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SOME PROPERTIES OF GENERALIZED THERMAL POTENTIALS RELATED TO A CERTAIN PARABOLIC EQUATION OF ORDER 2p

1. Introduction

Let R^n_T be the zone $R^n \times (0,T\rangle$, where $n \geqslant 2$, $0 < T < \infty$ and let us define the following operator

$$(1.1)\quad L^{(u)} \equiv \sum_{j=0}^{p-1} (-1)^j \binom{p}{j} \sum_{i_1 \dots i_k = 1}^{n} A_{i_1 \dots i_k} (D^{\alpha m} u) \frac{\partial^{k+j}}{\partial x_{i_1} \dots \partial x_{i_k} \partial t^j} + (-1)^p \frac{\partial^p}{\partial t^p}$$

where

$$D^{\alpha m} = \frac{\partial^{|\alpha|+m}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \partial t^m} \quad \left(|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad 0 \leqslant |\alpha| + 2m \leqslant 2p-1\right)$$

and

$$A_{i_1 \dots i_k} = a_{i_1 i_2} \cdot \dots \cdot a_{i_{k-1} i_k} \quad (k=2(p-j)) \text{ with } a_{ij} = a_{ji}.$$

In this paper we shall examine the regular continuity of some integrals related to the equation

$$(1.2)\qquad L^{(u)}\left[u(X,t)\right] = 0.$$

We assume that a_{ij} $(i,j=1,\dots,n)$ are continuous and bounded functions of X,t and $z^{\alpha m}$ for

$$(1.3)\qquad (X,t)\in \bar{R}^n_T,\ |z^{\alpha m}|<\infty \quad (0\leq |\alpha|+2m\leq 2p-1,\ X=(x_1,\dots,x_n))$$

and satisfy the Hölder condition

$$(1.4)\qquad \left|a_{ij}(X,t,z^{\alpha m}) - a_{ij}(\bar{X},\bar{t},\bar{z}^{\alpha m})\right| \leq$$
$$\leq \text{const}\left\{|X\bar{X}|^{h'} + |t-\bar{t}|^{h''} + \right.$$
$$\left. + \sum_{|\alpha|+2m=0}^{2p-1}\left[\exp(-b|O\dot{X}|)\,|z^{\alpha m}-\bar{z}^{\alpha m}|\right]^{h^*}\right\}$$

where $|OX|\leq|O\bar{X}|$; $h', h'', h^*\in(0,1\rangle$, $b\geq 0$.

Moreover, we assume that the characteristic form $\sum_{i,j=1}^{n} a_{ij}(X,t,z^{\alpha m})\lambda_i\lambda_j$ is positive-definite in the domain (1.3) and satisfies the inequality

$$(1.5)\qquad \sum_{i,j=1}^{n} a_{ij}(X,t,z^{\alpha m})\lambda_i\lambda_j \geq C_0\sum_{k=1}^{n}\lambda_k^2 \quad (C_0>0).$$

Let us note that if the functions $a_{ij}(X,t,z^{\alpha m})$ are constant (i.e. $a_{ij}(X,t,z^{\alpha m}) = a_{ij}$ for $i,j=1,\dots,n$) then the operator (1.1) is p-th iterate of the operator

$$L_0 \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} - \frac{\partial}{\partial t}.$$

The results of this paper will be used in our next paper concerning the Cauchy problem for a certain system of integro-differential equations of even order.

2. A fundamental solution

We shall construct the fundamental solution of equation (1.1) applying the idea of W.Pogorzelski presented in paper [3] and basing on the results of A.Borzymowski obtained in paper [1].

Let $u(X,t)$ be a real function defined and possessing the derivatives $D^{\alpha m}u(X,t)$ $(0 \leq |\alpha|+2m \leq 2p-1)$ in $\bar{R}^n_t$, satisfying the conditions

$$(2.1) \qquad \left|D^{\alpha m}u(X,t)\right| \leq \text{const}\ \exp(b|OX|),$$

$$(2.2) \qquad \left|D^{\alpha m}u(X,t)-D^{\alpha m}u(X,t)\right| \leq \text{const}\ \exp(b|OX|)\ \left[|X\bar{X}|^h+|t-\bar{t}|^{\frac{1}{2}h}\right],$$

where $|OX| \leq |O\bar{X}|$, $0<h\leq 1$ and b is the constant appearing in (1.4).

