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Jan Straburzyński

EXISTENCE OF SOLUTIONS OF THE GOURSAT PROBLEM FOR SOME FUNCTIONAL-DIFFERENTIAL EQUATIONS

0. Introduction

The present paper deals with the question of the existence of solutions of the Goursat problem for the equation

$$\begin{cases} z(x,y) = \varphi(x,y) & \text{for } (x,y) \in G \\ z''_{xy} = f(x,y,z,z'_x,z'_y) & \text{for almost all } (x,y) \in \bar{D} \end{cases}$$

by assumption that f satisfies Carathéodory and some Volterra type conditions and φ, G, $\bar{D}$ are defined in [1].

The results presented here are some generalizations of those obtained in paper [5] for functional-differential equations

$$\begin{cases} z(x,y) = \varphi(x,y) & \text{for } (x,y) \in G \\ z''_{xy}(x,y) = f(x,y,A_1(x,y,z),A_2(x,y,z'_x(x,\cdot)), \\ \qquad A_3(x,y,z'_y(\cdot,y)) & \text{for almost all } (x,y) \in D \end{cases}$$

with initial-boundary conditions of Darboux type.

We will use definitions and notations introduced in [5].

1. Notations, assumptions and lemmas

For given positive numbers α, β, a, b and non-decreasing functions $y = g(x)$, $x = h(y)$ of C' class defined on $[0,a]$, $[0,b]$ and such that $g(0) = h(0) = 0$, $0 \leq g(x) \leq b$, $0 \leq h(y) \leq a$ let $P = [-\alpha,a] \times [-\beta,b]$, $D = \{(s,t) : h(t) < s \leq a, g(s) < t \leq b\}$, $D_{xy} = \{(s,t) : h(t) < s \leq x, g(s) < t \leq y$ for $x \in [0,a]$ and $y \in [0,b]\}$.

Furthermore let $G = P \setminus D$, $\bar{D}_x = \{y \in [g(x),b] : (x,y) \in \bar{D}\}$ and $\bar{D}_y = \{x \in [h(y),a] : (x,y) \in \bar{D}\}$.

Let us denote by R^n an n-dimensional Euclidean space with the norm $\|x\| = \max(|x_1|,\dots,|x_n|)$ and by $C_0(P)$ the space of all continuous functions $u : P \to R^n$ with the norm $\|u\|_0 = \max_P (\|u(x,y)\|)$. By $C_1(P)$ we will mean the space of equivalence classes of all functions $v : P \to R^n$, such that the function

$$v(\cdot,y) : [-\alpha,a] \ni x \to v(x,y) \in R^n$$

is continuous for almost all $y \in [-\beta,b]$ and

$$v(x,\cdot) : [-\beta,b] \ni y \to v(x,y) \in R^n$$

is measurable for $x \in [-\alpha,a]$ and such that

$$\|v\|_1 = \int_{-\beta}^{b} \max\{\|v(x,y)\| : x \in [-\alpha,a]\}\, dy < \infty .$$

Similarly, by $C_2(P)$ we denote the space of equivalence classes of all functions $w : P \to R^n$ such that the function

$$w(\cdot,y) : [-\alpha,a] \ni x \to w(x,y) \in R^n$$

is measurable and

$$w(x,\cdot) : [-\beta,b] \ni y \to w(x,y) \in R^n$$

is continuous for almost all $x \in [-\alpha, a]$ and such that

$$\|w\|_2 = \int_{-\alpha}^{a} \max\left\{\|w(x,y)\| \ : \ y \in [-\beta, b]\right\} dx < \infty .$$

As in [1] we can verify that $(C_1(P), \|\cdot\|_1)$ and $(C_2(P), \|\cdot\|_2)$ are Banach spaces.

