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APPLICATION OF POLYNOMIALS TO INVESTIGATIONS OF MÖBIUS ALGEBRAS

The main objects of investigations in the branch of combinatorial theory dealing with ordered sets are real-valued functions of one variable x or two variables x,y satisfying $x \leqslant y$ ranging over an ordered set under consideration. Two-variable functions form the incidence algebra of the set (see [1]). One-variable functions form the Möbius algebra (see [2]) and they are the subject of this article. We propose to introduce a method of investigating such functions by means of polynomials.

Let $(P, \leqslant)$ be a locally finite partially ordered set with the least element ϑ. To any real-valued function $f\colon P \to R$ we assign the following family of polynomials

$$W_p(x) = \sum_{r \leqslant p} f(r)\, x^{\varrho(r,p)},$$

where the index p ranges over the set P (we might also add f as a second index), and ϱ denotes the rank function ($\varrho(r,p)$ is the maximal length of a chain from r to p minus 1; we may notice that st $W_p \leqslant \varrho(\vartheta, p)$).

To show the possibilities of such an idea we shall present two theorems.

T h e o r e m (the inverse formula for introduced polynomials). Let the ordered set $(P, \leqslant)$ have the property

$$\varrho(p,q) + \varrho(q,r) = \varrho(p,r)$$

for any $p \leqslant q \leqslant r$ and $f:P \to R$ be a real-valued function on P. For any $p \in P$

$$f(p) = \sum_{r \leqslant p} \mu(r,p) x^{\varrho(r,p)} W_r(x),$$

where $W_r(x)$ are polynomials defined above for function f.

P r o o f . We denote

$$V_r(x) = x^{\varrho(r)} W_r\left(\frac{1}{x}\right)$$

- another family of polynomials, $\varrho(r) = \varrho(\vartheta, r)$. For any $p \in P$ we have

$$V_p(x) = x^{\varrho(p)} \sum_{r \leqslant p} f(r) \left(\frac{1}{x}\right)^{\varrho(p)-\varrho(r)} = \sum_{r \leqslant p} f(r) x^{\varrho(r)}$$

for $\varrho(r,p) = \varrho(p) - \varrho(r)$. We apply the Möbius inverse formula and obtain

$$f(p) x^{\varrho(p)} = \sum_{r \leqslant p} \mu(r,p)\, V_r(x) = \sum_{r \leqslant p} \mu(r,p) x^{\varrho(r)} W_r\left(\frac{1}{x}\right).$$

Putting $\frac{1}{x}$ instead of x we get

$$f(p) = \sum_{r \leqslant p} \mu(r,p) x^{\varrho(p)-\varrho(r)} W_r(x)$$

and use the equality $\varrho(p) - \varrho(r) = \varrho(r,p)$ again.

In the following theorem we shall use the denotation: for $t, r \in P$

$t \to r \Leftrightarrow t < r$ and the interval $[t,r]$ is a chain.

T h e o r e m . Let the ordered set $(P, \leqslant)$ have the properties

$$\varrho(p,q) + \varrho(q,r) = \varrho(p,r)$$

for $p \leqslant q \leqslant r$ and

$$\left|\{t : q \leqslant t \to r\}\right| = \varrho(q,r)$$

for $q \leqslant r$. For any function $f:P \to R$ and $p \in P$

$$W_p'(x) = \sum_{r \to p} x^{\varrho(r,p)-1}\, W_r(x)$$

(where $'$ denotes the derivative).

P r o o f . We have

$$\sum_{r \to p} x^{\varrho(r,p)-1} W_r'(x) = \sum_{r \to p} x^{\varrho(r,p)-1} \sum_{s \leqslant r} f(s) x^{\varrho(s,r)} =$$

$$= \sum_{s<p} \Big(\sum_{s \leqslant r \to p} 1 \Big) f(s) x^{\varrho(s,r)+\varrho(r,p)-1} =$$

$$= \sum_{s<p} \varrho(s,p) f(s) x^{\varrho(s,p)-1} = \Big(\sum_{s \leqslant p} f(s) x^{\varrho(s,p)} \Big)' = W_p'(x).$$

Special cases of this theorem are the following formulas

$$W_n'(x) = \sum_{k=0}^{n-1} x^{n-k-1} W_k(x),$$

where

$$W_n(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

for $n = 0,1,2,\ldots$ and

$$W'_A(x) = \sum_{a\in A} W_{A-\{a\}}(x),$$

where

$$W_A(x) = \sum_{B\subset A} f(B)x^{|A-B|}$$

for any finite subset A of a set C.

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