Consider now the equation

$$(2.3) \qquad L^{(u)}\left[v(X,t)\right] = 0$$

and introduce the functions

$$(2.4) \qquad \vartheta^{P,\zeta}_{(u)}(X,Y) = \sum_{i,j=1}^{n} a^{ij}(P,\zeta,D^{\alpha m}u(P,\zeta))(x_i-y_i)(x_j-y_j)$$

and

$$(2.5) \qquad w^{P,\zeta}_{(u)}(X,t;Y,\tau) = (t-\tau)^{-\frac{n}{2}+p-1}\exp\left[-\frac{\vartheta^{P,\zeta}_{(u)}(X,Y)}{4(t-\tau)}\right],$$

where $X=(x_1,\dots,x_n)$ and $Y=(y_1,\dots,y_n)$ are two points of R^n, (P,ζ) is a fixed point of $\bar{R}^n_T$ and $a^{ij}(P,\zeta,D^{\alpha m}u(P,\zeta))$ $(i,j=1,\dots,n)$ denote the elements of the inverse matrix to $\left[a_{ij}(P,\zeta,D^{\alpha m}u(P,\zeta))\right]$.

From the boundedness of a_{ij} and the inequality (1.5) we deduce the following inequalities

(2.6) $$C_0|XY|^2 \leqslant \vartheta^{P,\eta}_{(u)}(X,Y) \leqslant C'_0|XY|^2 \quad (C'_0 > C_0).$$

From (2.6) it follows that the function (2.5) and its derivatives satisfy the inequalities (see [2], p.24 and [4], p.147-148 and 153)

(2.7) $$\left|D^{\nu k}\omega^{P,\eta}_{(u)}(X,t;Y,\tau)\right| \leqslant \text{const}(t-\tau)^{-\frac{n+2+|\nu|+2k-2p}{2}}\exp\left(-\frac{C|XY|^2}{4(t-\tau)}\right) \leqslant$$

$$\leqslant \text{const}(t-\tau)^{-\mu}|XY|^{-(n+2+|\nu|+2k-2p-2\mu)}\exp(-C'|XY|),$$

(2.8) $$\left|D^{\nu k}\omega^{P,\eta}_{(u)}(X,t;Y,\tau) - D^{\nu k}\,\omega^{\bar{P},\eta}_{(u)}(X,t;Y,\tau)\right| \leqslant$$

$$\leqslant \text{const}\ |P\bar{P}|^{h'_0}(t-\tau)^{-\frac{1}{2}(n+2+|\nu|+2k-2p)}\exp\left(-\frac{C|XY|^2}{4(t-\tau)}\right) \leqslant$$

$$\leqslant \text{const}\ |P\bar{P}|^{h'_0}(t-\tau)^{-\mu}|XY|^{-(n+2+|\nu|+2k-2p-2\mu)}\exp(-C'|XY|),$$

where $h'_0 = \min(h', h\cdot h^*)$, $C < C_0$, $C' > 0$ and $\mu < \min\left(1, \frac{n+|\nu|}{2} + 1 + k - p\right)$.

Let us note that if the operator (1.1) acts on the function $\omega^{Y,\tau}_{(u)}(X,t;Y,\tau)$ then the arguments $D^{\alpha m}u(Y,\tau)$ are not differentiated and hence the construction of the fundamental solution of equation (2.3) is analogous as in [1] (see sections II and IV).

The fundamental solution of equation (2.3) is of the form

(2.9) $$\Gamma_{(u)}(X,t;Y,\tau) = \omega^{Y,\tau}_{(u)}(X,t;Y,\tau) +$$

$$+ \int_\tau^t \int_{R^n} \omega^{Z,\zeta}_{(u)}(X,t;Z,\zeta)\,\Phi_{(u)}(Z,\zeta;Y,\tau)\,dZ\,d\zeta,$$

where $\Phi_{(u)}$ is a solution of the Volterra integral equation (see (71) in [1])

$$(2.10)\quad \Phi_{(u)}(X,t;Y,\tau) = \frac{1}{(2\sqrt{\pi})^n(p-1)!}\left[\det\left|a^{ij}(X,t,D^{\alpha m}u(X,t))\right|\right]^{\frac{1}{2}} \cdot\left\{L^{(u)}\left[\omega_{(u)}^{Y,\tau}(X,t;Y,\tau)\right]+\int_{\tau}^{t}\int_{R^n} L^{(u)}\left[\omega_{(u)}^{Z,\zeta}(X,t;Z,\zeta)\right]\Phi_{(u)}(Z,\zeta;Y,\tau)dZd\zeta\right\}.$$

3. The quasi-potential of spatial charge

In the present section we consider the integral

$$(3.1)\qquad V_{(u)}(X,t) = \int_{\tau}^{t}\int_{R^n} \omega_{(u)}^{Y,\tau}(X,t;Y,\tau)\,\varrho(Y,\tau)\,dY$$

assuming that (2.1) and (2.2) are fulfilled.

