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ON A GENERALIZATION OF HOSSZÚ THEOREM

1. Introduction

Post has shown (cf. [11]) that to each (n+1)-group G there exists a related group G^* (called the free covering group) and its certain invariant subgroup G_0 (called the associated group of G). That associated group G_0 serves to reproduce the former (n+1)-group G by itself and a certain automorphism of G_0 (cf. [15], [11]). A continuation of Post's research is the paper of Hosszú [5] where it occurs that an (n+1)-group can be reconstructed by using an appropriate group (the so-called binary retract) and some automorphism of it. That expression is in a sense unique (cf. [14]). It is also known that binary retracts of some classes of (n+1)-groups are isomorphic (cf. [13], [3]).

In our paper we show that Hosszú Theorem can be generalized to the (k+1)-ary case (i.e. that instead of the binary retract an appropriate (k+1)-retract can be used). We show also that such (k+1)-ary retracts (also in the case of $k = 1$) of a given (n+1)-group are isomorphic. The assignement of (k+1)-ary retracts to (n+1)-groups is functorial.

2. Some notions and notation

We use the usual notations which may be found in papers on n-groups, in particular in [2], [4], [8].

As in [11], n-groups are called also polyadic groups, especially when the arity of the operations is not crucial.

Similarly a sequence $a_1,\dots,a_m$ of elements of an $(n+1)$-group G is called (after Post) a polyad (shortly an m-ad). Post has shown that the equivalence class of such a polyad (with respect to an appropriate relation) is an element of the free covering group G^* of G (cf. [11]). Therefore we also adopt the terminology of group theory to polyads, i.e. we use terms like $(n+1)$-adic identity, the m-ad inverse to a given $(n-m)$-ad, central m-ads and so on.

We use the following abbreviated notation:

$$f(x_1,\dots,x_k,x_{k+1},\dots,x_{k+q},x_{k+q+1},\dots,x_{n+1}) = f\Big(x_1^k,\overset{(q)}{x},x_{k+q+1}^{n+1}\Big),$$

whenever $x_{k+1} = \dots = x_{k+q} = x$ (x_i^j is the empty symbol for $j < i$ and for $i > n+1$; also $\overset{(q)}{x}$ is the empty symbol for $q = 0$).

For an $(n+1)$-ary operation f one can define a new $(un+1)$-ary operation $f_{(u)}$ by

$$f_{(u)}(x_1^{un+1}) = \underbrace{f(f(\dots f(f}_{u}(x_1^{n+1}),x_{n+2}^{2n+1}),\dots),x_{(u-1)n+2}^{un+1}).$$

The operation $f_{(u)}$ has been called by Dörnte (cf. [1]) the long product. If f is an $(n+1)$-group operation, then $f_{(u)}$ is a $(un+1)$-group operation, too (cf. [1]). In certain situations, when the arity of such operation does not play a crucial role, or when it will differ depending on additional assumptions, we will write $f_{(\cdot)}$, to mean $f_{(u)}$ for some $u = 1,2,\dots$.

Introducing the empty n-group appears to be convenient, when considering the category $\mathbf{Gr}_n$ of n-groups, because the empty n-group is an initial object of that category (cf. [6]). Therefore, when defining the functor ret in section 3 we emphasize where the empty n-group appears. Otherwise an n-group is always understood to be a nonempty n-group (as it was used by Dörnte in [1]).

3. (k+1)-ary retracts

Let (G,f) be an $(n+1)$-quasigroup. The $(k+1)$-ary operation g defined by

$$g(x_1^{k+1}) = f\left(a_1^{p_1}, x_1, a_{p_1+1}^{p_2}, x_2, \ldots, a_{p_k+1}^{p_{k+1}}, x_{k+1}, a_{p_{k+1}+1}^{n-k}\right),$$

where $k \leqslant n$ and $a_1,\ldots,a_{n-k}$ are fixed elements of G, is an $(k+1)$-quasigroup operation. In general, g is not a $(k+1)$-group operation, even if (G,f) is an $(n+1)$-group. If $p_1 = 0$, $p_{k+1} = n-k$, $n = k(p_2+1)$, $p_2-p_1 = p_3-p_2 = \ldots = p_{k+1}-p_k$ and $a_{p_1+i} = a_{p_2+i} = \ldots = a_{p_k+i}$ $(i=1,\ldots,p_2)$, then g is of the form $g(x_1^{k+1}) = f(x_1, a_1^r, x_2, a_1^r, \ldots, a_1^r, x_{m+1})$, where $n = sk$, $r = s-1$. To avoid repetitions we assume througout the whole text $n = sk$ (admitting $s = 1$ or $n = 1$), $r+1 \equiv 0 \pmod s$ (exactly $r = s-1$ or $r = su-1$).

The above-defined $(k+1)$-grupoid (G,g) will be called a $(k+1)$-ary retract of (G,f) with respect to $a_1,\ldots,a_r$ and it will be denoted by $\mathrm{ret}^s_{a_1,\ldots,a_r}(G,f)$ (or simply $\mathrm{ret}_{a_1,\ldots,a_r}(G,f)$). According to that definition, by $\mathrm{ret}^1(G,f)$ we shall mean nothing but the $(n+1)$-group (G,f).

Since in an $(n+1)$-group (G,f) to every $a \in G$ there exists a unique element $\bar{a} \in G$ such that $f\left(\overset{(i-1)}{a}, \bar{a}, \overset{(n-i)}{a}, x\right) = f\left(x, \overset{(i-1)}{a}, \bar{a}, \overset{(n-i)}{a}\right) = x$ for every $x \in G$ and $i = 1,\ldots,n$ (cf. [1]), then $\mathrm{ret}^s_{a_1,\ldots,a_r}(G,f) = \mathrm{ret}^s_{a,\ldots,a,b}(G,f)$ where $b = f\left(\overset{(n-r)}{a}, \bar{a}, a_1^r\right)$.

Immediately from the definition of a polyadic group we get

L e m m a 1 (cf. [5]). The $(k+1)$-groupoid $\mathrm{ret}^s_{a_1,\ldots,a_r}(G,f)$ is a $(k+1)$-group.

Notice that retracts of commutative (semi-commutative) n-groups are commutative (semi-commutative). Binary retracts of semi-commutative n-groups are commutative groups, but retracts of autodistributive n-groups (cf. [3]) are not necessarily autodistributive.

The following lemma is needed for the proof of functoriality of ret.

L e m m a 2. Let $h:(A,f) \longrightarrow (B,f)$ be a homomorphism of $(n+1)$-groups. If a polyad $a_{t+1},\ldots,a_n \in A$ is an inverse of a polyad $a_1,\ldots,a_t \in A$, then the polyad $h(a_{t+1}),\ldots$ $\ldots,h(a_n) \in B$ is an inverse of the polyad $h(a_1),\ldots,h(a_t) \in B$.

