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COMPACTNESS AND UPPER SEMICONTINUITY OF SOLUTION SET OF GENERALIZED DIFFERENTIAL EQUATION IN SEPARABLE BANACH SPACE

1. Introduction

In this paper we study the generalized differential equation

$$(1)\qquad \begin{aligned} &\dot{x}(t) \in F(t,x(t)) \quad \text{for almost every} \quad t \in [0,T] \\ &x(t_0) = x_0;\ t_0 \in [0,T], \end{aligned}$$

where $F(t,x)$ is a compact convex subset of a separable Banach space. This paper is related to the previous paper (Ref. [1]) of this author, where the existence of solutions of (1), with compact, but not necessarily convex, right-hand side, have been considered. We will consider the equation (1) by the assumption that F satisfies the Carathéodory conditions and that there exists a Kamke function $\omega : [0,T]\times R^+ \to R^+$ such that

$$(2)\qquad \chi(F(T,b)) \leqslant \omega(t,\chi(B))$$

for each bounded set $B \subset X$ and almost all $t \in [0,T]$, where $\chi(A)$ denotes the ball measure of noncompactness of a bounded set $A \subset X$. We will prove that the set of all solutions of (1) is a compact subset of the Banach space C_T. From this it will follow that the solution set of (1) is upper semicontinuous with respect to $(t_0,x_0) \in M$, where M is a compact subset of $[0,T] \times X$. Compactness and upper semicontinuity of the solution set of the generalized differential equation in a Banach space have been considered by Tolsonogov (Ref. [2]). But he considered these problems, among others, by the assumption that for every $\varepsilon > 0$ there exists a closed set $E_\varepsilon \subset [0,T]$ with Lebesgue measure $\mu([0,T] \setminus E_\varepsilon) \geq \varepsilon$ such that

$$\lim_{\delta \to 0} \chi(F(T_{t,\delta},B)) \leq \omega(t,\chi(B))$$

for almost all $t \in E_\varepsilon$ and bounded $B \subset X$, where $T_{t,\delta} = (t-\delta,t+\delta) \cap [0,T]$.

The results of this paper generalized some results of J.L. Davy's (Ref. [3]).

2. Notations and fundamentals lemmas

Let $(X,|\cdot|)$ be a separable Banach space, μ the Lebesgue measure on the real line and let Conv(X) denote the metric space of nonempty compact convex subsets of X with the Hausdorff metric h defined by $h(A,B) = \max\{\sup_{x\in B} \alpha(x,A), \sup_{x\in A} \alpha(x,B)\}$, where $\alpha(x,C)$ denotes the distance of x from $C \in \mathrm{Conv}(X)$.

Let C_T and L_T denote, respectively the Banach space of all continuous or Bochner integrable mappings of $[0,T]$ into X with the usual norms $\|\cdot\|$ and $|\cdot|$.

By $\chi(B)$ we denote the ball measure of noncompactness of a bounded set $B \subset X$, defined by $\chi(B) = \inf\{r > 0 : B$ can be covered by finitely many balls of radius $\leq r\}$. It is a measure of noncompactness equivalent to the measure of noncompactness introduced by Kuratowski (Ref. [4] and [5]).

The measurability of single-valued and multivalued mappings we will mean as strong measurability. The following lemma was proved in the author paper (Ref. [1], Lemma 2.2).

L e m m a 2.1. Let $(X,|\cdot|)$ be a separable Banach space, (x_n) an integral bounded sequence of measurable functions of $[0,T]$ into X. Then $\psi(t) = \chi(\{x_n(t) : n \geq 1\})$ is measurable and

$$\chi\left(\left\{\int_E x_n(t)dt : n \geq 1\right\}\right) \leq \int_E \psi(t)dt$$

for each measurable set $E \subset [0,T]$.

We will need the following results presented by J.L.Davy (Ref. [3], Theorem 2.8) and Tolstonogov (Ref. [6], Lemma 2.2).