T h e o r e m 1. If the function $\varrho(X,t)$ is continuous in R^n_T and satisfies the conditions

$$(3.2)\qquad |\varrho(X,t)| \leqslant M_\varrho t^{-\mu_\varrho}\exp(b_\varrho|OX|),$$

$$(3.3)\qquad |\varrho(X,t)-\varrho(\bar{X},\bar{t})| \leqslant M_\varrho t^{-\mu_\varrho}\exp(b_\varrho|OX|)(|X\bar{X}|^{h_\varrho}+|t-\bar{t}|^{\frac{1}{2}h_\varrho})$$

where $|OX| \geqslant |O\bar{X}|$, $t \leqslant \bar{t}$, $0 \leqslant \mu_\varrho < 1$, $0 < h_\varrho \leqslant 1$; $M_\varrho, M'_\varrho > 0$, $b_\varrho \geqslant 0$ then the equality

$$(3.4)\quad L^{(u)}\left[V_{(u)}(X,t)\right] = -(2\sqrt{\pi})^n(p-1)!\left[\det\left|a^{ij}(X,t,D^{\alpha m}u(X,t))\right|\right]^{-\frac{1}{2}}\varrho(X,t)+ + \int_{\tau}^{t}\int_{R^n} L^{(u)}\left[\omega_{(u)}^{Y,\tau}(X,t;Y,\tau)\right]\varrho(Y,\tau)dYd\tau$$

holds in R^n_T and the derivatives $D^{\nu k}V_{(u)}(X,t)$ for $|\nu|+2k=2h$, satisfy the following estimates

$$(3.5)\qquad \left|D^{\nu k}V_{(u)}(X,t)\right| \leqslant (C_1M_\varrho+C_2M'_\varrho)t^{1-\mu-\mu_\varrho}\exp(b_\varrho\,|OX|).$$

$$(3.6)\qquad \left|D^{\nu k}V_{(u)}(X,t)-D^{\nu k}V_{(u)}(\bar{X},\bar{t})\right| \leqslant$$

$$\leqslant (C'_1M_\varrho+C'_2M'_\varrho)t^{-\mu_\varrho}\exp(b_\varrho\,|OX|)(|X\bar{X}|^{\bar{h}}+|t-\bar{t}|^{\frac{1}{2}\bar{h}})$$

where $|OX|\geqslant|O\bar{X}|$, $t\leqslant\bar{t}$, $1-\frac{1}{2}\min(h_0,h'_0)<\mu<1$, $\bar{h}=\min(h_\varrho,\theta h'_0)$ (h'_0 is the constant appearing in (2.8) while $\theta\in(0,1)$) and C_1, C_2, C'_1, C'_2 are positive constants.

P r o o f . Because the relation (3.4) is an extension of the thesis of Theorem 18 in [1] to the case when the coefficients of the equation (1.2) depend on $D^{\alpha m}u(X,t)$ and the density $\varrho(X,t)$ satisfies the inequality (3.2), thus the proof is analogous to that of Theorem 18 in [1].

In order to prove the inequalities (3.5) and (3.6) first we consider the case $0\leqslant k<p$ and decompose the derivatives $D^{6k}V_{(u)}(X,t)$, where $|6|+2k=2p-1$, as follows

$$(3.7)\qquad D^{6k}V_{(u)}(X,t)=\int_0^t\varrho(P,\tau)\int_{R^n}D^{6k}\omega^{P,\tau}_{(u)}(X,t;Y,\tau)\,dY\,d\tau+$$

$$+\int_0^t\varrho(P,\tau)\int_{R^n}\left[D^{6k}\omega^{Y,\tau}_{(u)}(X,t;Y,\tau)-D^{6k}\omega^{P,\tau}_{(u)}(X,t;Y,\tau)\right]dY\,d\tau+$$

$$+\int_0^t\int_{R^n}D^{6k}\omega^{Y,\tau}_{(u)}(X,t;Y,\tau)\left[\varrho(Y,\tau)-\varrho(P,\tau)\right]dY\,d\tau,$$

where P is an arbitrary point of R^n.