Let $n > 1$. Given $(n+1)$-groups (A,f), (B,f), (C,f), let r-ads (here $r = s-1$) $a_1,\ldots,a_r \in A$, $b_1,\ldots,b_r \in B$, $c_1,\ldots,c_r \in C$ be arbitrary but fixed. Consider the $(k+1)$-groups $(A,g) = \mathrm{ret}_{a_1,\ldots,a_r}(A,f)$, $(B,g) = \mathrm{ret}_{b_1,\ldots,b_r}(B,f)$, $(C,g) = \mathrm{ret}_{c_1,\ldots,c_r}(C,f)$. If $h:(A,f) \longrightarrow (B,f)$ is a homomorphism of $(n+1)$-groups, then we may define the mapping $\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r} h:A \longrightarrow B$ by the formula

$$\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r} h(x) = f(h(x),h(a_1),\ldots,h(a_r),b_{r+1}^n)$$

where the $(n-r)$-ad $b_{r+1},\ldots,b_n \in B$ is an inverse of the r-ad $b_1,\ldots,b_r$.

Let $x_1,\ldots,x_{k+1} \in A$. Then

$$\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r} h(g(x_1^{k+1})) =$$

$$= f(h(f(x_1,a_1^r,x_2,\ldots,x_k,a_1^r,x_{k+1})),h(a_1),\ldots,h(a_r),b_{r+1}^n) =$$

$$= f_{(\cdot)}(h(x_1),h(a_1),\ldots,h(a_r),h(x_2),\ldots,h(x_k),h(a_1),\ldots$$

$$\ldots,h(a_r),h(x_{k+1}),h(a_1),\ldots,h(a_r),b_{r+1}^n) =$$

$$= f_{(\cdot)}(f(h(x_1),h(a_1),\ldots,h(a_r),b_{r+1}^n),b_1^r,f(h(x_2),h(a_1),\ldots$$

$\ldots,h(a_r),b_{r+1}^n),\ldots,b_1^r,f(h(x_{k+1}),h(a_1),\ldots,h(a_r),b_{r+1}^n)) =$

$= g(\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r}h(x_1),\ldots,\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r}h(x_{k+1})).$

This shows that $\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r}h:(A,g) \to (B,g)$ is a homomorphism of (k+1)-groups.

It is easy to see that

$$\mathrm{ret}_{a_1,\ldots,a_r;a_1,\ldots,a_r}\mathrm{id}_{(A,f)} = \mathrm{id}_{\mathrm{ret}_{a_1,\ldots,a_r}(A,f)}.$$

Now. let $h_1:(A,f) \to (B,f)$ and $h_2:(B,f) \to (C,f)$ be homomorphisms of the corresponding (n+1)-groups, Then

$\mathrm{ret}_{b_1,\ldots,b_r;c_1,\ldots,c_r}h_2 \circ \mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r}h_1(x) =$

$= \mathrm{ret}_{b_1,\ldots,b_r;c_1,\ldots,c_r}h_2(f(h_1(x),h_1(a_1),\ldots,h_1(a_r),b_{r+1}^n)) =$

$= f(h_2(f(h_1(x),h_1(a_1),\ldots,h_1(a_r),b_{r+1}^n)),h_2(b_1),\ldots$

$\ldots,h_2(b_r),c_{r+1}^n) = f_{(2)}(h_2h_1(x),h_2h_1(a_1),\ldots,h_2h_1(a_r),h_2(b_{r+1}),\ldots$

$\ldots,h_2(b_n),h_2(b_1),\ldots,h_2(b_r),c_{r+1}^n) =$

$= f(h_2h_1(x),h_2h_1(a_1),\ldots,h_2h_1(a_r),c_{r+1}^n) =$

$= \mathrm{ret}_{a_1,\ldots,a_r;c_1,\ldots,c_r}h_2h_1(x).$

Choosing in each (n+1)-group a sequence of $r = s-1$ elements (perhaps with repetitions) and assuming additionally that for the empty (n+1)-group A let $\mathrm{ret}^s A$ be the empty (k+1)-group for $s < n$ and the one-element group for $s = n$, we obtain a functor (in fact a class of functors depending on the choice of elements) $\mathrm{ret}^s:\mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ from the category of all (n+1)-groups to the category of all (k+1)-groups. We shall show that all these functors are naturally equivalent.

Consider firstly the particular case of sequences of the form $a_1 = a_2 = \ldots = a \in A$. We write $\mathrm{Ret}^s_a(A,f)$ (or simply $\mathrm{Ret}_a(A,f)$) instead of $\mathrm{ret}^s_{a_1,\ldots,a_r}(A,f)$ and Ret^s for ret^s. Note that the homomorphism $\mathrm{Ret}^s_{a;b}h = \mathrm{ret}^s_{a,\ldots,a;b}b$: $:\mathrm{Ret}_a(A,f) \longrightarrow \mathrm{Ret}_b(B,f)$ is given by the formula

$$\mathrm{Ret}^s_{a;b}h(x) = f(h(x),\overset{(r)}{h(a)},\overset{(n-1-r)}{b},\bar{b}).$$

T h e o r e m 1. The functors $\mathrm{ret}^s : \mathbf{Gr}_{n+1} \longrightarrow \mathbf{Gr}_{k+1}$ and $\mathrm{Ret}^s : \mathbf{Gr}_{n+1} \longrightarrow \mathbf{Gr}_{k+1}$ are naturally equivalent.

P r o o f . In each (n+1)-group choose a sequence of r elements (perhaps with repetitions) and additionally a single element. So we get two functors: ret^s and Ret^s. Let the mapping

$$\lambda_A : \mathrm{ret}_{a_1,\ldots,a_r}(A,f) \longrightarrow \mathrm{Ret}_a(A,f) \text{ be given by } \lambda_A(x) = f(x,a_1^r,\overset{(n-1-r)}{a},\bar{a}).$$

From the earlier considerations it follows that λ_A is a homomorphism of (k+1)-groups. Moreover, λ_A is even an isomorphism.