L e m m a 2.2. Let X be a metric space and Y a normed linear space. Suppose $F : X \to \mathrm{Comp}(Y)$ is upper semicontinuous at $x_0 \in X$, where $\mathrm{Comp}(Y)$ denotes the space of nonempty compact subsets of Y. If (x_k) is a sequence of X convergint to x_0, then

$$\bigcup_{i=1}^{\infty} \overline{\mathrm{co}} \bigcap_{k=i}^{\infty} F(x_k) \subseteq \overline{\mathrm{co}}F(x_0),$$

where $\overline{\mathrm{co}}A$ denotes the closed convex hull of $A \subset Y$.

L e m m a 2.3. Let X be a Banach space and $\Gamma:[0,T] \to \mathrm{Comp}(X)$ measurable multivalued mapping such that $h(\Gamma(t),\{0\}) \leq m(t)$ for almost every $t \in [0,T]$, where $m:[0,T] \to R$ is a Lebesgue integrable function. Then for every measurable set $E \subset [0,T]$ we have

$$\overline{\int_E \Gamma(t)dt} = \int_E \overline{\mathrm{co}}\, \Gamma(t)dt,$$

where the integral is meant in Aumann's sense.

We will now prove the following lemma.

L e m m a 2.4. Let $(X,|\cdot|)$ be a Banach space and suppose that (u_n) is an integral bounded sequence of measurable mappings from $[0,T]$ into X. Then

$$\int_E \bigcap_{i=1}^{\infty} \overline{co} \bigcup_{k=i}^{\infty} u_k(t)dt \subseteq \bigcap_{i=1}^{\infty} \overline{co} \bigcup_{k=i}^{\infty} \int_E u_k(t)dt$$

for each measurable set $E \subset [0,T]$.

P r o o f . Let $\Gamma_i(t) = \{u_i(t), u_i'(t), \dots\}$ for $i=1,2,\dots$ and $t \in [0,T]$. By the definition of the Aumann integral, we have $\int_E \Gamma_i(t)dt = \bigcup_{k=i}^{\infty} \int_E u_k(t)dt$ for $i \geq 1$. In virtue of Lemma 2.3, we have

$$\int_E \overline{co}\, \Gamma_i(t)dt = \overline{\int_E \Gamma_i(t)dt} = \overline{co} \int_E \Gamma_i(t)dt.$$

Hence, we get

$$\bigcap_{i=1}^{\infty} \int_E \overline{co}\, \Gamma_i(t)dt = \bigcap_{i=1}^{\infty} \overline{co} \int_E \Gamma_i(t)dt.$$

Since

$$\int_E \bigcap_{i=1}^{\infty} \overline{co}\, \Gamma_i(t)dt \subseteq \bigcap_{i=1}^{\infty} \int_E \overline{co}\, \Gamma_i(t)dt,$$

then

$$\int_E \bigcap_{i=1}^{\infty} \overline{co}\, \Gamma_i(t)dt \subseteq \bigcap_{i=1}^{\infty} \overline{co} \int_E \Gamma_i(t)dt$$

which completes the proof.

Now, we present the following extension of Aumann's result (Ref. [7]).

L e m m a 2.5. Let $(X, |\cdot|)$ be a Banach space and let (x_k) be a sequence of absolutely continuous functions $x_k : [0,T] \to X$ such that

(i) $x_k(t) \to x(t)$ as $k \to \infty$, where $x : [0,T] \to X$,
(ii) $|\dot{x}_k(t)| \leqslant m(t)$ for almost every $t \in [0,T]$,
where $m : [0,T] \to R$ is a Lebesgue measurable function. Then x is absolutely continuous and

$$\dot{x}(t) \in \bigcap_{i=1}^{\infty} \overline{co} \bigcup_{k=i}^{\infty} \dot{x}_k(t)$$

for almost every $t \in [0,T]$.