It is easy to see that the first integral in (3.7) is equal to zero. Thus, differentiating (3.7) with respect to x_i and substituting $P = X$, we obtain the formula[*)]

$$(3.9)\qquad D^{\nu k}V_{(u)}(X,t) = \int_0^t \varrho(X,t) \int_{R^n} \left[D^{\nu k}\omega^{Y,\tau}_{(u)}(X,t;Y,\tau) - D^{\nu k}\omega^{X,\tau}_{(u)}(X,t;Y,\tau)\right] dY d\tau + \int_0^t \int_{R^n} D^{\nu k}\omega^{Y,\tau}_{(u)}(X,t;Y,\tau)\left[\varrho(Y,\tau)-\varrho(X,\tau)\right] dY d\tau = I_1(X,t) + I_2(X,t).$$

Making use of assumption (3.3), inequality (2.7) (where we take $C' > b_\varrho$) and relation $|OY| \leqslant |OX| + |XY|$, we obtain the following estimate of the integral $I_2(X,t)$

$$|I_2(X,t)| \leqslant \text{const } M'_\varrho \int_0^t \tau^{-\mu_\varrho}(t-\tau)^{-\mu} d\tau \cdot \int_{R^n} |XY|^{-(n+2-2\mu-h_\varrho)} \exp(b_\varrho |OX|) \exp(-C' |XY|) dY \leqslant$$

$$\leqslant \text{const } M'_\varrho t^{1-\mu-\mu_\varrho} \exp(b_\varrho |OX|) \cdot \int_{R^n} |XY|^{-(n+2-2\mu-h_\varrho)} \exp(-(C'-b_\varrho) |XY|) dY \leqslant$$

$$\leqslant \text{const } M'_\varrho t^{1-\mu-\mu_\varrho} \exp(b''_\varrho |OX|), \qquad \text{where } 1 - \tfrac{1}{2} h_\varrho < \mu < 1.$$

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*) $D^{\nu k}$ denotes $\frac{\partial}{\partial x_i} D^{6k}$.

By a similar argument we get for $I_1(X,t)$ the inequality

$$|I_1(X,t)| \leq \text{const}\ M_\varrho t^{1-\mu-\mu_\varrho} \exp(b_\varrho |OX|),$$

where $1 - \frac{1}{2} h'_0 < \mu < 1$ and combining the above-obtained results we arrive at the estimate (3.5).

We shall prove Hölder's condition (3.6). (It is enough to consider the case $2|X\bar{X}| < r_0$, $\sqrt{\bar{t}-t} < r_0$, where r_0 denotes a fixed positive number, since in the opposite case the validity of (3.6) follows from (3.5)).

Basing on the formula (3.9) we can write

$$|I_1(X,t) - I_1(\bar{X},t)| \leq$$

$$\leq \int_0^t \varrho(X,\tau) \int_{R^n} \Big| \Big[D^{\nu k} \omega^{Y,\tau}_{(u)}(X,t;Y,\tau) - D^{\nu k} \omega^{X,\tau}_{(u)}(X,t;Y,\tau) \Big] -$$

$$- \Big[D^{\nu k} \omega^{Y,\tau}_{(u)}(\bar{X},t;Y,\tau) - D^{\nu k} \omega^{\bar{X},\tau}_{(u)}(\bar{X},t;Y,\tau) \Big] \Big| \, dY d\tau +$$

$$+ \int_0^t |\varrho(X,\tau) - \varrho(\bar{X},\tau)| \int_{R^n} \Big| D^{\nu k} \omega^{Y,\tau}_{(u)}(\bar{X},t;Y,\tau) -$$

$$- D^{\nu k} \omega^{\bar{X},\tau}_{(u)}(\bar{X},t;Y,\tau) \Big| \, dY d\tau = \bar{I}_1 + \tilde{I}_1.$$

By virtue of (2.8) and (3.3) we get for $\tilde{I}_1$ the estimate

$$\tilde{I}_1 \leq \text{const}\ M'_\varrho t^{1-\mu-\mu_\varrho} \exp(b_\varrho |OX|)\, |X\bar{X}|^{h_\varrho},$$

where $1 - \frac{1}{2} h_\varrho < \mu < 1.$

Next we break the integral $\bar{I}_1$ into three components with the integration over K_0, K_0-K and R^n-K_0, where K_0 and K denote the balls with the centers at X and the radii r_0 and $2|X\bar{X}|$ respectively. We estimate the integral over K by the sum of the appropriate integrals and in order to estimate the integrals over K_0-K and R^n-K_0 we apply the mean--value theorem to the function

$$F(X,t;Y,\tau;P) = D^{\nu k}\omega^{Y,\tau}_{(u)}(X,t;Y,\tau) - D^{\nu k}\omega^{P,\tau}_{(u)}(X,t;Y,\tau) \quad (P \in R^n)$$

and we use the relations (2.7) and (2.8). As a result we obtain

$$\bar{I}_1 \leqslant \text{const } M_\varrho t^{-\mu_\varrho}\exp(b_\varrho |OX|)\ |X\bar{X}|^{\theta h'_0}, \quad \theta\in(0,1).$$