Now we show that for every homomorphism $h:(A,f) \longrightarrow (B,f)$ the following diagram

$$\begin{array}{ccc}
\mathrm{ret}_{a_1,\ldots,a_r}(A,f) & \xrightarrow{\quad\lambda_A\quad} & \mathrm{Ret}_a(A,f) \\
\Big\downarrow \mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r}h & & \Big\downarrow \mathrm{Ret}_{a;b}h \\
\mathrm{ret}_{b_1,\ldots,b_r}(B,f) & \xrightarrow{\quad\lambda_B\quad} & \mathrm{Ret}_b(B,f)
\end{array}$$

is commutative. In fact

$$\mathrm{Ret}_{a;b}h\circ\lambda_A(x)=\mathrm{Ret}_{a;b}h(f(x,a_1^r,\overset{(n-1-r)}{a},\bar{a}))=$$

$$=f(h(f(x,a_1^r,\overset{(n-1-r)}{a},\bar{a})),\overset{(r)}{h(a)},\overset{(n-1-r)}{b};\bar{b})=$$

$$=f_{(\cdot)}(h(x),h(a_1),\ldots,h(a_r),\overset{(n-1-r)}{h(a)},h(\bar{a}),\overset{(r)}{h(a)},\overset{(n-1-r)}{b},\bar{b})=$$

$$=f(h(x),h(a_1),\ldots,h(a_r),\overset{(n-1-r)}{b},\bar{b})=$$

$$=f(f(h(x),h(a_1),\ldots,h(a_r),b_{r+1}^n),b_1^r,\overset{(n-1-r)}{b},\bar{b})=$$

$$=\lambda_B(f(h(x),h(a_1),\ldots,h(a_r),b_{r+1}^n))=$$

$$=\lambda_B\circ\mathrm{ret}_{a_1,\ldots,a_r;b_1,\ldots,b_r}h(x),$$

which proves that $\lambda:\mathrm{ret}^s\to\mathrm{Ret}^s$ is a natural equivalence.

So we have the well-defined up to a natural equivalence the functor $\mathrm{ret}^s:\mathbf{Gr}_{n+1}\to\mathbf{Gr}_{k+1}$. In spite of the fact that ret^s is independent of the choice of elements in (n+1)-groups, in concrete situations one has to make such choice in one way or another.

Timm has proved in [13] that binary retracts of a commutative polyadic group are isomorphic commutative groups. Dudek has proved in [3] that also binary retracts of an autodistributive polyadic group are isomorphic. Theorem 1 is a generalization of those results.

C o r o l l a r y 1. All binary retracts of an (n+1)-group are isomorphic.

C o r o l l a r y 2. If (G,f) is an (n+1)-group, then all (k+1)-ary retracts of the (un+1)-group $(G,f_{(u)})$, where u is fixed, are isomorphic.

Note that dealing with (k+1)-ary retracts of (un+1)-groups $(G,f_{(u)})$ is in fact considering the functor $\mathrm{ret}^{su}\Psi_u$:

$\mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ which is the composition of the forgetful functor $\Psi_u:\mathbf{Gr}_{n+1} \to \mathbf{Gr}_{un+1}$ (cf. [6], [7]) with the functor $ret^{su}:\mathbf{Gr}_{un+1} \to \mathbf{Gr}_{k+1}$. This functor will be denoted briefly by $ret^{s,u}$ (i.e. $ret^{s,u} = ret^{su}\Psi_u$ and $Ret^{s,u} = Ret^{su}\Psi_u$). According to that notation, $ret^{s,1} = ret^{s}$. A restriction to the case of $u \leqslant k$ is possible since, as it is easy to check, for $u \equiv t \pmod k$ the $(k+1)$-ary retracts of the $(un+1)$-group $(G,f_{(u)})$ and the $(tn+1)$-group $(G,f_{(t)})$ are isomorphic. Henceforth throughout the paper we assume always $u \leqslant k$.

One may ask whether the functors $ret^{s,t}$ and $ret^{s,u}$ are naturally equivalent for $t \not\equiv u \pmod k$. If we consider $ret^{s,t}$ and $ret^{s,u}$ as functors from $\mathbf{Gr}_{n+1}$ to $\mathbf{Gr}_{k+1}$ they are not in general naturally equivalent.

As an example consider the 7-groupoid (G,f) where $G = Z_4$, $f(x_1^7) \equiv x_1-x_2+x_3-x_4+x_5-x_6+x_7 \pmod 4$. Then (G,f) is noncommutative, idempotent 7-group. Let

$$(G,g) = Ret_a^3(G,f) \quad \text{and} \quad (G,g') = Ret_a^6(G,f_{(2)}) = Ret_a^{6,2}(G,f).$$

Hence

$$g(x_1^3) = f(x_1,\overset{(2)}{a},x_2,\overset{(2)}{a},x_3) \equiv x_1-x_2+x_3 \pmod 4,$$

$$g'(x_1^3) = f_{(2)}\, x_1,\overset{(5)}{a},x_2,\overset{(5)}{a},x_3 \equiv x_1+x_2+x_3-2a \pmod 4.$$

Notice that (G,g) is a noncommutative, idempotent 3-group, while (G,g') has only two idempotent elements and (G,g') is a commutative 3-group. Therefore (G,g) and (G,g') are not isomorphic.

The situation simplifies considerably in the case of the functor $ret^{n,u}:\mathbf{Gr}_{n+1} \to \mathbf{Gr}_2$ to the category of groups. In this case $k = 1$, whence $u = 1$. Therefore all functors $ret^{n,u}$

(for arbitrary u) are naturally equivalent and there appears the functor $\mathrm{ret}^n: \mathbf{Gr}_{n+1} \to \mathbf{Gr}_2$ which is unique up to a natural equivalence. In contrast, for $k > 1$ one has to consider different functors $\mathrm{ret}^{s,u}: \mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ for different u. To have a natural equivalence of them it is necessary to restrict both functors to an appropriate subcategory of $\mathbf{Gr}_{n+1}$.

The case of $k = 1$ was studied in [10]. Pop considered two functors from $\mathbf{Gr}_{n+1}$ to $\mathbf{Gr}_2$ (for $n > 1$). One of them was, in our notation, the functor $\mathrm{ret}^n_{a,\ldots,a,\bar{a}}$, the other was the functor which had appeared de facto in the Post's construction of a free covering group (cf. [11], [10]). From [6], Theorem 1 it follows that the latter is naturally equivalent to the functor Ret^n, which in turn implies, by Theorem 1, the main theorem of [10].

In [6] (cf. also [8]) there was introduced a functor $\Phi_s: \mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ assigning to each $(n+1)$-groups its free covering $(k+1)$-group. That functor does not preserve projective limits, in particular the cartesian product (cf. [7]). In contrast with Φ_s, the functor $\mathrm{ret}^s: \mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ preserves the cartesian product.

