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SOME PROPERTIES OF CERTAIN PROCESSES WITH MEMORY

1. Introduction

In the paper [2] the notion of (Ω,k)-system and (Ω,k)-process is introduced and also some properties of (Ω,k)-processes are examined. Now we recall the basic definitions from this paper.

Let k be an arbitrary positive number, Ω - a real function defined on the interval $\langle 0;k\rangle$ and having a bounded variation on this interval. We assume that

$$(1) \qquad \underset{t^*\in(0;k\rangle}{\exists} \ \underset{\langle k-t^*,k\rangle}{\mathrm{Var}} \ \Omega(s) < 1.$$

Let $\mathbb{C}_{\langle 0;k\rangle}$ ($\mathbb{C}_{\langle 0;+\infty)}$) be the set of all real continuous functions, having as a common domain $\langle 0;k\rangle$ ($\langle 0;+\infty)$ respectively). Moreover, let $\tilde{\mathbb{C}}_{\langle 0;k\rangle}$ be the set of all functions $f\in\mathbb{C}_{\langle 0;k\rangle}$ satisfying the condition

$$(2) \qquad \int_0^k f(s)\,d\Omega(s) = f(k).$$

The integral on the left-hand side of the above condition (and also other integrals in this paper) is understood as a Riemann-Stieltjes integral.

D e f i n i t i o n . By an (Ω,k)-system we mean an operator

$$M : \tilde{\mathbb{C}}_{\langle 0;k\rangle} \to \mathbb{C}_{\langle 0;\, +\infty)}$$

such that for an arbitrary function $f \in \tilde{\mathbb{C}}_{\langle 0;k\rangle}$ the following conditions are satisfied:

$$(3) \qquad (Mf)|_{\langle 0;k\rangle} = f$$

and

$$(4) \qquad (Mf)(t+k) = \int_0^k (Mf)(t+s)d\Omega(s), \quad (t \geqslant 0).$$

As it is known ([2]), the operator M is well-defined.

Any function belonging to the range of an (Ω,k)-system is said to be an (Ω,k)-process.

Let x be an (Ω,k)-process. We can write the equation (4) in the form

$$(5) \qquad x(t+k) = \int_0^k x(t+s)d\Omega(s).$$

For each $t_0 \geqslant 0$ the function $x|_{\langle t_0;t_0+k\rangle}$ is said to be a state of the (Ω,k)-system. In particular, the state $x|_{\langle 0;k\rangle}$ is called the initial state of (Ω,k)-system.

Let Q be a set of functions $f : \langle 0;k\rangle \to \mathbb{R}$ such that
1) f is a non-decreasing function on the interval $\langle 0;k\rangle$
2) there exists a point $c \in (0;k)$ such that $f(0) < f(c)$
3) there exists a point $t^* \in (0;k)$ such that $f(k)-f(t^*) < 1$.

L e m m a 1. If $\Phi \in Q$ then there exists exactly one real number λ such that

$$(6) \qquad \int_0^k e^{\lambda(s-k)}d\Phi(s) = 1.$$

r o o f . The function $h(\lambda) = \int_0^k e^{\lambda(s-k)} d\Phi(s)$ is continuous and decreasing on $\mathbb{R}$. Moreover, $\lim_{\lambda \to -\infty} h(\lambda) = +\infty$ and, by condition 3), $h(\lambda) < 1$ for large positive λ.

Note that

$$\int_0^k d\Phi(s) = 1 \Longrightarrow \lambda = 0$$

$$\int_0^k d\Phi(s) < 1 \Longrightarrow \lambda < 0$$

$$\int_0^k d\Phi(s) > 1 \Longrightarrow \lambda > 0.$$

R e m a r k 1. Denote $w = \int_0^k d\Phi(s)$. If λ is a real solution of the equation (6), then

(7) $$|\lambda| \geqslant \left|\frac{1}{k} \ln w\right|.$$

R e m a r k 2. Let $\Phi_i \in Q$ $(i=1,2)$. If λ_i is a real solution of the equation (6) with $\Phi = \Phi_i$ and $(\Phi_2 - \Phi_1)$ is a non-decreasing function then $\lambda_1 \leqslant \lambda_2$. If, moreover, $(\Phi_2 - \Phi_1) \in Q$, then $\lambda_1 < \lambda_2$.