Proceeding analogously as in the examination of the difference $|I_1(X,t)-I_1(\bar{X},t)|$ we get for $I_2(X,t)$ the inequality

$$|I_2(X,t)-I_2(\bar{X},t)| \leqslant \text{const } M'_\varrho t^{-\mu_\varrho} \exp(b_\varrho |OX|)|X\bar{X}|^{h_\varrho}.$$

The proof of the Hölder condition with respect to t is similar to that above and is based on the inequality

$$\left|D^{\nu k}V_{(u)}(X,t)-D^{\nu k}V_{(u)}(X,\bar{t})\right| \leqslant$$

$$\leqslant \int_t^{\bar{t}} \varrho(X,\tau) \int_{R^n} \left|D^{\nu k}\omega^{Y,\tau}_{(u)}(X,\bar{t};Y,\tau) - D^{\nu k}\omega^{X,\tau}_{(u)}(X,\bar{t};Y,\tau)\right| dYd\tau \, +$$

$$+ \int_t^{\bar{t}} \int_{R^n} |D^{\nu k}\omega^{Y,\tau}_{(u)}(X,\bar{t};Y,\tau)|\ |\varrho(Y,\tau)-\varrho(X,\tau)|\ dYd\tau \, +$$

$$+\int_0^t |\varrho(X,\tau)| \int_{R^n} \Big| \Big[D^{\nu k}\omega_{(u)}^{Y,\tau}(X,t;Y,\tau) - D^{\nu k}\omega_{(u)}^{X,\tau}(X,t;Y,\tau) \Big] -$$

$$- \Big[D^{\nu k}\omega_{(u)}^{Y,\tau}(X,\bar{t};Y,\tau) - D^{\nu k}\omega_{(u)}^{X,\tau}(X,\bar{t};Y,\tau) \Big] \Big| \, dY d\tau +$$

$$+ \int_0^t \int_{R^n} \Big| D^{\nu k}\omega_{(u)}^{Y,\tau}(X,t;Y,\tau) - D^{\nu k}\omega_{(u)}^{Y,\tau}(X,\bar{t};Y,\tau) \Big| \cdot |\varrho(Y,\tau) - \varrho(X,\tau)| \, dY d\tau .$$

The examination of the first two integrals is based on the inequalities (2.7), (2.8), (3.2), (3.3) and on the following decomposition $R^n = K_0 \cup (R^n - K_0)$.

The remaining integrals can be estimated similarly as the corresponding integrals in the proof of Hölder's condition with respect to X, replacing the ball K by the ball K_1 with the center at X and radius $\sqrt{\bar{t}-t}$.

As a consequence we obtain the following result

$$\Big| D^{\nu k} V_{(u)}(X,t) - D^{\nu k} V_{(u)}(X,\bar{t}) \Big| \leqslant$$

$$\leqslant \text{const}\ t^{-\mu_\varrho} \exp(b_\varrho |OX|) \Big(M_\varrho |\bar{t}-t|^{\frac{1}{2}h'_0} + M'_\varrho |\bar{t}-t|^{\frac{1}{2}h_\varrho} \Big) .$$

Thus, the proof of inequality (3.6) in the case $k < p$ is completed. The validity of the estimates (3.5) and (3.6) for $k = p$ follows from the formula (3.4), the assumptions (1.3), (2.2) and from the results proved above for $k < p$.

Note that the estimates of the integral in (3.4) do not cause any difficulty, due to the weak singularity of the integrand (comp. (72) in [1]).

T h e o r e m 2. If the density $\varrho(X,t)$ is continuous in R^n_T and the inequality (3.2) is valid with $0 \leqslant \mu_\varrho < \frac{1}{2}$, then the derivatives $D^{\nu k}V_{(u)}(X,t)$ $(0 \leqslant |\nu| + 2k \leqslant 2p-1)$ satisfy in $\bar{R}^n_T$ the following conditions

(3.10) $$|D^{\nu k}V_{(u)}(X,t)| \leqslant \text{const}\ M_\varrho t^{1-\mu-\mu_\varrho} \exp(b_\varrho |OX|),$$

(3.11) $$|D^{\nu k}V_{(u)}(X,t) - D^{\nu k}V_{(u)}(\bar{X},\bar{t})| \leqslant$$
$$\leqslant \text{const}\ M_\varrho \bar{t}^{\theta_0} \exp(b_\varrho |OX|)\left(|X\bar{X}|^{\tilde{h}} + |\bar{t}-t|^{\frac{1}{2}\tilde{h}}\right)$$

where $|OX| \geqslant |O\bar{X}|$, $\bar{t} \geqslant t$, $\frac{1}{2} < \mu < 1-\mu_\varrho$, $0 < \tilde{h} < 1-2\mu_\varrho$, $0 < \theta_0 < \frac{1-\tilde{h}}{2} - \mu$.

