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SOME THEOREMS ON GENERALIZED MANIFOLDS

The generalized manifolds have been studied by Mishra and Kushwaha [3]. Upadhyay and Kumar [4] studied the generalized Hermitian spaces. The present paper is devoted to the study of generalized manifolds in almost Kählerian, quasi-Kählerian, semi-Kählerian, nearly Kählerian and Hermitian manifolds.

1. Preliminaries

Let M be a C^∞ real differentiable generalized Riemannian manifold of dimension n , $\mathcal{F}(M)$ the ring of real valued differentiable functions on M and $H(M)$ the module of derivations of $\mathcal{F}(M)$. Then $H(M)$ is a Lie algebra over real numbers and the elements of $H(M)$ are vector fields.

Every Riemannian metric g associated with the generalized Riemannian manifold M defines an inner product in $H(M)$, which we write as $g(X, Y)$ for $X, Y \in H(M)$. Let $\underline{g}$ denote the symmetric part of $g(X, Y)$ for $X, Y \in H(M)$. Further we define $\bar{g}(X, Y)$ such that its j^{th} and k^{th} components $\bar{g}^{jk} = \bar{g}(dx^j, dx^k)$ are related to the i^{th} and j^{th} components $\underline{g}_{ij} = \underline{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ of $\underline{g}(X, Y)$ by the relation

$$(1.1) \quad \underline{g}_{ij} \bar{g}^{jk} = \delta_i^k,$$

corresponding to each vector field $X \in H(M)$. We define the covector X_C of X by the relation $\langle Y, X_C \rangle = \underline{g}(X, Y)$, where Y is an element of $H(M)$.

An almost complex manifold is a differentiable manifold M equipped with a $(1,1)$ tensor H [H can also be regarded as $\mathcal{F}(M) \rightarrow 0$ a linear map $H:H(M) \rightarrow H(M)$] which satisfies [3]

$$(1.2) \quad H^2 = -I,$$

where I denotes the identity map.

From (1.2) it follows that the generalized manifold M is even dimensional.

D e f i n i t i o n 1.1. An almost generalized Hermitian manifold M is an almost complex manifold having the Riemannian metric $\langle \cdot, \cdot \rangle$, such that

$$(1.3) \quad \langle X, Y \rangle = \langle HX, HY \rangle$$

for all $X, Y \in H(M)$.

A g r e e m e n t 1.1. All the equations which follow, hold for vector fields belonging to $H(M)$.

To describe the geometry of an almost generalized Hermitian manifold M , it is useful to consider two special tensors. The first is a 2-form $F(X, Y)$ defined as

$$(1.4) \quad F(X, Y) = \langle HX, Y \rangle \quad \text{for all } X, Y \in H(M),$$

since F is skew-symmetric; it is in fact a differential form. The second is the Nijenhuis tensor $N(X, Y)$ which is a tensor of the type $(1,2)$ defined as

$$(1.5) \quad N(X, Y) = [HX, HY] - H[HX, Y] - H[X, HY] + H^2[X, Y],$$

which in view of (1.2) becomes

$$(1.6) \quad N(X, Y) = [HX, HY] - H[HX, Y] - H[X, HY] - [X, Y].$$

It can be easily shown that

$$(1.7) \quad N(X, Y) = -N(Y, X)$$

and

$$(1.8) \quad N(HX, Y) = N(X, HY) = -H.N(X, Y).$$

If we extend the Riemannian connection ∇_X of M to be a derivation on the tensor algebra of M , then we have [6]

$$(1.9) \quad (\nabla_X H)(Y) = \nabla_X(HY) - H\nabla_X Y,$$

$$(1.10) \quad (\nabla_X F)(Y, Z) = \langle (\nabla_X H)(Y), Z \rangle.$$

From (1.2), (1.3), (1.9) and (1.10), we obtain

$$(1.11) \quad (\nabla_X F)(Y, Z) = -(\nabla_X F)(HY, HZ)$$

and

$$(1.12) \quad (\nabla_X F)(HY, Z) = (\nabla_X F)(Y, HZ).$$

Let ∇ be the Riemannian connection determined by [1]

$$(1.13) \quad \nabla_X Y - \nabla_Y X = [X, Y]$$

and

$$(1.14) \quad X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.$$

From (1.2) and (1.6), we have

$$(1.15) \quad N(HX, HY) = -N(X, Y).$$

From (1.15), it is clear that N is hybrid in its covariant slots.