T h e o r e m 2. If $\left[\prod_{t\in T} G_t; \left\{\pi_t: \prod_{t\in T} G_t \to G_t\right\}_{t\in T}\right]$ is the cartesian product of a family of $(n+1)$-groups $\{(G_t,f)\}_{t\in T}$, then $\left[\mathrm{ret}^s \prod_{t\in T} G_t; \left\{\mathrm{ret}^s\pi_t : \mathrm{ret}^s \prod_{t\in T} G_t \to \mathrm{ret}^s G_t\right\}_{t\in T}\right]$ is the cartesian product of the family of $(k+1)$-groups $\{\mathrm{ret}^s(G_t,f)\}_{t\in T}$.

P r o o f . Let $(G,f) = \prod_{t\in T} (G_t,f)$ together with projections $\{\pi_t: G \to G_t\}_{t\in T}$ be the cartesian product of the nonempty family of $(n+1)$-groups $\{(G_t,f)\}_{t\in T}$. In each $(n+1)$-group G_t choose an element $a_t \in G_t$. Let $(G_t,g) = \mathrm{Ret}^s_{a_t}(G_t,f)$, $(G,g) = \mathrm{Ret}^s_a(G,f)$ (where $a = (a_t)_{t\in T} \in G$) and $(G,g') = \prod_{t\in T} (G_t,g)$. Take arbitrary elements $x_i = (x_{i,t})_{t\in T} \in G$ where for $i = 1,\ldots,k+1$ we have $x_{i,t} \in G_t$. Then

$$g'(x_1^{k+1}) = (g(x_{1,t},\dots,x_{k+1,t}))_{t\in T} =$$

$$= (f(x_{1,t},\overset{(r)}{a_t},\dots,a_t,x_{k+1,t}))_{t\in T} =$$

$$= f(x_1,\overset{(r)}{a},\dots,x_{k+1}) = g(x_1^{k+1}).$$

Moreover,

$$\mathrm{ret}^s\pi_t((x_t)_{t\in T}) = f(\pi_t((x_t)_{t\in T}),\overset{(r)}{\pi_t(a)},\overset{(n-1-r)}{a_t},\bar{a}_t) =$$

$$= f\left(x_t,\overset{(r)}{a_t},\overset{(n-1-r)}{a_t},\bar{a}_t\right) = x_t,$$

which proves that $\left[\mathrm{ret}^s G;\left\{\mathrm{ret}^s\pi_t:\mathrm{ret}^s G \to \mathrm{ret}^s G_t\right\}_{t\in T}\right]$ is the cartesian product of the nonempty family $\left\{\mathrm{ret}^s G_t\right\}_{t\in T}$. Further note that the functor ret^s preserves final objects (that is the cartesian product of the empty family) and initial objects (that is the cartesian products of families of (n+1)-groups containing the empty (n+1)-group). This completes the proof of Theorem 2.

We end this section with the following

P r o p o s i t i o n 1. Let an (n+1)-group (G,f) be given. If there exist elements $a_{k+2},\dots,a_{n+1} \in G$ such that for all $x_1,\dots,x_{k+1} \in G$ we have $f\left(x_1^k,a_{k+2}^{n+1},x_{k+1}\right) = f\left(x_1^{k+1},a_{k+2}^{n+1}\right)$, then G together with the operation $g\left(x_1^{k+1}\right) = f\left(x_1^{k+1},a_{k+2}^{n+1}\right)$ is a (k+1)-group.

P r o o f . Simple calculations show that the operation g is (1,2)-associative. Obviously it is also a (k+1)-quasigroup operation. Hence (G,g) is a (k+1)-group (cf. [2], [4]).

C o r o l l a r y 3. If a_i $(i = k+2,\dots,n+1)$ are central elements of an (n+1)-group (G,f), then (G,g), where g is defined as in Proposition 1, is a (k+1)-group.

Note, however, that for distinct choices of the sequence $a_{k+2},\dots,a_{n+1}$, the formerly defined (k+1)-groups are not necessarily isomorphic.

As an example consider the 7-group (Z_4,f) defined as above. 3-groups (Z_4,g) and (Z_4,g'), where $g(x_1^3) = f(x_1^3,3,1,1,1,1)$ and $g'(x_1^3) = f(x_1^3,1,1,1,1)$, are not isomorphic.

4. $\langle\gamma,c_1^k\rangle$-derived (n+1)-groups

For a given (binary) group $(G,\cdot)$ Timm (cf. [14]) has formed with the aid of a sequence of bijective mappings $\gamma_1,\dots,\gamma_n$ of G and an element $c\in G$ an (n+1)-ary operation f (in [14] it was n-ary, in fact) defined by $f(x_1^{n+1}) = x_1\cdot\gamma_1(x_2)\cdot\dots\cdot\gamma_n(x_{n+1})\cdot c$. The (n+1)-groupoid defined by this way was called the $\langle\gamma_i,c\rangle$-derived (n+1)-groupoid of the group $(G,\cdot)$. Depending on the choice of $\gamma_1,\dots,\gamma_n$ and $c\in G$ the (n+1)-groupoid (G,f) is a permutationally associative (n+1)-groupoid, an (n+1)-semigroup, an (n+1)-group and so on. In particular Timm has proved that $\langle\gamma_i,c\rangle$-derived (n+1)-groupoid of a group $(G,\cdot)$ is an (n+1)-group if and only if the sequence of mappings $\gamma_1,\dots,\gamma_n$ and the element c fulfil the Hosszú condition (cf. [5], [11], p.245), that is: γ_1 is an automorphism of $(G,\cdot)$ such that $\gamma_1(c) = c$, $\gamma_i = (\gamma_1)^i$ $(i=1,\dots,n)$ and γ_n is an inner automorphism given by $\gamma_n(x) = c\cdot x\cdot c^{-1}$. By analogy we say that an automorphism γ of a (k+1)-group (G,g) and elements $c_1,\dots,c_k\in G$ fulfil the Hosszú condition, if $\gamma(c_i) = c_i$ $(i = 1,\dots,k)$ and $g(\gamma^n(x),c_1^k) = g(c_1^k,x)$ (i.e. γ^n is an inner automorphism). Define an (n+1)-ary operation f on the set G by

$$f(x_1^{n+1}) = g_{(s+1)}(x_1,\gamma(x_2),\dots,\gamma^n(x_{n+1}),c_1^k).$$

If γ and $c_1,\dots,c_k$ fulfil the Hosszú condition, we say that the (n+1)-groupoid (G,f) is $\langle\gamma;c_1^k\rangle$-derived from the (k+1)-group (G,g) and it will be denoted by $\mathrm{der}^s_{\gamma;c_1,\dots,c_k}(G,g)$.