Let us note that inequality (3.10) is given in [4] p.197 (comp. (174)) and Hölder's condition (3.11) is a modification of the condition (175) in [4] (p.197) (both of these results concern the parabolic systems).

4. The Fourier-Poisson integral

In paper [1] there was considered the integral of the form

(4.1) $$\mathcal{K}_i(X,t) = \int_{R^n} v_i(X,t;Y)\, g(Y)dY, \quad (i=1,\dots,p),$$

where

(4.2) $$v_i(X,t;Y) = t^{-\frac{n}{2}+p-1} \exp\left(-\frac{|XY|^2}{4t}\right).$$

The integral (4.1) satisfies the p-th iterate of the heat equation.

In the present section we prove some theorems concerning the derivatives $D^{\nu k}\mathcal{K}_i(X,t)$ $(0 \leqslant |\nu| + 2k \leqslant 2p;\ i=1,\dots,p)$, analogous to those obtained in previous section for the integral (3.1).

Theorem 3. If the function $g(X)$ has in R^n continuous partial derivatives of all orders including the $(2p-2i+1)$-th which satisfy the inequality

$$(4.3)\qquad |D^\alpha g(X)| \leqslant M_g \exp(b_g |OX|),$$

where $0 \leqslant |\alpha| \leqslant 2p-2i+1$; $D^\alpha = D^{\alpha 0}$ and, besides that the derivatives of order $2p-2i+1$ satisfy in R^n the Hölder's condition of the form

$$(4.4)\qquad |D^\delta g(X) - D^\delta g(\bar{X})| \leqslant M_g' \exp(b_g |OX|)\ |X\bar{X}|^{h_g}$$

where $|\delta| = 2p-2i+1$; $|OX| \geqslant |O\bar{X}|$; $0 < h_g \leqslant 1$, $M_g > 0$, $M_g' > 0$, $b_g \geqslant 0$, then the derivatives $D^{\nu k}\mathcal{H}_i(X,t)$ $(|\nu|+2k=2p)$ of the integral (4.1) fulfil the following inequalities

$$(4.5)\qquad |D^{\nu k}\mathcal{H}_i(X,t)| \leqslant \text{const}(M_g + M_g')t^{-\mu} \exp(b_h |OX|),$$

$$(4.6)\qquad |D^{\nu k}\mathcal{H}_i(X,t) - D^{\nu k}\mathcal{H}_i(\bar{X},\bar{t})| \leqslant$$
$$\leqslant \text{const}(M_g+M_g')t^{-\frac{1-(1-\theta)h_g}{2}} \exp(b_g |OX|)\ |X\bar{X}|^{\theta h_g} + |\bar{t}-t|^{\frac{1}{2}\theta h_g}$$

where (X,t), $(\bar{X},\bar{t}) \in R_T^n$; $t \leqslant \bar{t}$, $\frac{1-h_g}{2} < \mu < 1$ and $\theta \in (0,1)$.

Proof. This theorem is a modification of Theorem 17 in [1]. Let $|\nu|+2k=2p$. We shall consider two cases: (i) $k \leqslant i-1$, and (ii) $i-1 < k \leqslant p$.

In the case (i) we make use of the following formula

$$(4.7)\qquad D^{\nu k}\mathcal{H}_i(X,t) = (i-1)!\sum_{j=0}^{k} \frac{1}{(i-1-k+j)!}\binom{k}{j} t^{i-1-k+j} D^{\nu 0}\Delta^j \mathcal{H}_1(X,t),$$

where Δ^j denotes the j-th iterate of the Laplace operator.

In the case (ii), basing on the formula (228) in [1], we obtain

$$(4.8)\qquad D^{\nu k}\mathcal{H}_1(X,t) = k!\sum_{j=0}^{k-1}\binom{i-1}{j}\frac{1}{(k-j)!}\,t^{i-1-j}D^{\nu^0}\Delta^{k-j}\mathcal{H}_1(X,t).$$

We shall estimate the expression $t^{i-1-k+j}D^{\nu^0}\Delta^j\mathcal{H}_1(X,t)$ $(j=0,\dots,k)$ appearing in (4.7) and $t^{i-1-j}D^{\nu^0}\Delta^{k-j}\mathcal{H}_1(X,t)$ $(j=0,\dots,i-1)$ in (4.8).

As we shall further see, both of these expressions have the estimates of the same order of singularity.