2. Generalized manifolds

In this section, we shall give some definitions and theorems relating to the generalized manifolds.

From (1.2), (1.3) and (1.4), we have

$$(2.1) \quad F(X, HY) = \langle HX, HY \rangle = \langle X, Y \rangle = -F(HX, Y),$$

$$(2.2) \quad F(HX, HY) = -\langle X, HY \rangle = \langle HX, Y \rangle = F(X, Y)$$

and

$$(2.3) \quad F(X, Y) = -F(Y, X).$$

From equations (2.2) and (2.3) it follows that F is skew-symmetric and hybrid in its covariant slots X and Y .

Generalized Kähler space has been defined by Mishra and Kushwaha [3], which satisfies

$$(2.4) \quad \nabla_X H = 0,$$

together with (1.2) and (1.3).

D e f i n i t i o n 2.1. An almost generalized Hermitian manifold is called almost generalized Kählerian if and only if $dF = 0$, where

$$(2.5) \quad dF(X, Y, Z) \stackrel{\text{def}}{=} (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y).$$

D e f i n i t i o n 2.2. An almost generalized Hermitian manifold is called nearly generalized Kählerian if and only if

$$(2.6) \quad (\nabla_X H)(Y) + (\nabla_Y H)(X) = 0$$

for all $X, Y \in H(M)$.

D e f i n i t i o n 2.3. An almost generalized Hermitian manifold is called quasi-generalized Kählerian if and only if

$$(2.7) \quad (\nabla_X H)(Y) + (\nabla_{HX} H)(HY) = 0$$

for all $X, Y \in H(M)$.

D e f i n i t i o n 2.4. An almost generalized Hermitian manifold is called semi-generalized Kählerian if and only if $\delta F = 0$, where

$$(2.8) \quad \delta F(X) = \sum_{i=1}^m \left\{ \nabla_{E_i} (F)(E_i, X) + \nabla_{HE_i} (F)(HE_i, X) \right\}$$

and $\{E_1, \dots, E_m; HE_1, \dots, HE_m\}$ is a frame field on an open subset of M .

D e f i n i t i o n 2.5. An almost generalized Hermitian manifold is called generalized Hermitian if and only if $N = 0$, where N is the Nijenhuis tensor.

T h e o r e m 2.1. An almost generalized Hermitian manifold is nearly generalized Kählerian if and only if

$$(2.9) \quad \nabla_X (HY) + \nabla_Y (HX) = H(\nabla_X Y + \nabla_Y X).$$

P r o o f . Let the almost generalized Hermitian manifold be nearly generalized Kählerian. In consequence of (1.9), we have

$$\nabla_X(HY) + \nabla_Y(HX) = (\nabla_X H)(Y) + H\nabla_X Y + (\nabla_Y H)(X) + H\nabla_Y X,$$

which in view of (2.6) yields (2.9).

The converse is obvious.

T h e o r e m 2.2. An almost generalized Hermitian manifold is quasi-generalized Kählerian if and only if

$$(2.10) \quad \nabla_X(H(Y)) + \nabla_{HX}(H(HY)) = H(\nabla_X Y + \nabla_{HX} HY).$$

P r o o f . Let the almost generalized Hermitian manifold be quasi-generalized Kählerian. In consequence of (1.9), we have

$$\nabla_X(H(Y)) + \nabla_{HX}(H(HY)) = (\nabla_X H)(Y) + H\nabla_X Y + (\nabla_{HX} H)(HY) + H\nabla_{HX} HY,$$

which in view of (2.7) yields (2.10).

The converse follows in a straightforward manner.

T h e o r e m 2.3. A quasi-generalized Kählerian manifold is generalized Kählerian if and only if

$$(2.11) \quad \nabla_{HX}(H(HY)) - H\nabla_{HX}(HY) = 0.$$

P r o o f . Let the quasi generalized Kählerian manifold be generalized Kählerian. For a quasi-generalized Kählerian manifold we have

$$(\nabla_X H)(Y) + (\nabla_{HX} H)(HY) = 0,$$

which by virtue of (2.4) reduces to

$$(\nabla_{HX} H)(HY) = 0.$$

The above equation in view of (1.9) yields (2.11).

The converse follows easily.