The set of all absolutely continuous functions $\varphi : G \to R^n$ possessing derivatives $\varphi'_x \in C_2(G)$ and $\varphi'_y \in C_1(G)$ is denoted by Φ. We introduce in Φ the norm by the formula

$$\|\varphi\|_\Phi = \sup_G \|\varphi(x,y)\| + \sup_G \|\varphi'_x(x,y)\| + \sup_G \|\varphi'_y(x,y)\| .$$

For given $\varphi \in \Phi$ let

$$C_0^\varphi(P) = \left\{ u \in C_0(P) \ : \ u(x,y) = \varphi(x,y) \text{ for } (x,y) \in G \right\}.$$

Similarly, we define the spaces $C_1^{\varphi'_y}(P)$ and $C_2^{\varphi'_x}(P)$. Furthermore by C_x and C_y we will denote Banach spaces of all continuous vector-valued functions on $[-\beta, b]$ and $[-\alpha, a]$, respectively, with supremum norms $|\cdot|_1$ and $|\cdot|_2$.

As usual we shall say that $f: \bar{D} \times C_0^\varphi \times C_x \times C_y \to R^n$ satisfies the Carathéodory type conditions if

(i) $f(\cdot,\cdot,z,p,q) : \bar{D} \to R^n$ is measurable for fixed $(u,v,w) \in C_0^\varphi \times C_x \times C_y$

(ii) $f(x,y,\cdot,\cdot,\cdot) : C_0^\varphi \times C_x \times C_y \to R^n$ is continuous for fixed $(x,y) \in \bar{D}$

(iii) there exists a Lebesgue integrable function $m: \bar{D} \to R$ such that

$$\|f(x,y,z,p,q)\| \le m(x,y)$$

for $(x,y) \in \bar{D}$ and $(z,p,q) \in C_0^\varphi \times C_x \times C_y$.

Furthermore, we shall say that $f : \bar{D} \times C_0^\varphi \times C_x \times C_y \to R^n$ has the property of Volterra if for $(x,y) \in P$, $z_1, z_2 \in C_0^\varphi(P)$,

$p_1, p_2 \in C_x$ and $q_1, q_2 \in C_y$ such that $z_1(s,t) = z_2(s,t)$, $p_1(t) = p_2(t)$ and $q_1(s) = q_2(s)$ for $(s,t) \in P(x,y)$ it follows that

$$f(x,y,z_1,p_1,q_1) = f(x,y,z_2,p_2,q_2)$$

where $P(x,y) = [-\alpha,x] \times [-\beta,y]$ for $(x,y) \in \overline{D}$.

By F we shall denote the set of all functions $f : \overline{D} \times C_0^{\varphi} \times C_x \times C_y \to R^n$ satisfying the Carathéodory and Volterra conditions with the equivalence relation $\sim$ defined by

$$(f_1 \sim f_2) \Leftrightarrow f_1(x,y,z,p,q) = f_2(x,y,z,p,q)$$

for almost all $(x,y) \in \overline{D}$.

Let us introduce in F a metric ϱ defined by

$$\varrho_F(f_1,f_2) = \|f_1 - f_2\|_F$$

where $\|f\|_F = \iint_{\overline{D}} \sup\left\{\|f(x,y,z,p,q)\| : (z,p,q) \in C_0^{\varphi} \times C_x \times C_y\right\} dxdy$ for $f, f_1, f_2 \in F$.

We call the sequence $(w_n) \in C_1^{\varphi'_y}(P)$ almost uniformly bounded on each $[-\beta,b]$, if for every $\varepsilon > 0$ there is a set $\Omega_\varepsilon \subset [-\beta,b]$ and a constant $K_\varepsilon > 0$ such that $|[-\beta,b] \setminus \Omega_\varepsilon| < \varepsilon$ and $\|w_n(x,y)\| \leq K_\varepsilon$ for all $y \in \Omega_\varepsilon$, $n = 1,2,\ldots$

We introduce the following two uniqueness assumptions:

<u>Assumption E_1</u>. If $\varphi \in \Phi$, $u \in C_0^{\varphi}(P)$ and $(w_n) \in C_1^{\varphi'_y}(P)$ is a sequence almost uniformly bounded on $[-\beta,b]$ for all $x \in [0,a]$ then the equation

$$(1.1)\quad v(y) = \begin{cases} \varphi'_x(x,y) & \text{for } y \in [-\beta, g(x)] \\ \varphi'_x(x,g(x)) + \lim\limits_{n\to\infty} \displaystyle\int_{g(x)} f(x,t,u,v,w_n(\cdot,t))dt & \text{for } y \in \overline{D}_x \end{cases}$$

has for almost all fixed $x \in [0,a]$ at most one solution $v(y)$ continuous on $\bar{D}_x$. Moreover, the exceptional null set $N_1 \subset [0,a]$ is independent of u and (w_n).

Assumption E_2. If $\varphi \in \Phi$, $u \in C_0^{\varphi}(P)$, $v \in C_2^{\varphi'_x}(P)$, then the equation

$$(1.2)\quad w(x) = \begin{cases} \varphi'_x(x,y) & \text{for } x \in [-\alpha, h(y)] \\ \varphi'_y(h(y),y) + \int\limits_{h(y)}^{x} f(s,t,u,v(s,\cdot),w)ds & \text{for } x \in \bar{D}_y \end{cases}$$

has for almost all fixed $y \in [0,b]$ at most one solution $w(x)$ continuous on $\bar{D}_y$ and the exceptional null set $N_2 \subset [0,b]$ is independent of u and v.

The object of our investigation is the functional-differential equation of the form

$$(I)\quad \begin{cases} z(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in G \text{ and } \varphi \in \Phi \\ z''_{xy}(x,y) = f(x,y,z,z'_x(x,\cdot),z'_y(\cdot,y)) \\ \qquad \text{for almost all } (x,y) \in \bar{D} \text{ and } f \in F. \end{cases}$$

By the solution of (I) we mean a function $z : P \to R^n$ absolutely continuous possessing derivatives z'_x, z'_y and z''_{xy} almost everywhere on $\bar{D}$ and satisfying (I).

In a similar manner as in paper [6] we can prove the following Lemmas:

Lemma 1. Suppose $f \in F$, $\varphi \in \Phi$ and (E_1) is satisfied. If $u \in C_0^{\varphi}(P)$ and $w \in C_1^{\varphi'_y}(P)$, then the equation

$$(1.3)\quad v(x,y) = \begin{cases} \varphi'_x(x,y) & \text{for } (x,y) \in G \\ \varphi'_x(x,g(x)) + \int\limits_{g(x)}^{y} f(x,t,u,v(x,\cdot),w(\cdot,t))dt \\ \qquad \text{for almost all } x \in [0,a] \text{ and } y \in \bar{D}_x \end{cases}$$

has exactly one solution $v \in C_2^{\varphi'_x}(P)$.

Lemma 2. Let $f \in F$, $\varphi \in \Phi$ and (E_2) be satisfied. If $u \in C_0^{\varphi}(P)$ and $v \in C_2^{\varphi'_x}(P)$, then the equation

$$(1.4)\qquad w(x,y) = \begin{cases} \varphi'_y(x,y) & \text{for } (x,y) \in G \\ \varphi'_y(h(y),y) + \int\limits_{h(y)}^{x} f(x,y,u,v(s,\cdot),w(\cdot,y))ds & \\ \text{for } x \in \bar{D}_y \text{ and almost all } y \in [0,b], & \end{cases}$$

has exactly one solution $w \in C_1^{\varphi'_y}(P)$.