2. Absolute value estimation of an arbitrary (Ω,k)-process

Assume that

(8) $$\exists_{c \in (0;k)} \quad \Omega(0) - \Omega(c) \neq 0.$$

Denote

$$(9)\qquad \Phi(t) = \begin{cases} 0 & \text{for} \quad t = 0 \\ \operatorname*{Var}_{\langle 0;t\rangle} \Omega(s) & \text{for} \quad t \in (0;k\rangle . \end{cases}$$

The function Φ is non-decreasing on $\langle 0;k\rangle$ and satisfies the condition 2), 3) from the definition of the set Q. Hence we have $\Phi \in Q$.

T h e o r e m 1. Suppose that the function Ω satisfies the condition (8). If x is (Ω,k)-process and λ^* is a real solution of the equation (6) with the function Φ defined by (9), then

$$(10)\qquad |x(t)| \leqslant e^{\lambda^* t}\cdot\beta, \quad \text{for} \quad t \geqslant 0$$

where $\beta = \max\limits_{\langle 0;k\rangle} e^{-\lambda^* t}\cdot|x(s)|$.

P r o o f. Denote $y(t) = e^{-\lambda^* t}x(t)$ and $M = \max\limits_{\langle 0;k\rangle}|y(t)|$. Hence we have to show that $|y(t)| \leqslant M$ for every $t > k$.

Suppose that this is not true. Then there exists $t_1 > k$ such that $|x(t_1)| > M$. Denoting by τ the lower bound of the set of all $t \geqslant 0$ such that $|x(t)| > |x(t_1)|$ we have $|x(\tau)| = |x(t_1)|$. Hence we have $\tau > k$ and

$$|x(t)| < |x(\tau)| \quad \text{for} \quad t < \tau .$$

Since

$$(11)\qquad y(\tau) = \int_0^k y(\tau-k+s)\cdot e^{\lambda^*(s-k)}d\Omega(s)$$

then

$$|y(\tau)|\cdot\int_0^k e^{\lambda^*(s-k)}d\Phi(s) = |y(\tau)| \leqslant \int_0^k |y(\tau-k+s)|\cdot e^{\lambda^*(s-k)}d\Phi(s).$$

Hence we have

$$\int_0^k e^{\lambda^*(s-k)}\ (|y(\tau)| - |y(\tau-k+s)|)d\Phi(s) \leqslant 0.$$

This result contradicts the fact that

$$0 < \int_0^c e^{\lambda^*(s-k)}\cdot(|y(\tau)| - |y(\tau-k+s)|)d\Phi(s) \leqslant$$

$$\leqslant \int_0^k e^{\lambda^*(s-k)}\cdot(|y(\tau)| - |y(\tau-k+s)|)d\Phi(s),$$

where $c \in (0;k)$ is a number from the condition (8).

Let us note that if the function Ω does not satisfy the condition (8), then the equation (6) has no real solutions and $x(t) = 0$ for $t \geqslant k$. In this case the estimation (10) holds for an arbitrary real number λ^*.

