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Radosław Godowski

STATES ON ORTHOMODULAR LATTICES

1. Introduction

In this paper we show some methods of construction of states on products and homomorphic images of orthomodular lattices. We show how one can describe the set of all states on the homomorphic image of L, if one knows the description of the set of all states on L. As an application of these methods it is proved that the classes SFSS and FSS of orthomodular lattices with a strongly full (respectively full) set of states are not closed under ultraproducts.

2. Basic definitions and properties

As in [1], an orthomodular lattice (abbreviated oml) is considered as an universal algebra $(L; \wedge, \vee, ', 0, 1)$ with the binary lattice operations $\wedge$ and $\vee$, the unary orthocomplementation operation $'$, and the two nullary operations (constants) 0 and 1, the smallest and largest element of the lattice. If some subalgebra of L is a Boolean algebra, then we call it a Boolean subalgebra. We write $a \perp b$, if $a \leq b'$ and aCb if a and b commute (i.e. the subalgebra generated by set $\{a,b\}$ is a Boolean subalgebra).

A state on an oml L is a map $m : L \to \langle 0;1\rangle \subseteq \mathbf{R}$ such that $m(1) = 1$ and if $a,b \in L$, $a \perp b$ then $m(a \vee b) = m(a) + m(b)$. If m is a state, then $a \leq b$ implies $m(a) \leq m(b)$. A set $\{m_t \mid t \in T\}$ of states on L is said to be full (strongly full) if for any $a,b \in L$

$$\left[\bigwedge_{t\in T} m_t(a) \leqslant m_t(b)\right] \Longrightarrow a \leqslant b$$

$$\left[\bigwedge_{t\in T} (m_t(a) = 1 \Longrightarrow m_t(b) = 1)\right] \Longrightarrow a \leqslant b, \quad \text{respectively .}$$

Any strongly full set of states is full. The converse is false (see [1]). The class of omls with a full set of states we denote by FSS. The class of omls with a strongly full set of states we denote by SFSS. Thus, by definition SFSS $\subsetneq$ FSS.

3. States on the homomorphic images

T h e o r e m 1. Let $a \twoheadrightarrow [a]$ be a canonical epimorphism from olm L to the factor-algebra $L_1 =: L/\varrho$, where ϱ is a congruence relation on L and let $J =: \{a \in L \mid a \,\varrho\, 0\}$.

I. Let m be a state on L such that $m(a) = 0$ for any $a \in J$. We define $\tilde{m}([x]) =: m(x)$. Then $\tilde{m}$ is a state on L_1.

II. Let m be a state on L_1. We define $\tilde{m}(x) =: m([x])$. Then $\tilde{m}$ is a state on L and $\tilde{m}(a) = 0$ for any $a \in J$.

P r o o f . I. It is easy to show that $\tilde{m}(1) = \tilde{m}([1]) = m(1) = 1$. Now, let $[x] \perp [y]$. Then $[x] = [x \wedge y']$, $[y] = [x' \wedge y]$. But $x \wedge y' \perp x' \wedge y$. Therefore $\tilde{m}([x] \vee [y]) = \tilde{m}([x\wedge y'] \vee [x' \wedge y]) = \tilde{m}([(x \wedge y') \vee (x' \wedge y)]) = m((x \wedge y') \vee (x' \wedge y)) = m(x \wedge y') + m(x' \wedge y) = \tilde{m}([x \wedge y']) + \tilde{m}([x' \wedge y]) = \tilde{m}([x]) + \tilde{m}([y])$.

II. Observe that $\tilde{m}(1) = m([1]) = 1$. Now let $x \perp y$. Then $[x] \perp [y]$ and $\tilde{m}([x] \vee [y]) = \tilde{m}([x]) + \tilde{m}([y])$. Therefore $m(x \vee y) = \tilde{m}([x \wedge y]) = \tilde{m}([x \wedge y]) = \tilde{m}([x]) + \tilde{m}([y]) = m(x) + m(y)$. Thus $\tilde{m}$ is a state on L. Now, let $a \in J$, i.e. $[a] = [0]$. Then $m(a) = \tilde{m}([a]) = \tilde{m}([0]) = 0$.

C o r o l l a r y . Let L, L_1, J be as above. Let M be a set of all states m on L such that $m(a) = 0$ for any $a \in J$. Then any state on L_1 is of the form $\tilde{m}([x]) =: m(x)$ for some $m \in M$.

4. States on the products of omls

If $\{L_t | t \in T\}$ is a family of omls and if m_u is a state on L_u, for some $u \in T$, then $m(a) =: m_u(a(u))$ is a state on the product $\prod_{t\in T} L_t$. We give some generalization of this method of construction of states on products of omls.

T h e o r e m 2. Let $\{L_t \mid t \in T\}$ be a family of omls; $L =: \prod_{t\in T} L_t$; F be an ultrafilter on T and let m_t be a state on L_t for any $t \in T$. We define for any $a \in L$, $\varepsilon > 0$ and $r \in \langle 0;1\rangle$:

$$D(a,r,\varepsilon) =: \{t \in T \mid |m_t(a(t)) - r| < \varepsilon\}$$

$m(a) = r \equiv: \bigwedge_{\varepsilon>0} D(a,r,\varepsilon) \in F.$

Then:

1) m is well defined (i.e. there exists unique r),

2) m is a state,

3) $a \underset{F}{\sim} b \Rightarrow m(a) = m(b)$,

P r o o f . 1) We define, by induction two sequences (a_n) and (b_n), such that $a_n < b_n$ and $W_n \in F$ for any natural number n, where $W_n =: \{t \in T \mid m_t(a(t)) \in \langle a_n, b_n\rangle\}$. Let $c_n =: \frac{a_n + b_n}{2}$. Now we define

$$a_{n+1} = \begin{cases} a_n & \text{if } \{t \in T \mid m_t(a(t)) \in \langle a_n, c_n\rangle\} \in F \\ c_n & \text{otherwise} \end{cases}$$

$$b_{n+1} =: \begin{cases} c_n & \text{if } a_{n+1} = a_n \\ b_n & \text{otherwise.} \end{cases}$$

It is easy to show that $a_{n+1} < b_{n+1}$ and W_{n+1} F. Now we see that there exists unique r, namely $r = \lim_{n\to\infty} a_n$.