By Lemmas 1 and 2 it follows that there exist mappings

$$\psi : C_0^{\varphi}(P) \times C_1^{\varphi'_y}(P) \ni (u,w) \longrightarrow \psi(u,w) = v \in C_2^{\varphi'_x}(P)$$

$$\Gamma : C_0^{\varphi}(P) \times C_2^{\varphi'_x}(P) \ni (u,v) \longrightarrow \Gamma(u,v) = w \in C_1^{\varphi'_y}(P).$$

Moreover, let us define the mapping $T : L_n(\bar{D}) \longrightarrow C_0^{\varphi}(P)$ by

$$(1.5)\qquad T(g)(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G \\ \lambda(x,y) + \iint\limits_{\bar{D}_{xy}} g(s,t)ds\,dt & \text{for } (x,y) \in \bar{D}, \end{cases}$$

where $\varphi \in \Phi$ and

$$(1.5.1)\qquad \lambda(x,y) = \varphi(0,0) + \int\limits_0^x \varphi'_x(s,g(s))ds + \int\limits_0^y \varphi'_y(h(t),t)dt.$$

We put now

$$K(g) = \psi\left[T(g),\ T'_y(g)\right] \quad \text{for} \quad g \in L_n(\bar{D})$$

and

$$\mathcal{H}(g) = \Gamma\left[T(g),\ K(g)\right] \quad \text{for} \quad g \in L_n(\bar{D}).$$

Let $H_\psi = \left\{ g \in L_n(\bar{D}) : \|g(x,y)\| \leq \psi(x,y) \right\}$ for $x \in \bar{D}_y$, $y \in [o,b]$ where $\psi \in L(\bar{D})$.

Similarly as in paper [6] we can prove that $T(H_\psi)$ is compact in $C_0(P)$ and T is continuous on H_ψ. Moreover $K(H_\psi)$, $\mathcal{H}(H_\psi)$ are conditionaly compact in $C_2^{\varphi'x}(P)$ and $C_1^{\varphi'y}(P)$, respectively, and $K, \mathcal{H}$ are continuous on H_ψ.

By the above notations an operation S can be defined as follows

(1.6) $$\left[S(g)\right](x,y) = f(x,y,T(g),K(g)(x,\cdot),\mathcal{H}(g)(\cdot,y))]$$

for almost all $(x,y) \in \bar{D}$.

2. Some properties of the operation S

T h e o r e m 2.1. Suppose $f \in F$, $\varphi \in \Phi$ and assumptions (E_1), (E_2) are satisfied. Then the operation S defined by (1.6) is continuous on the set H_ψ.

P r o o f . Let $g_0 \in H_\psi$ such tlat $|g_n - g_0| \to 0$ as $n \to \infty$, where $|g| = \iint_{\bar{D}} \|g(x,y)\|\, dxdy$. By the continuity of operations T, K and $\mathcal{H}$ it follows that $\|Tg_n - Tg_0\|_0 \to 0$, and

$$\|\mathcal{H} g_n - \mathcal{H} g_0\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, for every $n = 1,2,\ldots$

$$|S(g_n) - S(g_0)| =$$

$$= \iint_{\bar{D}} \|f(x,y,u_n,v_n(x,\cdot),w_n(\cdot,y)) - f(x,y,u_0,v_0(x,\cdot),w_0(\cdot,y))\|\, dxdy,$$

where $u_k = T(g_k)$, $v_k = K(g_k)$ and $w_k = \mathcal{H}(g_k)$ for $k=0,1,2,\ldots$

Hence, from the assumptions about f it follows that

$$|Sg_n - Sg_0| \to 0 \quad \text{as} \quad n \to \infty ,$$

which ends the proof.

T h e o r e m 2.2. Suppose that the assumptions of Theorem 2.1 are fulfilled. Then $S(H_\varphi)$ is conditionaly compact in the space $L_n(D)$. Then proof is analogous to the proof of Theorem 2.2 given in [5].

Now we prove that the operation S has a fixed point. We precede this by the following lemma.

L e m m a 2.1. The set $F \subset L_n(\bar{D})$ is conditionaly compact if and only if it is bounded and there exists a function $d_F : R^2 \to R_+$ such that

$$\lim_{\substack{h \to 0 \\ k \to 0}} d_F(h,k) = 0$$

and for every $f \in F$

$$\iint_{\bar{D}} \|\tilde{f}(x+h,y+k) - \tilde{f}(x,y)\| \, dxdy \leq d_F(h,k)$$

where

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{for} \quad (x,y) \subset \bar{D} \\ 0 & \text{for} \quad (x,y) \in \bar{D}^{\sim}, \ \bar{D}^{\sim} = R^2 \setminus \bar{D}. \end{cases}$$

This Lemma follows from Lemma given in [4] p.301.