The estimation (10) and the fact that $\int_0^k d\Phi(s) = \underset{\langle 0;k\rangle}{\mathrm{Var}}\ \Omega(s)$ imply the following properties of (Ω,k)-processes:

C o r o l l a r y 1. Denote $V = \underset{\langle 0;k\rangle}{\mathrm{Var}}\ \Omega(s)$.

a) Let $V = 1$. Then we have $\lambda^* = 0$. All (Ω,k)-processes are functions bounded on the interval $\langle 0;+\infty)$ and

$$|x(t)| \leqslant \max_{\langle 0;k\rangle} |x(s)|.$$

b) Let $V < 1$. Then we have $\lambda^* < 0$. All (Ω,k)-processes tend to zero, when $t \to \infty$. The rapidity of this convergence is at least $O(e^{\lambda^* t})$. Moreover, for (7) we have

$$|x(t)| \leqslant V^{\frac{t}{k}}\cdot\beta.$$

c) Let $V > 1$. Then we have $\lambda^* > 0$. The absolute value of an arbitrary (Ω,k)-process increase when $t \to \infty$ not faster then $O(e^{\lambda^* t})$.

3. Properties of (Ω,k)-processes if $\Omega \in Q$

Theorem 2. Let $\Omega \in Q$. If x is a (Ω,k)-process, $\bar{\lambda}$ is a real solution of the equation (6) with $\Phi = \Omega$, then for every $t_0 \geq 0$

$$(12) \qquad e^{\bar{\lambda}t}\cdot\alpha_1(t_0) \leq x(t) \leq e^{\bar{\lambda}t}\cdot\beta_1(t_0), \quad \text{for} \quad t \geq t_0,$$

where $\alpha_1(t_0) = \min\limits_{\langle t_0;t_0+k\rangle} e^{-\bar{\lambda}s}\cdot x(s)$, $\beta_1(t_0) = \max\limits_{\langle t_0;t_0+k\rangle} e^{-\bar{\lambda}s}x(s)$.

The proof of this theorem is analogical to the proof of Theorem 1.

Remark 3. It is easily seen that functions on the both sides of the inequality (12) are (Ω,k)-processes. This is the best estimation of (Ω,k)-processes.

From Theorem 2 we have

Corollary 2. If there exists a number $t_0 \geq 0$ such that the state $x|_{\langle t_0;t_0+k\rangle}$ is non-negative (non-positive), then the function x is non-negative (non-positive) on the interval $\langle t_0;+\infty)$.

Corollary 3. Let $V = \int_0^k d\Omega(s) > 1$. If there exists a number $t_0 \geq 0$ such that $x(t) \neq 0$ for $t \in \langle t_0;t_0+k\rangle$, then the rapidity of increase of $|x(t)|$ as $t \to +\infty$, is at least $O\left(V^{\frac{t}{k}}\right)$.

Denote $y(t) = e^{-\bar{\lambda}t}\cdot x(t)$. The function y satisfies the equation

$$(13) \qquad y(t+k) = \int_0^k y(t+s)e^{\bar{\lambda}(s-k)}d\Omega(s), \quad \text{for} \quad t \geq 0.$$

L e m m a 2. For an arbitrary $\tau \geq 0$ the equality

(14) $$\int_0^k e^{\bar{\lambda}s} d\Omega(s) \int_s^k y(t+\tau)dt = \int_0^k e^{\bar{\lambda}s} d\Omega(s) \int_s^k y(t)dt$$

holds.

P r o o f . From the equality (13) we have

(15) $$\int_0^\tau y(t+k)dt = \int_0^\tau dt \int_0^k y(t+s)e^{\bar{\lambda}(s-k)} d\Omega(s).$$

Since

$$\int_0^\tau dt \int_0^k y(t+s)e^{\bar{\lambda}(s-k)} d\Omega(s) = \int_0^k e^{\bar{\lambda}(s-k)} d\Omega(s) \int_0^\tau y(t+s)dt =$$

$$= \int_0^k e^{\bar{\lambda}(s-k)} d\Omega(s) \int_s^{s+\tau} y(t)dt = \int_0^k e^{\bar{\lambda}(s-k)} d\Omega(s) \left(\int_k^{k+\tau} y(t)dt + \right.$$

$$\left. + \int_s^k y(t)dt - \int_{s+\tau}^{k+\tau} y(t)dt \right) = \int_0^\tau y(t+k)dt + \int_0^k e^{\bar{\lambda}(s-k)} d\Omega(s) \int_s^k y(t)dt -$$

$$- \int_0^k e^{\bar{\lambda}(s-k)} d\Omega(s) \int_s^k y(t+\tau)dt,$$

then the equality (15) implies (14).

