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A BANACH SPACE REFORMULATION OF THE SPECTRAL THEOREM

1. Introduction

The well-known spectral theorem for self-adjoint operators is naturally formulated in the Hilbert space framework. As every Hilbert space is also a Banach space, there arises the question whether the spectral theorem can be reformulated in such a way that it uses only the language of Banach space theory. In this note we would like to show that such a reformulation is possible. Since the notions of self-adjoint operator and of projection operator are defined in Hilbert space with respect to the inner product, to obtain a Banach space reformulation of the spectral theorem first we have to redefine these notions in Banach space without any reference to the inner product structure. We will make use of quadratic forms. Namely, we will define an abstract counterpart of quadratic form and we will characterize the special class of projection quadratic forms. Finally we will prove an equivalent formulation of the spectral theorem in the framework of Banach space theory. Our attempt to restate the spectral theorem is motivated by some problems in the foundations of quantum mechanics. Namely, it is argued that the language of axioms of quantum mechanics should not make use of the Hilbert space notions, in particular, the reference to the inner product should be avoided. The Hilbert space structure should not be an assumption but rather a consequence of the axioms. This approach may lead to some generalizations of quantum mechanics.

2. Quadratic forms and the spectral theorem

Before we define the basic notions of the paper, we recall the formulation of the spectral theorem which will be used as a starting point for our generalization. We formulate the spectral theorem in terms of quadratic forms. For simplicity, we restrict our considerations to bounded operators, although all our results can be easily generalized to unbounded operators. We will also assume that linear spaces in questions are infinite dimensional.

Let X be a Hilbert space (a complete inner product vector space). With each bounded self-adjoint operator A on X $(A^* = A)$ we can associate a real-valued function f_A defined on X by

$$(2.1) \qquad f_A(x) = (Ax,x) \quad \text{for all} \quad x \in X.$$

Here $(.,.)$ denotes the inner product in X. The function defined by (2.1) is called the quadratic form corresponding to A. We would like to define quadratic forms on X without the necessity of using the operator A and the inner product $(.,.)$. To this aim we will use the following interesting theorem of S. Kurepa.

T h e o r e m 2.1. (S.Kurepa [1]). Let X be a complex vector space and $f : X \to C$ a complex-valued function such that

(i) $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x,y \in X$,

(ii) $f(cx) = |c|^2 f(x)$ for all $c \in C$ and $x \in X$.

Under these conditions, the function

$$h(x,y) = \frac{1}{4}\,(f(x+y) - f(x-y)) + \frac{1}{4}i(f(x+iy) - f(x-iy))$$

is linear in x and antilinear in y, that is, $h(x,y)$ is a sesquilinear form on X and we have $h(x,x) = f(x)$. If $f(x)$ is real-valued, then h is hermitean.

Using Kurepa's theorem, we can characterize quadratic forms on a Hilbert space as follows.

T h e o r e m 2.2. Let X be a Hilbert space and $f : X \to R$ a continuous real valued function defined on X such that

(i) $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x,y \in X$,

(ii) $f(\omega x) = f(x)$ for all $x \in X$ and all complex numbers with $|\omega| = 1$.

Then f is a quadratic form on X, i.e. there is a self-adjoint operator A such that $f(x) = f(Ax,x)$. Conversely, every quadratic form on X satisfies conditions (i) and (ii).