T h e o r e m 2.3. Let the assumptions of theorem 2.1 be fulfilled and let H_m be defined by

$$H_m = \left\{ g \in L_n(\bar{D}) : \|g(x,y)\| \leq m \right\}$$

for $x \in \bar{D}_y$ and for almost all $y \in [0,b]$, where $m \in L(\bar{D})$, and m satisfies the inequality

$$\|h(x,y,z,p,q)\| \leq m(x,y)$$

for $(x,y,z,p,q) \in \bar{D} \times C_0^{\varphi} \times C_x \times C_y$. Then the operation S defined by (1.6) has in the set H_m at least one fixed point.

P r o o f . In virtue of Theorem 2.2 the set $S(H_m)$ is conditionaly compact in $L_n(\bar{D})$. Also, by Lemma 2.1 there exist a constant $M > 0$ and a function $d_H: R^2 \to R_+$ such that $|x_g| \leq M$ and

$$\iint_{\bar{D}} \|\tilde{x}_g(x+h,y+k) - \tilde{x}_g(x,y)\| dxdy \leq d_H(h,k)$$

for $g \in H_m$ and $(h,k) \in R^2$, where

$$\tilde{x}_g = \begin{cases} x_g = S(g) & \text{for} \quad (x,y) \in \bar{D} \\ 0 & \text{for} \quad (x,y) \in \bar{D}^{\sim}. \end{cases}$$

Moreover,

$$\lim_{\substack{h\to 0^+ \\ k\to 0^+}} d_H(h,k) = 0.$$

Let $W \subset H_m$ be the set of all functions $g \in H_m$ for which $|g| \leq M$ and

$$\iint_{\bar{D}} \|\tilde{g}(x+h,y+k) - \tilde{g}(x,y)\| dxdy \leq d_H(h,k),$$

where

$$\tilde{g}(x,y) = \begin{cases} g(x,y) & \text{for} \quad (x,y) \in \bar{D} \\ 0 & \text{for} \quad (x,y) \in R^2 \setminus \bar{D}. \end{cases}$$

On account of $S(H_m) \subset W$ we see that $W \neq \phi$.

It follows from the definition that the set W is conditionaly compact, bounded, closed and convex in the space $L_n(\bar{D})$. Furthermore $S(W) \subset S(H_m) \subset W$. Consequently, from Schauder's fixed point theorem it follows that there exists a point $g_0 \in W$ such that $S(g_0) = g_0$. Thus the proof is completed.

3. The existence of solutions for the problem (I)

We precede the proof of the existence theorem by two lemmas.

L e m m a 3.1. If $f \in F$, $\varphi \in \Phi$, then the equation (I) is equivalent to the functional-differential-integral equation

$$\text{(II)} \quad z(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G \\ \lambda(x,y) + \iint\limits_{\bar{D}_{xy}} f(s,t,z,z'_x(s,\cdot),z'_y(\cdot,t))\,ds\,dt \end{cases}$$

for $(x,y) \in \bar{D}$ where $\lambda(x,y)$ is defined by (1.5.1).

P r o o f . By the assumptions concerning the function f it follows that $f(x,y,z,z'_x(x,\cdot),z'_y(\cdot,y))$ is a Lebesgue integrable function in $\bar{D}_{xy}$.