Note, that if $\Omega \in Q$, then $\int_0^k (k-s)e^{\bar{\lambda}s} d\Omega(s) \neq 0$.

Let g be a number defined by

(16) $$g = \frac{\int_0^k e^{\bar{\lambda}s} d\Omega(s) \int_s^k e^{-\bar{\lambda}t} x(t)dt}{\int_0^k (k-s)e^{\bar{\lambda}s} d\Omega(s)},$$

and $u(t) = e^{-\bar{\lambda}t}x(t)-g$ for $t \geq 0$. From (16) we have

$$\int_0^k e^{\bar{\lambda}s}\, d\Omega(s) \int_s^k u(t)dt = 0.$$

Hence, taking into account Lemma 2, we have for $\tau \geq 0$

$$(17) \qquad \int_0^k e^{\bar{\lambda}s}\, d\Omega(s) \int_s^k u(t+\tau)dt = 0.$$

We have

$$(18) \qquad 0 = \int_0^k dG(s) \int_s^k u(t+\tau)dt = -\int_0^k G(s)d\left[\int_s^k u(t+\tau)dt\right] =$$

$$= \int_0^k G(s)u(s+\tau)ds \qquad \text{for} \quad \tau \geq 0$$

where

$$(19) \qquad G(s) = \begin{cases} 0 & \text{for} \quad s = 0 \\ \int_0^s e^{\bar{\lambda}t}\, d\Omega(t) & \text{for} \quad s \in (0;k\rangle . \end{cases}$$

T h e o r e m 3. Let $\Omega \in Q$ and let g be the number defined by (16). If x is an (Ω,k)-process, $\bar{\lambda}$ is a real solution of the equation (6) with $\Phi = \Omega$, then the function $u(t) = e^{-\bar{\lambda}t}\cdot x(t)-g$ have zeros in each interval which length is k.

Moreover, if we assume that Ω is an increasing function on $\langle 0;k\rangle$, then the function u changes its sign in each interval having length k provided that it is not identically zero.

P r o o f of this theorem is obtained from the given above equality

$$\int_0^k G(s)u(s+\tau)ds = 0, \quad \text{for} \quad \tau \geq 0$$

and the following facts:

a) if $\Omega \in Q$, then $G(s) \geq 0$ for $s \in \langle 0;k\rangle$ and $G(s) > 0$ for $s > c$, where $c \in (0;k)$ is a number from the definition of the set Q,

b) if Ω is a function increasing on $\langle 0;k\rangle$, then $G(s) > 0$ for $s \in \langle 0;k\rangle$.

R e m a r k 4. If $\Omega \in Q$, then $\tilde{\lambda} = \lambda^*$.

4. Properties of (Ω,k)-processes on the interval on which they do not change sign

Let Ω_1, Ω_2 be the following non-decreasing functions on $\langle 0;k\rangle$

$$(20) \quad \Omega_1(s) = \frac{1}{2}\cdot\left[\operatorname*{Var}_{\langle 0;s\rangle} \Omega(z) + \Omega(s)\right], \quad \Omega_2(s) = \frac{1}{2}\cdot\left[\operatorname*{Var}_{\langle 0;s\rangle} \Omega(z) - \Omega(s)\right].$$

From the Jordan theorem we have

$$(21) \quad \Omega(s) = \Omega_1(s) - \Omega_2(s), \quad s \in \langle 0;k\rangle .$$

Taking into account the above equality we can write the equation (5) in the form

$$(22) \quad x(t+k) = \int_0^k x(t+s)d\Omega_1(s) - \int_0^k x(t+s)d\Omega_2(s).$$