In certain situations, when the form of γ and $c_1,\ldots,c_k$ does not play a crucial role, we call the $\langle\gamma;c_1^k\rangle$-derived (n+1)-groupoid simply a weakly derived (n+1)-groupoid. In the case when $\gamma=\mathrm{id}_G$ the $\langle \mathrm{id}_G;c_1^k\rangle$-derived (n+1)-groupoid will be called shortly the $\langle c_1^k\rangle$-derived (n+1)-groupoid and abbreviated by $\mathrm{der}^s_{c_1,\ldots,c_k}(G,g)$. Note that those notations are meaningful and nontrivial also for $s=1$.

T h e o r e m 3. The (n+1)-groupoid $\mathrm{der}^s_{\gamma;c_1,\ldots,c_k}(G,g)$ is an (n+1)-group.

P r o o f . The solvability of equations for the operation f is implied by the solvability of equations for the (k+1)-group operation g and bijectivity of the mapping γ^i $(i=1,\ldots,n)$. We show associativity of f. In fact,

$$f(f(x_1^{n+1}),x_{n+2}^{2n+1})=$$

$$=g_{(.)}(g_{(.)}(x_1,\gamma(x_2),\ldots,\gamma^n(x_{n+1}),c_1^k),\gamma(x_{n+2}),\ldots,\gamma^n(x_{2n+1}),c_1^k)=$$

$$=g_{(.)}(x_1,g_{(.)}(\gamma(x_2),\ldots,\gamma^n(x_{n+1}),c_1^k,\gamma(x_{n+2})),\gamma^2(x_{n+3}),\ldots,\gamma^n(x_{2n+1}),c_1^k)=$$

$$=g_{(.)}(x_1,\gamma(g_{(.)}(x_2,\ldots,\gamma^{n-1}(x_{n+1}),c_1^k,x_{n+2})),\gamma^2(x_{n+3}),\ldots,\gamma^n(x_{2n+1}),c_1^k)=$$

$$=g_{(.)}(x_1,\gamma(g_{(.)}(x_2,\ldots,\gamma^{n-1}(x_{n+1}),\gamma^n(x_{n+2}),c_1^k)),\gamma^2(x_{n+3}),\ldots$$

$$\ldots,\gamma^n(x_{2n+1}),c_1^k)=f(x_1,f(x_2^{n+2}),x_{n+3}^{2n+1}),$$

whence f is (1,2)-associative. Thus, by Sokolov's result (cf. [12]), f is associative, which completes the proof.

Recall that an (n+1)-group (G,f) is derived from a (k+1)-group (G,g) (i.e. $(G,f)=\Psi_s(G,g)$ in notation of [6]), if $f=g_{(s)}$ (cf. [1]). The relationship of the terms: derived and weakly derived (n+1)-group is not casual. Let $\gamma=\mathrm{id}_G$ and let a polyad $c_1,\ldots,c_k$ be an identity of a (k+1)-group (G,g). Then γ and $c_1,\ldots,c_k$ fulfil obviously the Hosszú condition and $\operatorname{der}^s_{\gamma;c_1,\ldots,c_k}(G,g) = \Psi_s(G,g)$, i.e. the (n+1)-group $\langle\gamma;c_1^k\rangle$-derived is simply the (n+1)-group derived in the sense of [1].

Under distinct choice of the automorphism γ and the elements $c_1,\ldots,c_k$ the (n+1)-groups $\operatorname{der}^s_{\gamma;c_1,\ldots,c_k}(G,g)$ may occur to be non-isomorphic. It happens even in the most simple case of $k = 1$. Consider the following example.

Let (G,g) be a group of exponent n $(n > 1)$ and let b be a central element of (G,g) different from the neutral one. Then the (n+1)-groups $\psi_n(G,g)$ and $\operatorname{der}^n_{\gamma;b}(G,g)$ are not idempotent).

Now, as in the case of derived (n+1)-groups (cf. [7]), we prove that the cartesian product of weakly derived (n+1)-groups of (k+1)-groups is the weakly derived (n+1)-group of the cartesian product of the (k+1)-groups.

T h e o r e m 4. Given a nonempty family of (k+1)-groups $\{(G_t,g)\}_{t\in T}$, let $c_i = (c_{i,t})_{t\in T}$ for $c_{i,t} \in G_t$ $(i=1,\ldots,k)$, $\gamma(x) = (\gamma_t(x_t))_{t\in T}$ for $x = (x_t)_{t\in T}$. Then

$$\operatorname{der}^s_{\gamma;c_1,\ldots,c_k} \prod_{t\in T} (G_t,g) = \prod_{t\in T} \operatorname{der}^s_{\gamma_t;c_{1,t},\ldots,c_{k,t}}(G_t,g).$$

P r o o f . Let

$$(G,g) = \prod_{t\in T} (G_t,g),\ (G_t,f) = \operatorname{der}^s_{\gamma_t;c_{1,t},\ldots,c_{k,t}}(G_t,g),$$

$$(G,f) = \operatorname{der}^s_{\gamma;c_1,\ldots,c_k}(G,g),\ (G,f') = \prod_{t\in T} (G_t,f).$$

It is easy to check that if γ_t and $c_{1,t},\ldots,c_{k,t}$ fulfil the Hosszú condition for every $t \in T$, then γ and $c_1,\ldots,c_k$ fulfil the Hosszú condition, too. Take arbitrary elements $x_i = (x_{i,t})_{t\in T} \in G$ $(i=1,\ldots,n+1)$. Then

$$f'(x_1^{n+1}) = (f(x_{1,t},\dots,x_{n+1,t}))_{t\in T} =$$

$$= (g_{(s+1)}(x_{1,t},\gamma_t(x_{2,t}),\dots,(\gamma_t)^n(x_{n+1,t}),c_{1,t},\dots,c_{k,t}))_{t\in T} =$$

$$= g_{(s+1)}(x_1,\gamma(x_2),\dots,\gamma^n(x_{n+1}),c_1^k) = f(x_1^{n+1}).$$

5. A generalization of Hosszú theorem

By a translation of an $(n+1)$-group (cf. [9]) we mean the mapping $\gamma(x) = f(a_1^{i-1},x,a_{i+1}^{n+1})$. It is always invertible. Its inverse is the mapping $\gamma^{-1}(x) = f(b_i^n,x,c_2^i)$ where the polyad $b_i,\dots,b_n$ is an inverse of the polyad $a_1,\dots,a_{i-1}$ and the polyad $c_2,\dots,c_i$ is an inverse of the polyad $a_{i+1},\dots,a_{n+1}$.