P r o o f . First we shall show that (i) and (ii) above imply condition (ii) of Theorem 2.1. Putting $x = y = 0$ in (i) we obtain $f(0) = 0$. Putting $x = y$ in (i) we obtain $f(2x) = 4f(x)$. By induction, we easily infer that $f(mx) = m^2 f(x)$ for all natural m and $x \in X$. From (ii) we have $f(-x) = f(x)$, hence $f(nx) = n^2 f(x)$ for every integer n. Putting $nx = y$ we infer that $f\left(\frac{1}{n} y\right) = \frac{1}{n^2} f(y)$ for all integer $n \neq 0$ and $y \in X$. Hence $f(rx) = r^2 f(x)$ for all rational r and $x \in X$. By continuity of f, we infer that $f(ax) = a^2 f(x)$ for all real a and $x \in X$. Now if c is any complex number, then $c = \omega a$ where $|\omega| = 1$ and a is a real number, hence $f(cx) = f(\omega ax) = f(ax) = a^2 f(x) = |c|^2 f(x)$, which shows that condition (ii) of Theorem 2.1 holds. Hence by this Theorem the function $h(x,y)$ defined above is a sesquilinear form on the Hilbert space X. Since f is continuous, h is also continuous and consequently bounded. By Riesz's representation theorem, there is a bounded self-adjoint operator A acting in X such that $h(x,y) = (Ax,y)$ for all $x,y \in X$. Hence we have $f(x) = h(x,x) = (Ax,x)$, i.e. f is a quadratic form. On the other hand, it is evident that every quadratic form on X satisfies conditions (i) and (ii) of Theorem 2.2. This ends the proof of this theorem.

Now we recall the standard formulation of the spectral theorem (see, e.g., Prugovecki [3]). To each bounded self-adjoint operator A on a Hilbert space X there corresponds a projection-valued measure $m^A : B(R) \to L(X)$ such that

(2.2) $$(Ax,y) = \int_R \lambda \, d(m^A(\{\lambda\})x,y) \quad \text{for all} \quad x,y \in X.$$

Putting in (2.2) $x = y$ we obtain the spectral theorem for quadratic forms

(2.3) $$(Ax,x) = \int_R \lambda \, d(m^A(\{\lambda\})x,x) \quad \text{for all} \quad x \in X.$$

For each Borel set $E \in B(R)$, $m^A(E)$ is a projection. Let us call every quadratic form (Px,x) where P is a projection, a projection quadratic form. Let us denote $f^A_\lambda(x) = (m^A(\{\lambda\})x,x)$. Hence for each A and $\lambda \in R$, $f^A(x)$ is a projection quadratic form. We can now write formula (2.3) as follows

(2.4) $$f(x) = \int_R \lambda \, df^A_\lambda(x).$$

This shows that every quadratic form on a Hilbert space X can be expressed, by means of an integral, in terms of projection quadratic forms. In particular, if the quadratic form f acts on a finite-dimensional Hilbert space, i.e. if A is a hermitean matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then the spectral expansion (2.4) for f takes the form

(2.5) $$f(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \dots + \lambda_n f_n(x)$$

where $f_1, \dots, f_n$ are suitable projection quadratic forms. If we think of a generalization of the spectral theorem beyond Hilbert space, the formulas (2.4) and (2.5) seem to be most suitable to be generalized. In the next section we will take advantage of this formulation of the spectral theorem.

3. Quadratic functionals

Before we generalize the spectral theorem, we have to generalize the notion of a quadratic form. We would like to define a quadratic form in an abstract way without using inner product. We will look upon quadratic forms as real-valued mappings defined on a vector space X with some special properties. When X becomes a Hilbert space, our generalized quadratic form should coincide with the standard Hilbert space quadratic form. Theorem 2.2 of the previous section shows us which properties of quadratic forms distinguish them from other real-valued mappings. Hence we can introduce the following definition.

D e f i n i t i o n 3.1. Let X be a topological linear space. A real-valued function on X $f : X \to R$ is said to be a quadratic functional on X if it is continuous and satisfies the following conditions:

(i) $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x,y \in X$,

(ii) $f(\omega x) = f(x)$ for all $x \in X$ and all complex numbers $|\omega| = 1$.

It is evident that the set of all quadratic functional on X forms a linear space. On the basis of Theorem 2.2 we can state the following corollary.

C o r o l l a r y 3.2. Every quadratic functional on a Hilbert space is a quadratic form.

Hence on a Hilbert space (and also on an inner product space) the notions of quadratic functional and quadratic form coincide.