Since

$$\iint\limits_{\bar{D}_{xy}} f(s,t,z,z'_x(s,\cdot),z'_y(\cdot,t))\,ds\,dt =$$

$$= \int_0^{h(y)} \left(\int_{g(s)}^{h^{-1}(s)} f(s,t,z,z'_x(s,\cdot),z'_y(\cdot,t))\,dt \right) ds +$$

$$+\int_{h(y)}^{x}\left(\int_{g(s)}^{y} f(s,t,z,z'_x(s,\cdot),z'_y(\cdot,t))dt\right)ds =$$

$$= \int_{0}^{g(x)}\left(\int_{h(t)}^{g^{-1}(t)} f(s,t,z,z'_x(s,\cdot),z'_y(\cdot,t))ds\right)dt +$$

$$+\int_{g(x)}^{y}\left(\int_{h(s)}^{x} f(s,t,z,z'_x(s,\cdot)z'_y(\cdot,t)ds\right)dt$$

we obtain

$$(3.1.1)\quad z'_x(x,y) = \begin{cases} \varphi'_x(x,y) & \text{for } (x,y) \in G \\ \varphi'_x(x,g(x)) + \int_{g(x)}^{y} f(x,t,z,z'_x(x,\cdot),z'_y(\cdot,t))dt \end{cases}$$

for almost all $x \in [0,a]$ and $y \in \bar{D}_x$, and

$$(3.1.2)\quad z'_y(x,y) = \begin{cases} \varphi'_y(x,y) & \text{for } (x,y) \in G \\ \varphi'_y(h(y),y) + \int_{h(y)}^{x} f(s,y,z,z'_x(s,\cdot),z'_y(\cdot,y))ds \end{cases}$$

for $x \in \bar{D}_y$ and for almost all $y \in [0,b]$.

Obviously we get

$$z''_{xy}(x,y) = f(x,y,z,z'_x(x,\cdot),z'_y(\cdot,y))$$

for almost all $(x,y) \in \bar{D}$.

L e m m a 3.2. Let $f \in F$, $\varphi \in \phi$ and let the assumptions (E_1), (E_2) be satisfied. Let Λ be the set of all fixed points of the operation S defined by (1.6), and let Z be the class of all solutions of the equation (II). Then the restriction of the operation T defined by (1.5) to the set Λ is a bijection of Λ on the set Z.

P r o o f . Let $g \in \Lambda$. Then $g = S(g)$ and by the definition of the operation S we have

$$(3.1.3)\qquad g(x,y) = f(x,y,T(g),(Kg)(x,\cdot),(\mathcal{K}g)(\cdot,y))$$

for almost all $(x,y) \in \bar{D}$, where $T(g)$ is defined by (1.5) and the functions $v = K(g)$, $w = \mathcal{K}(g)$ are the unique solutions of the equations

$$(3.1.4)\qquad v(x,y) = \begin{cases} \varphi'_x(x,y) & \text{for } (x,y) \in G \\ \varphi'_x(x,g(x)) + \int_{g(x)}^{y} f(x,t,z,z'_x(x,\cdot),z'_y(\cdot,t))dt \end{cases}$$

for almost all $x \in [0,a]$ and $y \in \bar{D}_x$

$$(3.1.5)\qquad w(x,y) = \begin{cases} \varphi'_y(x,y) & \text{for } (x,y) \in G \\ \varphi'_y(h(y),y) + \int_{h(y)}^{x} f(s,y,z,z'_x(s,\cdot),z'_y(\cdot,y))ds \end{cases}$$

for $x \in \bar{D}_y$ and for almost all $y \in [0,b]$.

Let $z = T(g)$. In virtue of (3.1.3) and (1.5) we obtain

$$(3.1.6)\qquad z_x(x,y) = \begin{cases} \varphi'_x(x,y) & \text{for } (x,y) \in G \\ \varphi_x(x,g(x)) + \int_{g(x)}^{y} f(x,t,z,z_x(x,\cdot),z'_y(\cdot,t))dt \end{cases}$$

for almost all $x \in [0,a]$ and $y \in \bar{D}_x$ and

$$(3.1.7)\quad z'_y(x,y) = \begin{cases} \varphi'_y(x,y) & \text{for} \quad (x,y) \in G \\ \varphi'_y(h(y),y) + \int\limits_{h(y)}^{x} f(s,y,z,z'_x(s,\cdot),z'_y(\cdot,y))\,ds \end{cases}$$

for $x \in \bar{D}_y$ and for almost all $y \in [0,b]$.