2) If $a = 1$, then $a(t) = 1$ and conequently $m_t(a(t)) = 1$ for any $t \in T$. Thus $r = 1$. Now let $a \perp b$, i.e. $a(t) \perp b(t)$

for any $t \in T$. Suppose that $m(a) = r_1$ and $m(b) = r_2$. Observe that for any $t \in T$: $|m_t((a \vee b)(t)) - (r_1 + r_2)| = |m_t(a(t) \vee b(t) - (r_1 + r_2))| = |(m_t(a(t)) - r_1) + (m_t(b(t)) - r_2)| \leqslant |m_t(a(t)) - r_1| + |m_t(b(t)) - r_2|$. Therefore for every $\varepsilon > 0$: $D(a \vee b, r_1+r_2, \varepsilon) \supseteq D(a, r_1, \frac{\varepsilon}{2}) \cap D(b, r_2, \frac{\varepsilon}{2})$ and hence $m(a \vee b = r_1 + r_2$.

3) Let $m(a) = r$ and $a \underset{F}{\sim} b$. Then for any $\varepsilon > 0$: $D(b,r,\varepsilon) \supseteq D(a,r,\varepsilon) \cap \{t \in T \mid a(t) = b(t)\}$ and consequently $D(b,r,\varepsilon) \in F$, hence $m(b) = r$.

Remark. Observe that if F is principal such that $\{u\} \in F$ then $m(a) = m_u(a(u))$ (the symbols are the same as in Theorem 2).

5. States on the ultraproducts of omls

An ultraproduct is a homomorphic image of product, therefore we can use Theorems 1 and 2 to describe states on ultraproduct.

Definition. A set $\{m_t \mid t \in T\}$ of states on L is said to be strongly separable if for any $a,b \in L$

$$\Big[\bigwedge_{t \in T} (m_t(a) = 1 \Longrightarrow m_t(b) \neq 0)\Big] \Longrightarrow a \leqslant b.$$

The class of omls with a strongly separable set of states will be denoted by SSFSS. Thus, by definition SSFSS $\subseteq$ SFSS $\subseteq$ FSS.

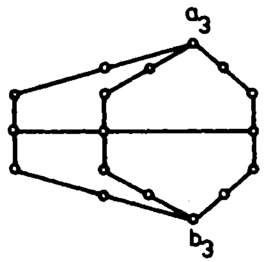
Recapitulation:

Name	$a \nleqslant b \Longrightarrow$ there exists $m \in M$ such that:
FSS	$m(a) > m(b)$
SFSS	$1 = m(a) > m(b)$
SSFSS	$1 = m(a) > m(b) = 0$

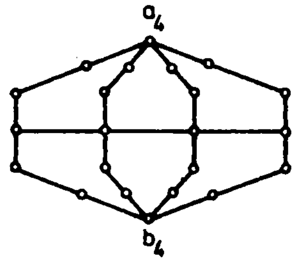
Theorem 3. I. The class SSFSS is closed under ultraproducts. II. The classes SFSS and FSS are not.

Proof. I. This part is a consequence of Theorems 1 and 2 of this paper. II. We construct the family of omls

$\{SNT_n \mid n=3,4,5,\ldots\}$ such that $SNT_n \in SFSS$ for any natural number $n \geq 3$, however the ultraproduct of SNT's does not belong to FSS. There is the Greechie diagram of SNT_n:



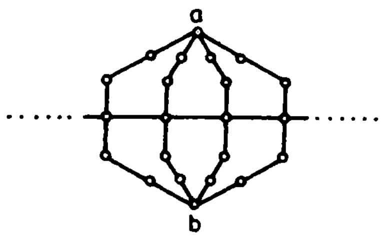
SNT_3



SNT_4

It is easy to prove that $SNT_n \in SFSS$ and if for some state m, $m(a_n) = 1$, then $m(b_n) \leq \frac{1}{n}$.

Now let F be a non-principal ultrafilter on the set $\{3,4,5,\ldots\}$. Then the ultraproduct $NFS =: \prod_{n=3} SNT_{n/F}$ can be schematically presented below



In this case, if m is a state of NFS then $m(a) + m(b) \leq 1$ though $a \not\perp b$. Thus $NFS \notin FSS$.

6. Problems

The class FSS, SFSS, SSFSS are closed under subalgebras and products. Let $\mathcal{H}(K)$ be the class of all homomorphic images of all algebras from the class K. Then we have: $SSFSS \subseteq \mathcal{H}(SSFSS) \subseteq \mathcal{H}(SFSS) \subseteq \mathcal{H}(FSS) \subsetneq OML$. We do not know if these classes are all different. In particular, we cannot prove or disprove the following conjecture:

(C) The class SSFSS forms a variety.

REFERENCES

[1] R. Godowski: Varieties of orthomodular lattices with a strongly full set of states, Demonstratio Math. 14 (1981) 725-733.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARS'
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