Let us consider now the ball K_0 introduced in p.9 and make the following decomposition

$$D^{\nu}\mathcal{H}_1(X,t) = \int_{K_0} D^{\nu}\Delta^j v_1(X,t;Y)g(Y)\,dY +$$

$$+\int_{R^n-K_0} D^{\nu}\Delta^j v_1(X,t;Y)g(Y)\,dY \qquad (|\nu| = 2p-2k).$$

Applying the Green theorem to the integral over K_0 we obtain

$$(4.9)\qquad \int_{K_0} D^{\nu}\Delta^j v_1(X,t;Y)g(Y)\,dY =$$

$$= \int_{K_0} D^{\nu^0}v_1(X,t;Y)D^{\nu^*}\Delta_Y^j g(Y)\,dY + R_1(X,t),$$

where $R_1(X,t)$ denotes a sum of certain bounded integrals over ∂K_0, $\nu^0 = (\nu_1^0,\dots,\nu_n^0)$ and $\nu^* = (\nu_1^*,\dots,\nu_n^*)$ satisfy the conditions $|\nu^0| = 2(i-1+j-k)+1$, $|\nu^*| = 2p-2i+1-2j$ and $\nu_m^0+\nu_m^* = \nu_m$ for $m=1,\dots,n$ $(\nu=(\nu_1,\dots,\nu_n))$.

Proceeding analogously as in the case (ii) we obtain

$$D^{\nu} \Delta^{k-j} v_1(X,t;Y) = \int_{K_0} D^{\nu} \Delta^{k-j} v_1(X,t;Y) g(Y) dY +$$

$$+ \int_{R^n - K_0} D^{\nu} \Delta^{k-j} v_1(X,t;Y) g(Y) dY,$$

where

$$(4.10) \qquad \int_{K_0} D^{\nu} \Delta^{k-j} v_1(X,t;Y) g(Y) dY =$$

$$= \int_{K_0} D^{\nu^0} \Delta^{i-1-j} v_1(X,t;Y) D^{\nu^*} \Delta^{k+1-j} g(Y) dY + R_2(X,t)$$

and $R_2(X,t)$ denotes a sum of the same type as in (4.9).

The indices ν^0 and ν^* satisfy the conditions $|\nu^0| = 1$, $|\nu^*| = 2p-2k+1$ and $\nu^0_m + \nu^*_m = \nu_m$ for $m=1,\dots,n$.

Basing on the estimate (see (2.7))

$$(4.11) \qquad \left| D^{\alpha m'} v_1(X,t;Y) \right| \leqslant \text{const}\ t^{-\frac{n+|\alpha|+2m}{2}} \exp\left(-\frac{\tilde{C}|XY|^2}{4t}\right) \leqslant$$

$$\leqslant \text{const}\ t^{-\mu} |XY|^{-(n+2+|\alpha|+2m-2p-2\mu)} \exp(-\tilde{C}_1 |XY|),$$

where $0 < \tilde{C} < 1$, $\tilde{C}_1 > 0$ and μ is the parameter choosen as in (2.7), we easily observe that the expression appearing in (4.7) and (4.8) has the analogous estimates as $t^1 \tilde{\mathcal{K}}(X,t)$, where

$$(4.12)\qquad \tilde{\mathcal{H}}(X,t) = \int_{K_0} D^{\eta} v_1(X,t;Y) D^{\sigma} g(Y)\, dY$$

$$(0 \leqslant l \leqslant i-1, \quad |\eta| = 2l+1, \quad |\sigma| = 2p-2i+1).$$

We shall consider only the integral taken over K_0, since the integrals appearing in $R_1(X,t)$ and $R_2(X,t)$ (see (4.9) and (4.10)) are bounded and the integrals over R^n-K_0 have the estimates of the form

$$\left| \int_{R^n-K_0} D^{\eta} v_1(X,t;Y) D^{\sigma} g(Y)\, dY \right| \leqslant \text{const}\ M_g \exp(b_g |OX|)$$

and satisfy Hölder's conditions with an arbitrary exponent from the interval $(0,1\rangle$ and the exponential coefficient of the same form as that in the right-hand side of the inequality written above.

Since the integral $\int_{K_0} D^{\eta} V_1(X,t;Y)\, dY$, where $|\eta| = 2l+1$ is equal to zero, the function (4.12) can be written in the form

$$(4.13)\qquad \tilde{\mathcal{H}}(X,t) = \int_{K_0} D^{\eta} v_1(X,t;Y) \left[D^{\sigma} g(Y) - D^{\sigma} g(X) \right] dY,$$

$$(|\eta| = 2l+1, \quad |\sigma| = 2p-2i+1).$$

In virtue of assumption (4.4), estimate (4.11) and the relations $|OY| \leqslant |OX| + |XY|$, $\exp(b_g |OX|) \leqslant$ const, satisfied for $Y \in K_0$, we have

$$|\tilde{\mathcal{H}}(X,t)| \leqslant \text{const}\ M_g' t^{-(1+\mu)} \int_{K_0} |XY|^{-(n+1-2\mu-h_g)} \exp(b_g|OY|)dY \leqslant$$

$$\leqslant \text{const}\ M_g' t^{-(1+\mu)} \exp(b_g|OX|),$$

where $\frac{1-h_g}{2} < \mu < 1$.