A special case of translations are translations of the form $\alpha_{a,b}^{(i)}(x) = f(x,\overset{(i-1)}{a},b,\overset{(n-i)}{a})$ and $\beta_{a,x}^{(j)}(x) = f(\overset{(j-1)}{a},c,\overset{(n-j)}{a},x)$. Dudek has proved (cf. [3]) that an $(n+1)$-semigroup (G,f) is an $(n+1)$-group if and only if for some $i,j = 1,\dots,n$ and all $a\in G$ there exist elements $b,c \in G$ such that $\alpha_{a,b}^{(i)}(x) = \beta_{a,c}^{(j)}(x) = x$. Obviously, if (G,f) is an autodistributive $(n+1)$-group, then all translations are automorphisms. Moreover, if the polyad $a_1,\dots$ $\dots,a_{i-1},a_{i+1},\dots,a_{n+1}$ is an identity of (G,f), then γ is an automorphism of (G,f). Hence γ is an isomorphism from $\mathrm{ret}^s_{c_1,\dots,c_r}(G,F)$ onto $\mathrm{ret}^s_{\gamma(c_1),\dots,\gamma(c_r)}(G,f)$. Therefore, if $\gamma(c_i) = c_i$ for $i = 1,\dots,r$, then γ is an automorphism of $\mathrm{ret}^s_{c_1,\dots,c_r}(G,f)$.

From the above remarks there follows immediately

L e m m a 3. A translation of the form $\gamma(x) = f(\bar{a},\overset{(n-r-1)}{a},x,\overset{(r)}{a})$ is an automorphism of the $(n+1)$-group (G,f) and of its $(k+1)$-ary retract $\mathrm{Ret}_a^s(G,f)$.

T h e o r e m 5. Let an $(n+1)$-group (G,f) be given. To every $(k+1)$-group $\mathrm{Ret}_a^{s,u}(G,f)$ there exist an automorphism γ of $\mathrm{Ret}_a^{s,u}(G,f)$ and elements $c_1,\dots,c_k \in G$ such that

$$\operatorname{der}^{s}_{\gamma;c_1,\ldots,c_k}\operatorname{Ret}^{s,u}_{a}(G,f) = (G,f).$$

Namely,

$$\gamma(x) = f\Big(\bar a, \overset{(n-su)}{a}, x, \overset{(su-1)}{a}\Big),\quad c_1 = f_{(\cdot)}\Big(\overset{((n+k)(n-su))}{a}, \overset{(n+1)}{\bar a}\Big),$$

$$c_2 = \ldots = c_k = \bar a.$$

P r o o f . Let $(G,g) = \operatorname{Ret}^{s,u}_{a}(G,f)$, where (G,f) is an $(n+1)$-group, and let $a \in G$ be a certain element of it. Define the mapping $\gamma : G \to G$ by the formula $\gamma(x) = f_{(\cdot)}\Big(\bar a, \overset{(n-1-r)}{a}, x, \overset{(r)}{a}\Big)$ where $r = s-1$. By Lemma 3 it follows that γ is an automorphism of (G,g). We prove that the automorphism γ^n is an inner automorphism. Let

$$c_1 = f_{(\cdot)}\Big(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar a}\Big),\quad c_2 = \ldots = c_k = \bar a.$$

Then

$$g(\gamma^n(x).c_1^k) =$$

$$= g\Big(f_{(\cdot)}\Big(\overset{(n)}{\bar a}, \overset{(n(n-1-r))}{a}, x, \overset{(nr)}{a}\Big), f_{(\cdot)}\Big(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar a}\Big), \overset{(k-1)}{\bar a}\Big) =$$

$$= f_{(\cdot)}\Big(f_{(\cdot)}\Big(\overset{(n)}{\bar a}, \overset{(n(n-1-r))}{a}, x, \overset{(nr)}{a}\Big), \overset{(r)}{a}, f_{(\cdot)}\Big(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar a}\Big),$$

$$\underbrace{\overset{(r)}{a}, \bar a, \ldots, \overset{(r)}{a}, \bar a}_{k-1}\Big) =$$

$$= f_{(\cdot)}\Big(\overset{(n)}{\bar a}, \overset{(n(n-1-r))}{a}, x, \overset{((n+k)(n-1))}{a}, \overset{(n+k)}{\bar a}\Big) =$$

$$= f_{(\cdot)}\Big(\overset{(n)}{\bar a}, \overset{(n(n-1-r))}{a}, x\Big).$$ At the same time $g(c_1^k, x) =$

$$= g\left(f_{(\cdot)}\left(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar{a}}\right), \overset{(k-1)}{\bar{a}}, x\right) =$$

$$= f_{(\cdot)}\left(f_{(\cdot)}\left(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar{a}}\right), \underbrace{\overset{(r)}{a}, \bar{a}, \ldots, \overset{(r)}{a}, \bar{a}}_{k-1}, \overset{(r)}{a}, x\right) =$$

$$= f_{(\cdot)}\left(\overset{(n)}{\bar{a}}, \overset{(n(n-1-r))}{a}, x\right),$$

which shows that $g(\gamma^n(x), c_1^k) = g(c_1^k, x)$.

One can easily verify that $\gamma(c_i) = c_i$ for $i = 1,\ldots,k$. Thus γ and $c_1,\ldots,c_k$ fulfil the Hosszú condition. Furthermore,

$$g_{(s+1)}(x_1, \gamma(x_2), \ldots, \gamma^n(x_{n+1}), c_1^k) =$$

$$= f_{(\cdot)}\left(x_1, \overset{(r)}{a}, \gamma(x_2), \overset{(r)}{a}, \ldots, \overset{(r)}{a}, \gamma^n(x_{n+1}), \overset{(r)}{a}, c_1, \ldots, \overset{(r)}{a}, c_k\right) =$$

$$= f_{(\cdot)}\left(x_1, \overset{(r)}{a}, \bar{a}, \overset{(n-1-r)}{a}, x_2, \overset{(r)}{a}, \overset{(r)}{a}, \overset{(2)}{\bar{a}}, \overset{(2(n-1-r))}{a}, x_3, \overset{(2r)}{a}, \overset{(r)}{a}, \ldots\right.$$

$$\ldots, \overset{(r)}{a}, \overset{(n)}{a}, \overset{(n(n-1-r))}{a}, x_{n+1}, \overset{(nr)}{a}, \overset{(r)}{a}, \overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar{a}},$$

$$\left.\underbrace{\overset{(r)}{a}, \bar{a}, \ldots, \overset{(r)}{a}, \bar{a}}_{k-1}\right) = f(x_1^{n+1}),$$

whence $(G,f) = \mathrm{der}^s_{\gamma; c_1,\ldots,c_k}(G,g)$, which was to be proved.

It turns out that forming a weakly derived (n+1)-group is a procedure converse to that of forming a retract. Note, however, that a special form of retract has been considered in the proof of Theorem 5, namely Ret. Dealing with arbitrary (k+1)-ary retracts the situation slightly complicates.

