B(X,t))\right|\right]^{\frac{1}{2}} \sum_{i=1}^{p} L_\beta^{(u_B)}\left[\varkappa_\beta^i(X,t)\right] -$$

$$- F_\beta(X,t,D^{\nu k}u_B(X,t), \int_0^t \int_{R^n} H_B(X,t;Y,\tau,D^{\nu k}u_B(Y,\tau))dYd\tau)$$

where $a^{\beta}_{ij}(X,t,D^{\alpha m}u_B(X,t))$ ($\beta \in B$; $i,j=1,\ldots,n$) denote the elements of the inverse matrix of $\left[a^{ij}_{\beta}(X,t,D^{\alpha m}u_B(X,t))\right]$.

2^0. $g^j_{\beta}(X)$ ($\beta \in B$; $j=1,\ldots,p$) are defined by the formula

$$(3.7)\qquad g^j_{\beta}(X) = \frac{1}{(2\sqrt{\pi})^n(p-1)!}\sum_{i=1}^{j-1}(-1)^i\binom{j-1}{i}\Delta^i f^{j-1-i}_{\beta}(X),$$

where Δ^i is the i-th iterate of Laplace operator.

3^0. $\omega_{\beta}^{(u_B),Y,\tau}(X,t;Y,\tau)$ ($\beta \in B$) are quasi-solutions of the eqations $L_{\beta}^{(u_B)}\left[v(X,t)\right] = 0$ (comp. (2.3) and (2.5) in [5]).

4^0. $v_i(X,t;Y)$ ($i=1,\ldots,p$) are given by the formula

$$v_i(X,t;Y) = t^{-\frac{n}{2}+i-1}\exp\left(-\frac{|XY|^2}{4t}\right).$$

In virtue of the estimate (72) in [1], the singularities of the kernels (3.5) of equation (3.4) are weak.

In order to examine the system of integro-differential equations (3.3), (3.4) with the unknown functions $u_{\beta}(X,t)$, $\varrho_{\beta}(X,t)$ ($\beta \in B$) let us consider the following system of integral equations

$$(3.8)\qquad u^{\sigma l}_{\beta}(X,t) = D^{\sigma l}V_{\beta}^{(u_B)}(X,t) + \sum_{i=1}^{p} D^{\sigma l}\mathcal{H}^i_{\beta}(X,t)$$

$$(3.9)\qquad \varrho_{\beta}(X,t) = G_{\beta}^{(u_B)}(X,t) + \int_0^t\int_{R^n} N_{\beta}^{(u_B)}(X,t;Y,\tau)\,\varrho_{\beta}(Y,\tau)\,dY d\tau$$

in which the functions $u^{\nu k}_{\beta}(X,t)$, $\varrho_{\beta}(X,t)$ $(0 \le |\nu| + 2k \le 2p)$ are unknown.

We shall prove the existence of the solution of the system (3.8), (3.9) using a topological method based on the following Schauder-Tikhonov theorem (see e.g. [2], p.227):

"In a locally convex Hausdorff space, every continuous transformation of a non-empty, convex and compact set into itself possesses at least one fixed point".

Let us consider the family of spaces Λ_β ($\beta \in B$), consisting of all systems $U_\beta = \left\{ u_\beta^{\nu k}(X,t), \varrho_\beta(X,t) \right\}$ ($0 \leq |\nu| + 2k \leq 2p$) of real functions defined and continuous in $\bar{R}_T^n$ (functions $u_\beta^{\nu k}$ for $0 \leq |\nu| + 2k \leq 2p-1$) and R_T^n (functions ϱ_β and $u_\beta^{\nu k}$ for $|\nu| + 2k = 2p$) respectively, and satisfying the conditions

$$
(3.10) \begin{cases} \mathcal{N}_\beta^{(1)} := \max\limits_{0 \leq |\nu| + 2k \leq 2p-1} \sup\limits_{(X,t) \in \bar{R}_T^n} \exp(-b' |OX|) \, |u_\beta^{\nu k}(X,t)| < \infty, \\ \mathcal{N}_\beta^{(2)} := \max\limits_{|\nu| + 2k = 2p} \sup\limits_{(X,t) \in R_T^n} t^{\mu+\varepsilon} \exp(-b' |OX|) \, |u_\beta^{\nu k}(X,t)| < \infty, \\ \mathcal{N}_\beta^{(3)} := \sup\limits_{(X,t) \in R_T^n} t^{\mu+\varepsilon} \exp(-b' |OX|) \, |\varrho_\beta(X,t)| < \infty, \end{cases}
$$

where b' is the number appearing in (2.1) and the parameters μ and ε satisfy the inequalities $\max\left(\mu_F, \frac{1-\theta h_F}{2}\right) < \mu < \frac{1}{2}$, $0 < \varepsilon < \frac{1}{2} - \mu$, $\theta \in (0,1)$.