P r o o f . Let us observe that $\Gamma(t) = \bigcap_{i=1}^{\infty} \overline{co} \bigcup_{k=i}^{\infty} \dot{x}_k(t)$ is measurable and integrally bounded. Then there exists a measurable selector f of Γ (Ref. [8]). In virtue of Lemma 2.4, for every t_0, $t \in [0,T]$; $t_0 < t$, we have

$$\int_{t_0}^{t} f(s)ds \in \int_{t_0}^{t} \Gamma(s)ds \subseteq \bigcap_{i=1}^{\infty} \overline{co} \bigcup_{k=i}^{\infty} \int_{t_0}^{t} \dot{x}_k(s)ds = x(t) - x(t_0),$$

because $x_k(t) \to x(t)$ as $k \to \infty$. Therefore, x is absolutely continuous and $\dot{x}(t) \in \Gamma(t)$ for almost every $t \in [0,T]$. This completes the proof.

3. Compactness of solution set

Let us assume that $F : [0,T]\times X \to Conv(X)$ satisfies the Carathéodory conditions, i.e. that $F(\cdot,x)$ is measurable for fixed $x \in X$, $F(t,\cdot)$ is continuous for fixed $t \in [0,T]$ and there exists a Lebesgue integrable function $m : [0,T] \to R$ such that $h(F(t,x), \{0\}) \leqslant m(t)$ for $x \in X$ and almost all $t \in [0,T]$. Furthermore, suppose that condition (2) is satisfied. It can be proved (Ref. [1]), that if $(X, |\cdot|)$ is a separable Banach space, then the above conditions imply that for every $(t_0,x_0) \in [0,T]\times X$, there exists at least one solution of (1). Denote by $H(t_0,x_0)$ the set of all solutions of (1) corresponding to $(t_0,x_0) \in [0,T]\times X$. Furthermore, for a given nonempty set $M \subset [0,T]\times X$, let $H(M) = \bigcup\{H(t_0,x_0) : (t_0,x_0) \in M\}$.

We shall now prove the following theorem.

T h e o r e m 3.1. Let $(X,|\cdot|)$ be a separable Banach space and suppose that $F:[0,T]\times X\to \mathrm{Conv}(X)$ satisfies the Carathéodory conditions. If furthermore, F satisfies the condition (2), then for every nonempty compact set $M\subset[0,T]\times X$, the set $H(M)$ is a compact subset of C_T.

P r o o f . Let us observe that for each $x\in H(M)$ there is $(t_o,x_o)\in M$ such that $x(t_o)=x_o$, $\|x\|\leqslant|x_o|+\int_0^T m(t)dt$ and $|\dot{x}(t)|\leqslant m(t)$ for almost all $t\in[0,T]$. Then $H(M)$ is a bounded and uniformly equicontinuous subset of C_T. Let (x_n) be a sequence of $H(M)$ and let $A=\{x_n : n\geqslant 1\}$. Since $A\subset H(M)$, A is a bounded and uniformly equicontinuous subset of C_T, too. Let $\{(t_o^n,x_o^n)\}$ be a sequence of M such that $x_n(t_o^n)=x_o^n$. By the compactness of M, there exists a subsequence of $\{(t_o^n,x_o^n)\}$, say again $\{(t_o^n,x_o^n)\}$ and $(t_o,x_o)\in M$, such that $|t_o^n-t_o|+|x_o^n-x_o|\to 0$ as $n\to\infty$. For each $n\geqslant 1$ and $t\in[0,T]$ we have $x_n(t)=x_o^n+\int_{t_o^n}^{t} \dot{x}_n(s)ds$. Then

$$\chi(\{x_n(t) : n\geqslant 1\})\leqslant\chi\left(\left\{\int_{t_o^n}^{t}\dot{x}_n(s)ds : n\geqslant 1\right\}\right)\leqslant$$

$$\leqslant\chi\left(\left\{\int_{t_o^n}^{t_o}\dot{x}_n(s)ds : n\geqslant 1\right\}\right)+\chi\left(\left\{\int_{t_o}^{t}\dot{x}_n(s)ds : n\geqslant 1\right\}\right)$$

for $t\in[0,T]$. In a similar way, as in the previous paper of the author (Ref. [1]), we can show that $|t_o^n-t_o|\to 0$, and $|\dot{x}_n(t)|\leqslant m(t)$ for almost all t $[0,T]$, imply that $\chi\left(\left\{\int_{t_o^n}^{t_o}\dot{x}_n(s)ds : n\geqslant 1\right\}\right)\to 0$ as $n\to\infty$. Therefore, we have