L e m m a 4. If $h:(A,g) \to (B,g)$ is an isomorphism of (k+1)-groups, then $h:\mathrm{der}^s_{\gamma_A; a_1,\ldots,a_k}(A,g) \to$

$\mathrm{der}^{s}_{\gamma_B;b_1,\ldots,b_k}(B,g)$, where $\gamma_B = h\gamma_A h^{-1}$, $b_i = h(a_i)$ $(i = 1,\ldots,k)$, is also an isomorphism of $(n+1)$-groups.

P r o o f .

$$h(f(x_1^{n+1})) = h(g_{(s+1)}(x_1,\gamma_A(x_2),\ldots,\gamma_A^n(x_{n+1}),a_1^k)) =$$

$$= g_{(s+1)}(h(x_1),h(\gamma_A(x_2)),\ldots,h(\gamma_A^n(x_{n+1})),h(a_1),\ldots,h(a_k)) =$$

$$= g_{(s+1)}(h(x_1),\gamma_B(h(x_2)),\ldots,\gamma_B^n(h(x_{n+1})),b_1^k) = f(h(x_1),\ldots,h(x_{n+1})).$$

P r o p o s i t i o n 2. Let an $(n+1)$-group (G,f) be given. To every $(k+1)$-group $\mathrm{ret}^{s,u}_{a_1,\ldots,a_r}(G,f)$ there exists an automorphism γ of $\mathrm{ret}^{s,u}_{a_1,\ldots,a_r}(G,f)$ and elements $c_1,\ldots,c_k \in G$ such that the $(n+1)$-groups $\mathrm{der}^{s}_{\gamma;c_1,\ldots,c_k}\ \mathrm{ret}^{s,u}_{a_1,\ldots,a_r}(G,F)$ and (G,f) are isomorphic. Namely,

$$\gamma(x) = f_{(\cdot)}\left(\bar{a},\overset{(n-1-r)}{a},x,a_1^r,\overset{(r)}{a},a_{r+1}^n\right),$$

$$c_1 = f_{(\cdot)}\left(\overset{((n+k)(n-1-r))}{a},\overset{(n+1)}{\bar{a}},\overset{(r)}{a},a_{r+1}^n\right),$$

$$c_2 = \ldots = c_k = f_{(\cdot)}\left(\bar{a},\overset{(r)}{a},a_{r+1}^n\right),$$

where a is an arbitrary element of G and the polyad $a_{r+1},\ldots,a_n$ is an inverse of $a_1,\ldots,a_r$.

P r o o f . According to Theorem 1, the mapping $\lambda:\mathrm{ret}^{s,u}_{a_1,\ldots,a_r}(G,f) \to \mathrm{Ret}^{s,u}_{a}(G,f)$ given by the formula $\lambda(x) = f\left(x,a_1^r,\overset{(n-1-r)}{a},\bar{a}\right)$ is an isomorphism of $(k+1)$-groups. As it is easy to check, the map $\mu = \lambda^{-1}$ is of the form

$\mu(x) = f(x, \overset{(r)}{a}, a_{r+1}^{n})$ where the polyad $a_{r+1},\dots,a_n$ is an inverse of $a_1,\dots,a_r$. In view of Theorem 5 it follows that by an appropriate choice of the automorphism (denoted by δ) and elements (denoted by $b_1,\dots,b_k$) the equality $\mathrm{der}^{s}_{\delta;b_1,\dots,b_k}\mathrm{Ret}^{s,u}_{a}(G,f) = (G,f)$ holds. Recall that in this case we have $\delta(x) = f(\bar{a}, \overset{(n-1-r)}{a}, x, \overset{(r)}{a})$, $b_1 = f_{(\cdot)}(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar{a}})$, $b_2 = \dots = b_k = \bar{a}$. Then, by Lemma 4, the mapping

$$\mu : \mathrm{der}^{s}_{\delta;b_1,\dots,b_k}\mathrm{Ret}^{s,u}_{a}(G,f) \to \mathrm{der}^{s}_{\gamma;c_1,\dots,c_k}\mathrm{ret}^{s,u}_{a_1,\dots,a_r}(G,f),$$

where $\gamma = \mu\delta\lambda$ and $c_i = \mu(b_i)$, is an isomorphism. Furthermore,

$$\begin{aligned}\gamma(x) &= \mu\delta(x, a_1^r, \overset{(n-1-r)}{a}, \bar{a}) = \\ &= \mu\left(f_{(\cdot)}(\bar{a}, \overset{(n-1-r)}{a}, x, a_1^r, \overset{(n-1-r)}{a}, a, \overset{(r)}{\bar{a}})\right) = \\ &= \mu\left(f_{(\cdot)}(\bar{a}, \overset{(n-1-r)}{a}, x, a_1^r)\right) = \\ &= f_{(\cdot)}(\bar{a}, \overset{(n-1-r)}{a}, x, a_1^r, \overset{(r)}{a}, a_{r+1}^n)\end{aligned}$$

and

$$c_1 = \mu(b_1) = f_{(\cdot)}(\overset{((n+k)(n-1-r))}{a}, \overset{(n+1)}{\bar{a}}, \overset{(r)}{a}, a_{r+1}^n),$$

$$c_2 = \mu(b_2) = f_{(\cdot)}(\bar{a}, \overset{(r)}{a}, a_{r+1}^n).$$

This completes the proof of Proposition 2.

If the (n+1)-group (G,f) satisfies additionally the condition: the equality $f\left(a_1^i, \overset{(r)}{x}, a_{i+1}^{n+1-r}\right) = b$ has a solution for arbitrary elements $a_1, \dots, a_{n+1-r}, b \in G$ (for $r = 1$ this is always true), then the element $a \in G$ in Proposition 2 can be chosen in such a way that the isomorphism μ becomes the identity. Then $\gamma = \delta$, $c_i = b_i$ $(i=1,\dots,k)$, whence
$\mathrm{der}^s_{\gamma; c_1,\dots,c_k} \mathrm{ret}^{s,u}_{a_1,\dots,a_r}(G,f) = (G,f)$.

It may happen that the (n+1)-groups $\mathrm{der}^s_{b,\dots,b}\mathrm{Ret}^s_b(G,f)$ and (G,f) are non-isomorphic. As an example take the 13-group $(G,f) = \mathrm{der}^{12}_b(G,+)$ where $(G,+) = (Z_{12},+)$ is the cyclic group of order 12 and $b = 3$. Then $f(x_1^{13}) = x_1 + \dots + x_{13}+3$. Consider the 5-group $(G,g) = \mathrm{Ret}^3_b(G,f)$, whence $g(x_1^5) = x_1+\dots+x_5+3$. Let the 13-group $(G,f') = \mathrm{der}^3_{b,\dots,b}(G,g)$. Then $f'(x_1^{13}) = g_{(4)}(x_1^{13}, \overset{(4)}{b}) = x_1 + \dots + x_{13}$. Thus the 13-group (G,f'), being idempotent, is not isomorphic, to the 13-group (G,f) which is not idempotent.