If we define in Λ_β in the usual manner linear operations, norm by the formula $\| U_\beta \| = \mathcal{N}_\beta^{(1)} + \mathcal{N}_\beta^{(2)} + \mathcal{N}_\beta^{(3)}$ and metrix of difference $U_\beta - \bar{U}_\beta$ ($U_\beta, \bar{U}_\beta \in \Lambda_\beta$), then it can be shown that the spaces Λ_β ($\beta \in B$) are linear, metric and complete (comp. [6], p.144).

Let Z_β, for each $\beta \in B$, denote the set of all points of Λ_β satisfying the conditions

$$
(3.11) \qquad \exp(-b |OX|) \, |u_\beta^{\alpha m}(X,t)| \leq R,
$$

$$
(3.12) \qquad \exp(-b |OX|) \, |u_\beta^{\alpha m}(X,t) - u_\beta^{\alpha m}(\bar{X},\bar{t})| \leq R(|X\bar{X}|^h + |\bar{t}-t|^{\frac{1}{2}h}),
$$

where $0 \leq |\alpha| + 2m \leq 2p-1$; $(X,t), (\bar{X},\bar{t}) \in \bar{R}_T^n$; $|OX| \geq |O\bar{X}|$,

$$
(3.13) \qquad t^\mu \exp(-b |OX|) \, |u_\beta^{\nu k}(X,t)| \leq R_1,
$$

$$(3.14)\quad t^{\mu}\exp(-b|0X|)\,|u_\beta^{\nu k}(X,t)-u_\beta^{\nu k}(\bar{X},\bar{t})|\le R_1\left(|X\bar{X}|^{h_1}+|\bar{t}-t|^{\frac{1}{2}h_1}\right),$$

where $|\nu|+2k=2p$; $(X,t),\in(\bar{X},\bar{t})\in R_T^n$; $|0X|\ge|0\bar{X}|$, $t<\bar{t}$,

$$(3.15)\quad t^{\mu}\exp(-b|0X|)\,|\varphi_\beta(X,t)|\le \varkappa,$$

$$(3.16)\quad t^{\mu}\exp(-b|0X|)\,|\varphi_\beta(X,t)-\varphi_\beta(\bar{X},\bar{t})|\le \varkappa_1\left(|X\bar{X}|^{h_1}+|\bar{t}-t|^{\frac{1}{2}h_1}\right),$$

where (X,t), $(\bar{X},\bar{t})\in R_T^n$; $|0X|\ge|0\bar{X}|$; $t\le\bar{t}$, $0<h_1<$ $<\min(h_0,\, h_F,\, hh'_F,\, h_H h'_F,\, (1-\theta)h_f)$ (with $h_0=\min(h',2h'',hh^*)$ and θ being the constant defined on the preceding page), b, h and μ are the constants appearing in (3.2) and (3.10) and R, R_1, $\varkappa$, $\varkappa_1$ are positive parameters.

It is easily seen that the sets Z_β $(\beta\in B)$ are non-empty and convex and their compactness can be proved by an argument similar to that in paper [6] (see p.147).

Further, we shall consider the space $\Lambda^*=\mathop{X}\limits_{\beta\in B}\Lambda_\beta$ with the Tikhonov topology (see e.g. [3], p.252). As it is known (see e.g. [2], p.31 and 116) this space is a locally convex Hausdorff space.

The set $Z^*=\mathop{X}\limits_{\beta\in B}Z_\beta$ of the elements of Λ^* is non-empty and compact as a cartesian product of non-empty, convex and compact sets.

In view of the system (3.8), (3.9), we map Z^* by the transformation

$$(3.17)\quad v_\beta^{\sigma l}(X,t)=D^{\sigma l}V_\beta^{(u_B)}(X,t)+\sum_{i=1}^{p}D^{\sigma l}\varkappa_\beta^{i}(X,t)$$

$$(3.18)\quad \varphi_\beta(X,t)-\int_0^t\int_{R^n}N_\beta^{(u_B)}(X,t;Y,\tau)\,\varphi_\beta(Y,\tau)\,dY\,d\tau=G_\beta^{(u_B)}(X,t),$$

where $\beta\in B$; $0\le|\sigma|+2l\le 2p$, $0\le|\nu|+2k\le 2p$, $0\le|\alpha|+2m\le 2p-1$.