T h e o r e m 4. Assume that $\Omega_1 \in Q$. Let x be an (Ω,k)-process, $\tilde{\lambda}$ a real solution of the equation (6) with $\Phi = \Omega_1$. If there exists $a \geqslant 0$ such that x is a non--decreasing function on $\langle a;+\infty)$, then for each $t_0 \geqslant a$ we have

$$(23) \qquad x(t) \leqslant e^{\tilde{\lambda}t}\cdot\beta(t_0) \quad \text{for} \quad t \geqslant t_0,$$

where $\beta(t_0) = \max\limits_{\langle t_0;t_0+k\rangle} e^{-\tilde{\lambda}s}x(s)$.

P r o o f of this theorem follows on substituting the condition (11) in the proof of Theorem 1 by the inequality

$$y(\tau) \leqslant \int_0^k y(\tau-k+s)e^{\tilde{\lambda}(s-k)}d\Omega_1(s), \quad \text{for} \quad t \geqslant a$$

where $y(t) = e^{-\tilde{\lambda}t}x(t)$.

T h e o r e m 5. If Ω_1, x and $\tilde{\lambda}$ satisfy suppositions of Theorem 4 and if there exists a finite limit

$$\lim_{t\to\infty} e^{-\tilde{\lambda}t}x(t) = g,$$

then there does not exist an interval $\langle t_0;t_0+k\rangle$ $(t_0 \geqslant a)$ such, that function $u(t) = e^{-\tilde{\lambda}t}x(t)-g$ is negative on this interval.

P r o o f of this theorem is obtained immediately from the estimation

$$u(t) \leqslant \max_{\langle t_0;t_0+k\rangle} u(s), \quad \text{for} \quad t \geqslant t_0,$$

where t_0 is an arbitrary number from the interval $\langle a;+\infty)$.

Let us now consider the case x is a non-positive function on the interval $\langle a,+\infty)$, where $a \geqslant 0$. Then the function $-x$, being also an (Ω,k)-process, is non-negative on $\langle a;+\infty)$. Taking into account Theorems 4 and 5 we have

T h e o r e m 6. Suppose that $\Omega_1 \in Q$. Let x be an (Ω,k)-process and $\tilde{\lambda}$ a real solution of the equation (6) with $\Phi = \Omega_1$. If there exists $a \geqslant 0$ such that x is a non-positive function on $\langle a,+\infty)$, then for each $t_0 \geqslant a$ we have

$$(24) \qquad e^{\tilde{\lambda}t}\cdot\alpha(t_0) \leqslant x(t), \quad \text{for} \quad t \geqslant t_0,$$

where $\alpha(t_0) = \min\limits_{\langle t_0;t_0+k\rangle} e^{-\tilde{\lambda}s}x(s)$.

T h e o r e m 7. If Ω_1, x and $\tilde{\lambda}$ satisfy the suppositions of theorem 6 and if there exists a finite limit

$$\lim_{t\to\infty} e^{-\tilde{\lambda}t}x(t) = g$$

then there does not exist an interval $\langle t_0;t_0+k\rangle$ $(t_0 \geqslant a)$ such that the function $u(t) = e^{-\tilde{\lambda}t}x(t)-g$ is positive on this interval.

R e m a r k 5. As it is known, the decomposition of the function Ω

$$\Omega(s) = \Omega_1(s) - \Omega_2(s)$$

where Ω_1, Ω_2 are non-decreasing functions, is not unique. The functions defined by (20) are the most slowly increasing among all functions Ω_1, Ω_2. This fact and Remark 2 implies that a real solution of the equation (6), with $\Phi = \Omega_1$ defined by (20), is the least number of this kind.