In Hilbert space, among quadratic forms we can distinguish the simplest quadratic forms - namely the projection quadratic forms. As we know from the spectral theorem, every quadratic form can be constructed from projection quadratic forms. There arises the question how to define a counterpart of the class of projection quadratic forms in the framework of quadratic functionals. We answer this question by using the notion of a projection system, useful also in quantum logic theory.

D e f i n i t i o n 3.2. Let S be a set and L a family of mappings from S into the real interval $[0,1]$. We say that L is a projection system if the following condition holds:

(i) For every sequence f_i of elements of L such that $f_i + f_j \leqslant 1$ for all $i \neq j$, there exists $g \in L$ such that $g + f_1 + f_2 + \dots = 1$.

We of course assume that L is non-empty. In the sequel we shall show that the set of all projection quadratic forms on a Hilbert space X (restricted to the unit sphere of X) is a projection system.

Observe that every projection system L is naturally partially ordered by the partial order relation between the real functions in L: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in S$. It is easy to show that $f \in L$ implies $1 - f \in L$. We also have $0 \in L$, $1 \in L$ (here 0 denotes the function taking the value 0 for all $x \in S$, similarly 1). Hence we can introduce the mapping $f' = 1 - f$ of L into L. It can be shown (see [2]) that $(L, \leqslant, ')$ is an orthomodular partially ordered set. We define an orthogonality relation in L by $f \perp g$ iff $f + g \leqslant 1$ (equivalently iff $f \leqslant g'$).

A map $m : B(R) \to L$ from the family $B(R)$ of Borel sets on the real line into L is said to be an L-valued measure if

$$(3.1) \qquad m(E_1 + E_2 + \dots) = m(E_1) + m(E_2) + \dots$$

whenever $E_i \cap E_j = \emptyset$ for $i \neq j$, and

$$(3.2) \qquad m(R) = 1.$$

It is easy to show that $E \cap F = \emptyset$ implies $m(E) \perp m(F)$.

If m is an L-valued measure, then for each fixed $u \in S$ the map

$$(3.3) \qquad m_u : E \to m(E)(u)$$

is a probability measure on $B(R)$. Assume that this probability measure is bounded, i.e. there exists a bounded Borel set E such that $m_u(E) = 1$. Then we can compute the integral of λ with respect to m_u (interpreted as the expected value of the random variable corresponding to the probability density m_u):

$$f(u) = \int_R \lambda \, dm_u = \int_R \lambda \, dm(\{\lambda\})(u). \tag{3.4}$$

In this way we define a function f from S into R. We shall say that f is generated by L. The set of all functions generated by L will be denoted by ϑ and called the set of all random variables on L or L-random variables. Observe that $L \subseteq \vartheta$.

4. Spectral theorem for quadratic functionals on Banach spaces

We now restrict our consideration to Banach spaces. Let X be a Banach space and S the unit sphere of X. Let $L(X)$ denote the set of all closed subspaces of X partially ordered by the inclusion relation. We shall say that X is an orthocomplemented Banach space if there exists a map $' : L(X) \to L(X)$ with the following properties:

$$M'' = M \quad \text{for all} \quad M \in L(X), \tag{4.1}$$

$$M \subseteq N \quad \text{implies} \quad N' \subseteq M' \quad \text{for all} \quad M, N \in L(X), \tag{4.2}$$

$$M \cap M' = 0 \quad \text{for all} \quad M \in L(X), \tag{4.3}$$

(4.4) if $N \in L(X)$ contains both M and M', then $N = X$.

We can now prove our main theorem.

T h e o r e m 4.1. Let X be an orthocomplemented Banach space and Q the set of continuous quadratic functionals defined on X. Then there exists a projection system $L \subseteq Q|S$ such that $Q|S$ is generated by L ($Q|S$ denotes the set of all functions in Q with domain restricted to S).