We shall denote by $V=\mathop{X}\limits_{\beta\in B}V_\beta$ the image of an arbitrary choosen element $U=\mathop{X}\limits_{\beta\in B}U_\beta$ (where $V_\beta=\left\{v_\beta^{\nu k}(X,t),\ \varphi_\beta(X,t)\right\}$,

$U_\beta = \left\{ u_\beta^{\nu k}(X,t),\ \varphi_\beta(X,t) \right\},\ 0 \le |\nu| + 2k \le 2p)$ of Z^* and by Z' the image of the whole set Z^*.

Now we are going to obtain some auxiliary estimates.

Let θ^* be a positive number satisfying the inequality

$$\theta^* < \min\left(\mu - \mu_F,\ \mu - \frac{1-\theta h_F}{2},\ \frac{h_f - h}{2},\ \frac{1-h}{2} - \mu,\ \frac{1}{2} h_1\right).$$

By Theorems 2 and 4 in [5], the functions $v_\beta^{\alpha m}(X,t)$ $(\beta \in B;\ 0 \le |\alpha| + 2m \le 2p-1)$ satisfy the following relations

$$\left|v_\beta^{\alpha m}(X,t)\right| \le C_1\, T^{\theta^*}(z + M_f + \overline{M}_f) \exp(b\,|OX|),$$

$$\left|v_\beta^{\alpha m}(X,t) - v_\beta^{\alpha m}(\overline{X},\overline{t})\right| \le C_1\, T^{\theta^*}(z + M_f + \overline{M}_f) \exp(b\,|OX|)\left(|X\overline{X}|^h + |\overline{t} - t|^{\frac{1}{2}h}\right)$$

where $(X,t),\ (\overline{X},\overline{t}) \in \overline{R}_T^n;\ |OX| \ge |O\overline{X}|$. Making use of theorems 1 and 3 in [5] we obtain the inequalities

$$\left|v_\beta^{\nu k}(X,t)\right| \le C_2\, T^{\theta^*}(z + z_1 + M_f + \overline{M}_f) \exp(b\,|OX|),$$

$$\left|v_\beta^{\nu k}(X,t) - v_\beta^{\nu k}(\overline{X},\overline{t})\right| \le C_2\left[z + z_1 + T^{\theta^*}(M_f + \overline{M}_f)\right] t^{-\mu} \exp(b\,|OX|)\left(|X\overline{X}|^{h_1} + |\overline{t} - \overline{t}|^{\frac{1}{2}h_1}\right)$$

where $|\nu| + 2k = 2p;\ (X,t),\ (\overline{X},\overline{t}) \in R_T^n,\ |OX| \ge |O\overline{X}|,\ t \le \overline{t}$.

The corresponding conditions for $\varphi_\beta(X,t)$ $(\beta \in B)$ can be obtained in virtue of formula (3.6), assumptions IV and V, Theorem 3 in [5] and the Hölder condition

$$\left|a_\beta^{ij}(X,t,u_B^{\alpha m}(X,t)) - a^{ij}(\overline{X},\overline{t},u_B^{\alpha m}(\overline{X},\overline{t}))\right| \le$$

$$\le \text{const}\ (1 + R^{h^*})\left(|X\overline{X}|^{h_0} + |\overline{t} - t|^{\frac{1}{2}h_0}\right),$$

where $h_0 = \min(h',\ 2h'',\ hh^*)$ that results from (2.1) and (3.12).

Namely we have

$$|\varphi_\beta(X,t)| \leq C_3(m_F R_1 + T^{\theta^*}\cdot M^*)t^{-\mu}\exp(b|OX|),$$

$$|\varphi_\beta(X,t) - \varphi_\beta(\bar{X},\bar{t})| \leq C_3\Big\{\big[m_F(1+R^{h^*}) + \bar{m}_F\big]R_1 +$$

$$+T^{\theta^*}\Big[M^*(1+R^{h^*})+\bar{M}_F(1+R^{h'_F}+\bar{M}_H^{h'_F})\Big]\Big\}\ t^{-\mu}\exp(b|OX|)\Big(|X\bar{X}|^{h_1}+|\bar{t}-t|^{\frac{1}{2}h_1}\Big),$$

where $\beta \in B$; $(X,t),(\bar{X},\bar{t}) \in R^n_T$, $|OX| \geq |O\bar{X}|$, $t \leq \bar{t}$
and

$$M^* = M_f + \bar{M}_f + M_F\Big\{1 + R^{r_F} + \big[M_H(1+R^{r_H}) + m_H R_1^{r_H}\big]^{r_F}\Big\}, \tag{3.19}$$

C_1, C_2, C_3 and C_4 in the above estimates are some positive constants.