Hosszú has proved in [5] that (using our notation) an (n+1)-groupoid (G,f) is an (n+1)-group if and only if $(G,f) = \mathrm{der}^n_{\gamma;c}(G,g)$ for some (binary) group (G,g). From Theorem 3 and Theorem 5 we can draw a corollary being a generalization of Hosszú Theorem. Note that in the case of $k = 1$ (hence $s = n$) Corollary 4 becomes Hosszú Theorem (cf. [5]).

C o r o l l a r y 4. An (n+1)-groupoid (G,f) is an (n+1)-group if and only if f is of the form

$f(x_1^{n+1}) = g_{(s+1)}(x_1, \gamma(x_2), \dots, \gamma^n(x_{n+1}), c_1^k)$, where (G,g) is a (k+1)-group and γ is an automorphism of (G,g) such that $g(\gamma^n(x), c_1^k) = g(c_1^k, x)$ and $\gamma(c_i) = c_i$ for $i = 1,\dots,k$.

From Theorem 5 it follows immediately

C o r o l l a r y 5. Every (n+1)-group is $\langle\gamma; c_1^k\rangle$-derived from a (k+1)-group.

In the proof of Theorem 5 the (k+1)-ary retract of the (n+1)-group (G,f) has been chosen to obtain the needed (k+1)-group (G,g). The question arises whether such a choice of the (k+1)-group is unavoidable. In other words - is the following theorem: "If an (n+1)-group is of the form $(G,f) = \operatorname{der}^{s}_{\gamma;c_1,\dots,c_k}(G,g)$, then the (k+1)-group (G,g) is isomorphic to the (k+1)-group $\operatorname{ret}^{s,u}_{a_1,\dots,a_r}(G,f)$". true for $k = 1,2,\dots$?

In the case of $k = 1$ the answer is positive (cf. [14]).

P r o p o s i t i o n 3. Let a (binary) group (G,g) be given. To every (n+1)-group $\operatorname{der}^{n}_{\gamma;c}(G,g)$ there exist elements $a_1,\dots,a_{n-1} \in G$ such that $\operatorname{ret}^{n}_{a_1,\dots,a_{n-1}}\operatorname{der}^{n}_{\gamma;c}(G,g) = (G,g)$. Nemaly, $a_1 = \dots = a_{n-2} = e$ (the neutral element of (G,g), $a_{n-1} = c^{-1}$ (the inverse of c in (G,g)).

P r o o f.

$$f(x,e,\dots,e,c^{-1},y) =$$

$$= g_{(n+1)}(x,\gamma(e),\gamma^{2}(e),\dots,\gamma^{n-2}(e),\gamma^{n-1}(c^{-1})\gamma^{n}(y),c) =$$

$$= g_{(n+1)}(x,e,\dots,e,c^{-1},c,y,c^{-1},c) = g(x,y),$$

which was to be proved.

It points out that in the case of $k = 1$ forming retracts is a procedure converse to that of forming weakly derived (n+1)-groups. It reminds of the situation in Theorem 5 and Proposition 2. But there the automorphism γ and elements $c_1,\dots,c_k$ had to be chosen in an appropriate way. Otherwise the (n+1)-groups der ret(G,f) and (G,f) appeared not necessarily isomorphic (see the example following Proposition 2). From Proposition 3 and Corollary 2 we can draw the following

C o r o l l a r y 6. The groups (G,g) and $\operatorname{ret}^{n}_{a_1,\dots,a_{n-1}}\operatorname{der}^{n}_{\gamma;c}(G,g)$ are always isomorphic (under an arbitrary choice of the elements $a_1,\dots,a_{n-1},c \in G$ and the automorhpism σ).

From Corollary 6 and the functoriality of retracts we obtain

P r o p o s i t i o n 4. If (n+1)groups $der^n_{\sigma_1;a_1}(G,g_1)$ and $der^n_{\sigma_2;a_2}(G,g_2)$ are isomorphic, then the groups (G,g_1) and (G,g_2) are isomorphic, too.

Now we show that in the case of $k > 1$ the answer is negative.

P r o p o s i t i o n 5. If an (n+1)-group (G,f) is derived from a (k+1)-group (G,g) (where $n = tk^2$, i.e. $s = tk$), then any (k+1)-ary retract of (g,f) has an idempotent element.

P r o o f . Given a (k+1)-group (G,g), let $(G,f) = \psi_s(G,g)$ (where $s = tk$), $(G,g') = Ret^s_a(G,f)$ (a being an arbitrary element of G). Let $\bar{a}$ denote the skew element to a in the (s+1)-group $\psi_t(G,g) = (G,g_{(t)})$. Then $g'\left(\overset{(k+1)}{\bar{a}}\right) = f\left(\bar{a}, \overset{(s-1)}{a}, \bar{a},\dots,\bar{a}, \overset{(s-1)}{a}, \bar{a}\right) = \bar{a}$, which proves that $\bar{a}$ is an idempotent element in (G,g'). By Theorem 1 all (k+1)-ary retracts of a given (n+1)-group are isomorphic. Thus every (k+1)-retract of the (n+1)-group $(G,f) = \psi_s(G,g)$ has an idempotent element.

P r o p o s i t i o n 6. If $(G,\cdot)$ is a group of exponent k, then for any element b different from the neutral one the (k+1)-group $(G,g) = der^k_b(G,\cdot)$ has no idempotent element.

P r o o f . Let $(G,g) = der^k_b(G,\cdot)$ and x be an arbitrary element of G. Then $g\left(\overset{(k+1)}{x}\right) = \underbrace{x\cdot\ldots\cdot x}_{k+1}\cdot b = x\cdot b$. Hence $g\left(\overset{(k+1)}{x}\right) \neq x$ (since, by assumption, $b \neq e$), which was to be proved.

From Proposition 5 and Proposition 6 we obtain

C o r o l l a r y 7. Let $k > 1$ and $n = tk^2$ (i.e. $s = tk$). Then there exists an (n+1)-group of the form

$(G,f) = \mathrm{der}^{s}_{b,\ldots,b,\bar{b}}(G,g)$ such that the $(k+1)$-groups (G,g) and $\mathrm{ret}^{s}_{a_1,\ldots,a_r}(G,f)$ are never isomorphic.

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