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ON STRICTLY POSITIVE MEASURES AND STRICTLY CONVEX NORMS

Given a compact Hausdorff space K, $C(K)$ denotes the Banach space of real-valued continuous functions on K with the usual sup norm, and denote by $M(K)$ the space of all regular finite real-valued Borel measures on K. A measure $\mu \in M(K)$ is called strictly positive if $\mu(G) > 0$ for all non-empty open $G \subset K$. We say that K carries a strictly positive measure if there is a strictly positive measure $\mu \in M(K)$.

Let $(X, \|\cdot\|)$ be a normed space. We say that the norm $\|\cdot\|$ is strictly convex provided $\|x+y\| < 2$ whenever $\|x\| = \|y\| = 1$ and $x \neq y$.

It is well known, [1], that if a compact Hausdorff space K carries a strictly positive measure, then $C(K)$ admits an equivalent strictly convex norm. This is simply the norm given by

$$(1)\quad \|f\| = \sup_{x\in K} |f(x)| + \left(\int_K |f(x)|^2 \mu(dx)\right)^{1/2}, \quad f \in C(K),$$

where μ is strictly positive measure on K.

S. Negropontis posed the problem whether the converse of this statement is true in a class of extremally disconnected compact Hausdorff spaces. The norm defined by (1) is of course a lattice norm (i.e. $|f| \le |g|$ implies $\|f\| \le \|g\|$ for $f, g \in C(K)$). The aim of this note is to give a partial answer to the problem of Negropontis by adding the assumption that an equivalent strictly convex norm on $C(K)$ is a lattice norm and dropping the assumption of extremal disconnectivity of K. Namely we shall prove the following

T h e o r e m . Let K be a compact Hausdorff space such that $C(K)$ admits an equivalent strictly convex lattice norm. Then K carries a strictly positive measure.

P r o o f . It is well known, [3]; §26, that if $\|\cdot\|$ is a strictly convex norm in X, then for each $x \in X$, $x \neq 0$ there exists a functional $x^* \in X^*$ (topological dual of X) exposing x, i.e. such that $x^*(x) > x^*(y)$ for every $y \in X$, $y \neq x$ and $\|y\| \leq \|x\|$.

Let $\|\cdot\|$ be a strictly convex lattice norm on $C(K)$ equivalent to the sup norm. Let us consider the constant function $f \in C(K)$ defined as

$$f(x) = \|1\|, \quad x \in K,$$

where 1 denotes constant function on K equal to 1. Let $\mu \in C(K)^*$ be the functional exposing f. We shall prove that the measure $\mu \in M(K)$ (we identify $C(K)^*$ with $M(K)$ by the Riesz representation theorem) is strictly positive.

Suppose to the contrary that G is non-empty open subset of K with $\mu(G) = 0$. Since every compact space is completely regular we can find a positive function $g \in C(K)$ so that $g \leq f$ and

$$\mu(g) = \int_K g(x)\mu(dx) = 0$$

(this follows easily from the very definition of a completely regular space; [2], p.117).

Since μ exposes f we have that

$$\mu(f) > \mu\left(\frac{\|f\|}{\|f-g\|}(f-g)\right) = \frac{\|f\|}{\|f-g\|}\mu(f).$$

But this is impossible because the norm $\|\ \|$ is a lattice norm and so $\|f-g\| \leq \|f\|$.

Other results on the existence of strictly positive measures are given in the paper [4].

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