T h e o r e m 2.4. If the generalized manifold satisfies any two of the following conditions, it satisfies the third one also

- (i) it is nearly generalized Kählerian,
- (ii) it is quasi-generalized Kählerian,
- (iii) it is a generalized manifold satisfying

$$(2.12) \quad (\nabla_Y H)(X) = (\nabla_{HX} H)(HY).$$

P r o o f . Let us put

$$(2.13) \quad L(X,Y) \stackrel{\text{def}}{=} (\nabla_X H)(Y) + (\nabla_Y H)(X),$$

$$(2.14) \quad M(X,Y) \stackrel{\text{def}}{=} (\nabla_X H)(Y) + (\nabla_{HX} H)(HY).$$

From (2.13) and (2.14) we obtain

$$(2.15) \quad L(X,Y) - M(X,Y) = (\nabla_Y H)(X) - (\nabla_{HX} H)(HY).$$

If (i) and (ii) are satisfied, then from (2.6), (2.7) and (2.15) we get (2.12).

Similarly, the results hold for (i) and (ii) also.

3. Some results

In this section, we shall give the formula for the exterior derivative of the generalized Kähler manifold from F , the condition for H to preserve connections and we shall establish some theorems.

T h e o r e m 3.1. For $X, Y, Z \in H(M)$, we have

$$(3.1) \quad dF(X,Y,Z) + dF(X,HY,HZ) + dF(Y,HZ,HX) + dF(Z,HX,HY) = \\ = 2 \left\{ (\nabla_{HX} F)(HY,Z) + (\nabla_{HY} F)(HZ,X) + (\nabla_{HZ} F)(HX,Y) \right\}.$$

P r o o f . The proof follows by virtue of the equations (1.11), (1.12) and (2.5).

L e m m a 3.1. For $X, Y \in H(M)$, we have

$$(3.2) \quad N(X,Y) = (\nabla_{HX} H)(Y) - H(\nabla_X H)(Y) - (\nabla_{HY} H)(X) + H(\nabla_Y H)(X).$$

P r o o f . The proof follows by virtue of the equations (1.2), (1.6), (1.9) and (1.13).

Theorem 3.2. The manifold M is generalized Hermitian if and only if

$$(3.3) \quad (\nabla_{HX}H)(Y) - (\nabla_{HY}H)(X) = H\left\{(\nabla_XH)(Y) - (\nabla_YH)(X)\right\},$$

for all $X, Y \in H(M)$.

Proof. The proof follows by virtue of the equation (3.3) and Definition 2.5.

Definition 3.1. H is said to be connection preserving if and only if

$$(3.4) \quad \nabla_{HX}HY = \nabla_XY$$

for all $X, Y \in H(M)$.

Theorem 3.3. A quasi-generalized Kählerian manifold in which H is connection preserving, is generalized Kählerian.

Proof. Let the generalized manifold be quasi-generalized Kählerian in which H is connection preserving, then from (1.9), (2.7) and (3.4), we obtain

$$\begin{aligned} (\nabla_XH)(Y) &= -(\nabla_{HX}H)(HY) = -\left\{\nabla_{HX}(H(HY)) - H\nabla_{HX}HY\right\} = \\ &= -\left\{\nabla_X(HY) - H\nabla_XY\right\} = -(\nabla_XH)(Y). \end{aligned}$$

This implies that

$$(\nabla_XH)(Y) = 0.$$

Hence the manifold is generalized Kählerian.

Theorem 3.4. In a generalized Kählerian manifold, the necessary and sufficient condition for H to preserve connections, is that

$$(3.5) \quad \nabla_{HX}Y = -H\nabla_XY.$$

Proof. Since the manifold is generalized Kählerian, therefore from (1.9) and (2.4), we have

$$(3.6) \quad \nabla_X(HY) - H\nabla_XY = 0.$$

Let (3.5) be satisfied. Then from (1.2) and (3.6), we obtain

$$\nabla_{HX}(HY) = -H\nabla_X(HY) = -H^2\nabla_XY = \nabla_XY.$$

Thus H is connection preserving.

Conversely, let H be connection preserving, then (3.4) is satisfied. Therefore

$$(3.7) \quad \nabla_{HX}(H(HY)) = \nabla_X(HY).$$

Hence from (1.2), (3.6) and (3.7), we have

$$\nabla_{HX}Y = -H\nabla_XY.$$

This proves Theorem.

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