R e m a r k 6. Let x be an (Ω,k)-process not changing its sign on interval $\langle 0;+\infty)$. From the estimation (23), (24) we have

$$|x(t)| \leqslant e^{\tilde{\lambda}t}\cdot\bar{\beta}, \quad \text{for} \quad t \geqslant 0,$$

where $\beta = \max\limits_{\langle 0;k\rangle} e^{-\tilde{\lambda}s}\cdot|x(s)|$. The above estimation of the function x on the interval $\langle 0;+\infty)$ is not worse (but frequently better) than the estimation (10). This follows from the following facts

a) $\tilde{\lambda} \leq \lambda^*$

b) if $\Omega_2 \in Q$, then $\tilde{\lambda} < \lambda^*$.

R e m a r k 7. If the function Ω can be given by an integral

$$\Omega(t) = a + \int_0^t \alpha(s)ds, \quad t \in \langle 0;k \rangle ,$$

where α is an integrable function (in Lebesque's sense) on $\langle 0;k\rangle$ and a is a constant, then

$$\operatorname*{Var}_{\langle 0;s\rangle} \Omega(s) = \int_0^s |\alpha(z)|\, dz, \quad \Omega_1(s) = \frac{1}{2}\int_0^s (|\alpha(z)| - \alpha(z))dz.$$

5. Limit of (Ω,k)-process if $t \to \infty$

It is proved in the paper [2] that if for at least one (Ω,k)-process x there exists a finite and non-zero limit $\lim_{t\to\infty} x(t) = g$ then

$$\int_0^k d\Omega(s) = 1. \tag{25}$$

T h e o r e m 8. Suppose that the function Ω satisfies the condition (25) and

$$\int_0^k (k-s)d\Omega(s) \neq 0. \tag{26}$$

If x is an (Ω,k)-process and there exists a finite limit $\lim_{t\to\infty} x(t) = g$, then

$$g = \frac{\int_0^k d\Omega(s) \int_s^k x(t)dt}{\int_0^k (k-s)d\Omega(s)} . \tag{27}$$

P r o o f . Writting the function x in the form $x(t) = g+\varepsilon(t)$ we have $\lim_{t\to\infty} \varepsilon(t) = 0$. The function x satisfies the equation (14) when $\bar{\lambda} = 0$. This implies

$$\int_0^k d\Omega(s) \int_s^k x(t)dt = g \int_0^k (k-s)d\Omega(s) + \int_0^k d\Omega(s) \int_s^k \varepsilon(t+\tau)dt, \quad \tau > 0.$$

Passing in above equality to the limit as $t \to \infty$, we get (27).

Suppose that the function Ω satisfies the conditions (25), (26). Let g be a number defined by (27). Taking into account results obtained in section 3 we conclude that the function $u(t) = x(t)-g$ satisfies the equation

$$\int_0^k u(s+t)d\left[\int_0^s (\Omega(z) - \Omega(0))dz\right] = 0, \quad t \geq 0.$$

Since

$$u(t+k) = \int_0^k u(s+t)d\Omega(s), \quad t \geq 0$$

then for an arbitrary real number ξ we have

$$u(t+k) = \int_0^k u(s+t)d\psi_\xi(s), \quad t \geq 0$$

where

$$(28) \qquad \psi_\xi(s) = \Omega(s) + \xi \int_0^s (\Omega(z)-\Omega(0))dz.$$

This fact and the Corollary 1b) imply

T h e o r e m 9. Suppose that function Ω satisfies the conditions (25), (26). If there exists a number ξ such that $\operatorname{Var}_{\langle 0;k\rangle} \psi_\xi(s) < 1$, then for every (Ω,k)-process x there exists a finite limit $\lim_{t\to\infty} x(t)$.