It is easily seen that if

$$C_2(\varkappa + \varkappa_1) \leq R_1,\quad C_3 m_F R_1 < \varkappa\ ,\quad C_4\big[m_F(1+R^{h^*}) + \bar{m}_F\big] < \varkappa_1$$

(it suffices to assume that m_F and $\bar{m}_F$ are small enough) and, moreover, the number T is sufficiently small, then the following system

$$\begin{aligned}
&C_1 T^{\theta^*}(\varkappa + M_f + \bar{M}_f) \leq R,\\
&C_2\big[\varkappa+\varkappa_1+T^{\theta^*}(M_F+\bar{M}_F)\big] \leq R_1,\\
&C_3(m_F R_1 + T^{\theta^*}M) \leq \varkappa\ ,\\
&C_4\Big\{\big[m_F(1+R^{h^*})+\bar{m}_F\big]R_1 + T^{\theta^*}\Big[M^*(1+R^{h^*}) + \bar{M}_F\Big(1+R^{h'_F}+\bar{M}_H^{h'_F}\Big)\Big]\Big\} \leq \varkappa_1,
\end{aligned} \tag{3.20}$$

implying the inclusion $Z' \subset Z^*$, is valid.

By an argument analogous to that in [4] and [7] one can prove that the transformation (3.17), (3.18) is continuous.

Taking into account the results obtained in the present section we can assert that all the assumptions of Schauder-Tikhonov theorem are satisfied and therefore there exists in Z^* at least one element $\tilde{U} = \underset{\beta \in B}{X} \tilde{U}_\beta$, where $\tilde{U}_\beta = \left\{ \tilde{u}_\beta^{\nu k}(X,t),\ \tilde{\varphi}_\beta(X,t) \right\}$ $(0 \leqslant |\nu| + 2k \leqslant 2p)$, being a fixed point of the transformation (3.17), (3.18). Hence, the functions $\tilde{u}_\beta^{\nu k}(X,t)$, $\tilde{\varphi}_\beta(X,t)$ $(\beta \in B;\ 0 \leqslant |\nu| + 2k \leqslant 2p)$ satisfy the system of integral equations (3.8), (3.9). It results from the properties of the integrals appearing in (3.3) (see [5]) that the functions $\tilde{u}_\beta^{\varrho o}(X,t)$, $\tilde{\varphi}_\beta(X,t)$ $(\beta \in B;\ |\varrho| = 0)$ satisfy the system of integro-differential equations (3.3), (3.4). It is easily seen that in virtue of Theorem 2 in [5] (comp. (3.10)) and formula (3.7) in the present paper, these functions fulfil (see (3.4)) the initial conditions (1.2) and satisfy the system (1.1).

Thus, we have proved the following theorem.

T h e o r e m . If assumptions I - V are satisfied and if the coefficients m_F, $\bar{m}_F$ and the number T are sufficiently small, so that the system (3.20) is valid, then there exists at least one solution $u_B(X,t)$ of the Cauchy problem (1.1), (1.2).

BIBLIOGRAPHY

[1] A. B o r z y m o w s k i : Some partial differential equations of even order, Demonstratio Math. 2 (1970) 5-85.

[2] R. E d w a r d s : Funkcjonal'nyj analiz, Moskwa 1969.

[3] K. K u r a t o w s k i : Wstęp do teorii mnogości i topologii, Warszawa 1966.

[4] W. L e k s i ń s k i , W. Ż a k o w s k i : Investigation of certain system of singular integral equations of arbitrary power, Ann. Polon. Math. 29 (1974) 373-378.

[5] P. O l s z e w s k i : Some properties of generalized thermal potentials related to a certain parabolic equation of order 2p, Demonstratio Math. 15 (1982) 833-850.

[6] W. P o g o r z e l s k i : Propriétés des intégrales généralisées de Poisson-Weierstrass et problème de Cauchy pour un système parabolique, Ann. Sci. l'Ecole Norm. Sup. 16 (1959) 125-149.

[7] W. S z n i t k o : The non-linear Hilbert problem for a system of arbitrary power, Demonstratio Math. 7 (1974) 257-277.

[8] M. T r y j a r s k a : Problème aux limites pour un système parabolique d'équations quasi-linéaires, Zeszyty Nauk. Politech. Warszaw. 141 (1966) 173-187.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
Received May 13, 1981.