P r o o f . Since X is an orthocomplemented Banach space, by a theorem of Kakutani-Mackey in [4] there is an inner product $(.,.)$ defined on X such that $(X,(.,.))$ is a Hilbert space and the Hilbert space topology of X coincides with the original Banach space topology of X. Now let $f \in Q$ be a continuous quadratic functional on X. By Theorem 2.2 f is a quadratic form on X, i.e. there exists a bounded self-adjoint operator on X such that $f(x) = f(Ax,x)$ for all $x \in X$. Let L_0 be the set of all orthogonal projections on X (i.e. for each $P \in L_0$ we have $P^2 = P = P^*$), and let for each $P \in L_0$ $f_P(x) = (Px,x)$ for all x X. It is clear that f_P is a quadratic functional. We shall show that $L = \{f_P|S : P \in L_0\}$ is a projection system. In fact, let f_i be a sequence of elements of L such that $f_i + f_j \leqslant 1$ for all $i \neq j$. Let $f_i(u) = (P_i u,u)$ for all $u \in S$. If $(P_i u,u) + (P_j u,u) \leqslant 1$ for all $u \in S$, then $(P_i u,u) \leqslant ((1 - P_j)u,u)$ for all $u \in S$. This implies $P_i \leqslant 1 - P_j = P_j^{\perp}$, i.e. P_i and P_j are orthogonal. Hence $P = P_1 + P_2 + \ldots$ exists an is a projection. Let $g(u) = (P^{\perp}u,u)$ for all $u \in S$. We have $g \in L$ and $g + f_1 + f_2 + \ldots = 1$. Hence property (i) of Definition 3.2 holds. This shows that L is a projection system.

Next we shall show that L generates $Q|S$. Let $f \in Q$ and let $f(x) = (Ax,x)$ for some self-adjoint operator A. By the spectral theorem, there exists a projection valued measure m^A such that

$$(4.5) \qquad (Ax,x) = \int_R \lambda \, d(m^A(\{\lambda\})x,x)$$

for all $x \in X$. For each Borel set $E \in B(R)$, $m^A(E)$ is a projection, hence $(m^A(E)u,u)$ with $u \in S$ is a member of L, and the map $m^f : E \to (m^A(E)u,u)$ is an L-valued measure. We can write (4.5) in the form

$$f(u) = \int_R \lambda \, dm^f(\{\lambda\})(u) \tag{4.6}$$

which shows that f is generated by L. This ends the proof of Theorem 4.1.

Theorem 4.1 shows that our original aim is achieved: the spectral theorem has been formulated entirely in terms of the theory of Banach spaces, without direct reference to inner product and self-adjoint operators. We shall now give another more probabilistic formulation of this theorem.

T h e o r e m 4.2. Let X be an orthocomplemented Banach space and Q the set of all continuous quadratic functionals on X. Then there exists a doubly-indexed family of probability measures

$$\{m_{f,u}\}_{f \in Q,\ u \in S} \tag{4.7}$$

such that the following conditions hold:

(i) For each $f \in Q$ and $E \in B(R)$ the map $u \to m_{f,u}(E)$ is a quadratic functional defined on S.

(ii) The family L of all maps defined in (i) is a projection system generating Q.

(iii) We have $f(u) = \int_R \lambda \, dm_{f,u}$ for all $f \in Q$, $u \in S$.

P r o o f . By Theorem 4.1 there exists a projection system $L \subseteq Q|S$ such that $Q|S$ is generated by L. Hence for each $f \in Q$ there is an L-valued measure $E \to m_f(E)$ such that for each $f \in Q$ we have

$$f(u) = \int_R \lambda \, dm_f(\{\lambda\})(u).$$

For each $f \in Q$ and each $u \in S$ the map $m_{f,u}$ from $B(R)$ into $[0,1]$ defined by $m_{f,u}(E) = m_f(E)(u)$ is a probability measure. It is easy to verify that the family $\{m_{f,u}\}_{f \in Q, u \in S}$ satisfies the conditions of the theorem.

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