Theorem 10. Suppose that the function Ω can be given in the form

$$\Omega(s) = a + \int_0^s \alpha(t)dt + \gamma(s),$$

where α is an integrable (in Lebesque's sense) function on $\langle 0;k\rangle$, a is a constant and γ is a function of jumps of the function $\Omega(\gamma(0) = 0)$. Let the function Ω satisfy the condition (25). If α is a function positive almost everywhere on the interval $\langle 0;k\rangle$ and γ is non-decreasing function on the interval $\langle 0;k\rangle$ then there exists a number ξ such that $\operatorname{Var}_{\langle 0;k\rangle} \psi_\xi(s) < 1$.

Proof. Writting the function $\psi_\xi(s)$ in the form

$$\psi_\xi(s) = \int_0^s \left[\alpha(t) + \xi\cdot\int_0^t \alpha(z)dz + \xi\cdot\gamma(t)\right] dt + \gamma(s) + a$$

we have

$$\operatorname{Var}_{\langle 0;k\rangle} \psi_\xi(s) = \int_0^k \left|\alpha(t)+\xi\cdot\int_0^t \alpha(z)dz + \xi\cdot\gamma(t)\right| dt + \gamma(k).$$

Consider now the function φ of real variable ξ defined by

$$\varphi(\xi) = \int_0^k \left|\alpha_\xi(s)\right| ds,$$

where $\alpha_\xi(s) = \alpha(s) + \xi\cdot\int_0^s \alpha(z)dz + \xi\cdot\gamma(s)$. We will show that this function has the derivative at the point $\xi = 0$ and

$$(29) \qquad \varphi'(0) = \int_0^k dt\left(\int_0^t \alpha(z)dz + \gamma(t)\right).$$

We have

$$(30) \qquad \frac{\varphi(\xi)-\varphi(0)}{\xi} = \int_0^k \frac{|\alpha_\xi(s)-\alpha_0(s)|}{\xi}\, ds$$

and for every $s \in \langle 0;k\rangle$

$$\left|\frac{|\alpha_\xi(s)|-|\alpha_0(s)|}{\xi}\right| \leqslant \int_0^s \alpha(z)dz + \gamma(s).$$

For almost all s from $\langle 0;k\rangle$ (namely for all s such that $\alpha(s) > 0$) we have

$$\lim_{\xi\to 0} \frac{|\alpha_\xi(s) - \alpha_0(s)|}{\xi} = \int_0^s \alpha(z)dz.$$

From (30) and Lebesque's theorem (on the passing to the limit under the integral) it follows that there exists $\varphi'(0)$ having the form (29).

From (29) it follows that $\varphi'(0) > 0$. Hence, we have

$$\operatorname*{Var}_{\langle 0;k\rangle} \varphi_\xi(s) = \varphi(\xi) + \gamma(k) < \varphi(0) + \gamma(k) = \int_0^k \alpha(t)dt+\gamma(k) = 1$$

for negative, sufficiently near zero numbers ξ.

R e m a r k 9. The suppositions about the function α from theorem 10 can be weakened. Let R be a can interval (sum of intervals) on which $\alpha(t) \equiv 0$. We have

$$\varphi(\xi) = \int_{\langle 0;k\rangle - R} \left| \alpha(t) + \xi \int_0^t \alpha(z)dz + \xi \cdot \gamma(t) \right| dt +$$

$$+ |\xi| \cdot \int_R \left| \int_0^t \alpha(z)dz + \gamma(t) \right| dt.$$

If we suppose that the function α is positive almost everywhere on $(\langle 0;k\rangle - R)$, then

$$\varphi'(0-) = \int_{\langle 0;k\rangle - R} dt \left(\int_0^t \alpha(z)dz + \gamma(t) \right) - \int_R dt \left(\int_0^t \alpha(z)dz + \gamma(t) \right).$$

This implies the condition $\varphi'(0-) > 0$.

R e m a r k 10. In particular case, if Ω is a function of jumps ((Ω,k)-process after reduction to the set of natural numbers is $(\vec{\alpha},k)$-computation), then the properties of (Ω,k)-processes presented in this paper are known (cf. [1], [3]).

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