Thus from (3.1.5) and (3.1.7) by the uniqueness of the solutions of equation (3.1.5) and the fact that $z = T(g)$ we get $z'_y(x,y) = w(x,y)$ for $x \in \bar{D}_y$ and for almost all $y \in [0,b]$. It is obvious that $z'_y = w$ on G.

Similarly, from (3.1.4), (3.1.6) we get

$$z'_x(x,y) = v(x,y)$$

for almost all $x \in [0,a]$ and $y \in \bar{D}_x$. Moreover, $z'_x = v$ on G.

By substituting $K(g) = z'_x$, $\mathcal{H}(g) = z'_y$ into (3.1.3) and next g into (1.5) we conclude that the function $z = T(g)$ is a solution of equation (II). Consequently, the restriction of the operation T to Λ transforms Λ on the set Z. Denoting by B the restriction $T|\Lambda$ we have $B : \Lambda \to Z$. The operation B is a one-to-one mapping from Λ to Z. In fact, suppose to the contrary that for $g_1 \neq g_2$ we have

$$Bg_1 = Bg_2.$$

Then by (1.5)

$$\iint\limits_{\bar{D}_{xy}} \left[g_1(s,t) - g_2(s,t)\right] ds\,dt = 0 \quad \text{for} \quad (x,y) \in \bar{D}$$

Hence $g_1 = g_2$ for almost all $(x,y) \in \bar{D}$, which contradicts the assumption that $g_1 \neq g_2$ in $L_n(D)$.

It remains to prove that $Z \subset B(\Lambda)$. Assume that $z \in Z$ and let

$$(3.1.9) \qquad g(x,y) = f(x,y,z,z'_x(x,\cdot),z'_y(\cdot,y))$$

for almost all $(x,y) \in \bar{D}$, and

$$(3.1.10) \quad z'_x(x,y) = \begin{cases} \varphi'_x(x,y) & \text{for} \quad (x,y) \in G \\ \varphi'_x(x,g(x)) + \int\limits_{g(x)}^{y} f(x,t,z,z'_x(x,\cdot),z'_y(\cdot,t))dt \end{cases}$$

for almost all $x \in [0,a]$ and $y \in \bar{D}_x$

$$(3.1.11) \quad z'_y(x,y) = \begin{cases} \varphi'_y(x,y) & \text{for} \quad (x,y) \in G \\ \varphi'_y(x,g(x)) + \int\limits_{h(y)}^{x} f(x,t,z,z'_x(x,\cdot),z'_y(\cdot,t))dt \end{cases}$$

for $x \in \bar{D}_y$ and for almost all $y \in [0,b]$.

By (3.1.7), the definition of ψ and K and the equalities $z = T(g)$ and $z'_y = T'_y(g)$ it follows that

$$z'_x = \psi\left[T(g),T'_y(g)\right] = K(g)$$

and similarly by the definition of the operations Γ, $\mathcal{H}$ we see that $z'_y = \mathcal{H}(g)$. Consequently,

$$f(x,y,z,z'_x(x,\cdot),z'_y(\cdot,y)) = f(x,y,T(g),K(g)(x,\cdot),\mathcal{H}(g)(\cdot,y)) = g(g)$$

which completes the proof.

Now we have

T h e o r e m 3.1. If the assumptions of Lemma 3.2 are satisfied, then there exists at least one solution of the problem (I).

P r o o f . From the Theorem 2.3 it follows that the class Λ of all fixed points of the operation S is a non-empty set. Let B be the mapping defined in the proof of Lemma 3.2. Then $B(\Lambda) = Z$ and $B(\Lambda) \neq \phi$. Consequently $Z \neq \phi$.

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INSTITUTE OF MATHEMATICS AND PHYSICS, ENGINEERING COLLEGE, ZIELONA GÓRA

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