$\chi(A(t)) \leqslant \int_0^t \omega(s,\chi(A(s))ds$. Then $\bar{A}$ is a compact subset of C_T. Suppose that (x_k) is a subsequence of (x_n) such that $\|x_k - x\| \to 0$ as $k \to \infty$, where $x \in C_T$. In virtue of Lemma 2.5, $\|x_k - x\| \to 0$ and $|\dot{x}_k(t)| \leqslant m(t)$ for almost every $t \in [0,T]$, imply that x is absolutely continuous and $\dot{x}(t) \in \bigcap_{i=1}^{\infty} \overline{co} \bigcup_{k=1}^{\infty} \dot{x}_k(t)$ for almost every $t \in [0,T]$. Hence and Lemma 2.2, it follows that $\dot{x}(t) \in F(t,x(t))$ for almost all $t \in [0,T]$. Since, $\|x_k - x\| \to 0$, $|t_0^k - t_0| + |x_0^k - x_0| \to 0$ as $k \to \infty$ and $x_k(t_0^k) = x_0^k$ for $k = 1,2,\dots$ imply that $x(t_0) = x_0$, then $x \in H(M)$ and the proof is complete.

4. Upper semicontinuity of the solution set

As a Corollary of Theorem 3.1, it follows that for every compact set $M \subset [0,T] \times X$, the mapping $H : M \to Comp(C_T)$ is upper semicontinuous. Let $S_r(z)$ be an open ball of C_T, centered at $z \in C_T$ and with the radius $r > 0$. Furthermore, for $A \subset C_T$ and $\varepsilon > 0$ let $A^\varepsilon = \bigcup_{x \in A} S_\varepsilon(x)$.

T h e o r e m 4.1. Let $(X, |\cdot|)$ be a separable Banach space and suppose that the assumptions of Theorem 3.1 are satisfied. Then for every nonempty compact set $M \subset [0,T] \times X$ the multivalued mapping $H : M \ni (t_0, x_0) \to H(t_0, x_0) \in Comp(C_T)$ is upper semicontinuous.

P r o o f . Let M be a given nonempty compact subset of $[0,T] \times X$ and $(t_0, x_0) \in M$. Assume that H is not upper semicontinuous at (t_0, x_0). Then there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, $H(S_\delta(t_0,x_0) \cap M) \not\subset H^{\varepsilon_0}(t_0,x_0)$. Choose $x_k \in H(S_{\frac{1}{k}}(t_0,x_0) \cap M)$ and $x_k \notin H^{\varepsilon_0}(t_0,x_0)$. Since $x_k \in H(M)$ for $k = 1,2,\dots$ and $H(M)$ is a compact subset of C_T, there exists a subsequence of (x_k), say again (x_k), and $x \in H(M)$ such that $\|x_k - x\| \to 0$ as $k \to \infty$. Furthermore, there is a sequence $\{(t_0^k, x_0^k)\}$ of $S_{\frac{1}{k}}(t_0,x_0) \cap M$ such that

$x_k(t_0^k) = x_0^k$ and $|t_0^k - t_0| + |x_0^k - x_0| \to 0$ as $k \to \infty$. Hence it is easy to see that $x(t_0) = x_0$. Thus, $x \in H(t_0, x_0)$. On the other hand, $x_k \notin H^{\varepsilon_0}(t_0, x_0)$ for each $k \geq 1$. Therefore, $x \notin H(t_0, x_0)$. From this contradiction we conclude that H is upper semicontinuous on M. This completes the proof.

R e m a r k 1. We can take in the above theorems, F such that $F(t,\cdot)$ is upper semicontinuous for fixed $t \in [0,T]$, instead of continuous.

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