Making use of the formula (4.13) and the equality $K_0 = K \cup (K_0 - K)$, where K is a ball with the center at X and the radius $2|X\bar{X}| < r_0$ we can write

$$|\mathcal{H}(X,t) - \mathcal{H}(\bar{X},t)| \leqslant \int_K |D^\eta v_1(X,t;Y)|\ |D^\sigma g(Y) - D^\sigma g(X)|dY +$$

$$+ \int_K |D^\eta v_1(\bar{X},t;Y)|\,|D^\sigma g(Y) - D^\sigma g(\bar{X})|\,dY + |D^\sigma g(\bar{X}) - D^\sigma g(X)| \int_{K_0-K} |D^\eta v_1(X,t;Y)dY +$$

$$+ \int_{K_0-K} |D^\eta v_1(X,t;Y) - D^\eta v_1(\bar{X},t;Y)|\,|D^\sigma g(Y) - D^\sigma g(\bar{X})|\,dY = \tilde{\mathcal{H}}^{(1)} + \dots + \tilde{\mathcal{H}}^{(4)}.$$

In order to estimate the integral $\tilde{H}^{(1)}$ we consider the inequality (4.11) introducing polar coordinates. As a consequence we have

$$(4.14) \qquad \tilde{\mathcal{H}}^{(1)} \leqslant \text{const}\ M_g' t^{-(1+\mu)} \exp(b_g|OX|)\ |X\bar{X}|^{h_g+2\mu-1},$$

where $\frac{1-h_g}{2} < \mu < 1 - \frac{1}{2}h_g$.

In a similar way we estimate $\tilde{\mathcal{H}}^{(2)}$. The component $\tilde{\mathcal{H}}^{(3)}$ is equal to zero. The integral $\tilde{\mathcal{H}}^{(4)}$ is examined by applying the mean-value theorem, the assumption (4.4) and, subsequently, the relations $|\bar{X}Y| < \frac{3}{2}|XY|$, $|X^*Y| > \frac{1}{2}|XY|$, where X^* is a point of the interior of a sector $\overline{X\bar{X}}$, $|OY| \leqslant |OX| + |XY|$ and $\exp(b_g|OX|) \leqslant \text{const}$.

Hence we obtain the inequality of the form (4.14) with $\frac{1-h_g}{2} < \mu < \frac{1}{2}$.

Joining the results obtained above and substituting $h_g+2-1 = \theta h_g$, where $0 < \theta < 1$, we get Hölder's condition (4.6) with respect to X.

The same condition with respect to t can be proved analogously by introducing a ball with a center at X and radius $\sqrt{\bar{t}-t}$ and breaking the domain of integration in $\tilde{\mathcal{X}}(X,t)$ into two domains.

T h e o r e m 4. Under the assumptions of the previous theorem (see p.12), the derivatives $D^{\alpha m}\mathcal{X}_i(X,t)$ $(0 \leqslant |\alpha| + 2m \leqslant 2p-1;\ i=1,\dots,p)$ satisfy in $\bar{R}^n_T$ the following inequalities

$$(4.15) \qquad \left|D^{\alpha m}\mathcal{X}_i(X,t)\right| \leqslant \text{const}\ M^* t^{\theta_0'} \exp(b_g|OX|),$$

$$(4.16) \qquad \left|D^{\alpha m}\mathcal{X}_i(X,t) - D^{\alpha m}\mathcal{X}_i(\bar{X},\bar{t})\right| \leqslant$$

$$\leqslant \text{const}\ M^* \bar{t}^{\frac{1}{2}(1-\theta)h_g} \exp(b_g|OX|)\left(|X\bar{X}|^{\tilde{h}} + |\bar{t}-t|^{\frac{1}{2}\tilde{h}}\right),$$

where $0 < \theta_0' < \frac{1}{2}h_g$, $M^* = \begin{cases} M_g & \text{for } 0 \leqslant |\alpha|+2m \leqslant 2p-3 \\ M_g' & \text{for } |\alpha|+2m = 2p-2, 2p-1, \end{cases}$

$\tilde{h} = \begin{cases} \theta h_g & \text{for } |\alpha|+2m = 2p-1 \\ 1 & \text{for } 0 \leqslant |\alpha|+2m \leqslant 2p-2 \end{cases}$ and $\theta \in (0,1)$.

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