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SOME CATEGORICAL PROPERTIES OF CONVEX PROCESSES

R.T. Rockafellar gave a definition of a convex process and studied this subject from mathematical and economic points of view [9], [10], [11], Some categorical properties of polyhedral convex processes are given in [13]. The purpose of this paper is to study the necessity of the assumption about polyhedrality of the convex processes, when we study them as a category.

We consider finite dimensional, real, linear spaces X,Y and their adjoint spaces X^*,Y^*. A convex cone in the space X is a set $G \subseteq X$ such that $G + G \subseteq G$ and $tG \subseteq G$ for any number $t \geq 0$ [10]. For any set $A \subseteq X$ by con A we denote a convex cone spanned over A, e.i:

$$\operatorname{con} A = \left\{ x \,\middle|\, x = \sum_{i=1}^{n} \alpha_i a_i, \quad a_i \in A,\ n \geq 1,\ \ \alpha_i \geq 0 \right\}.$$

A cone G is said to be a polyhedral convex cone if there exists a finite set $A \subseteq X$ such that $G = \operatorname{con} A$.

A multivalued mapping $T : X \to Y$ is said to be a convex (polyhedral convex) process if its graph:

$$G(T) = \left\{ (x,y) \,\middle|\, y \in T(x) \right\} \subseteq X \times Y$$

is a convex (polyhedral convex) cone [10], [11].

Note that every linear transformation is a polyhedral convex process.

If $G(T)$ is a closed convex cone, then we say that T is a closed convex process. Let us note that a polyhedral convex process is always a closed convex process.

In [13] we have introduced the category $\mathcal{P}$ whose the objects are finite dimensional real, linear spaces and the morphisms are polyhedral convex processes, defining the composition $ST : X \to Z$ of polyhedral convex processes $T : X \to Y$, $S : Y \to Z$ by

$$ST(x) = S(T(x)) = \bigcup_{y \in T(x)} S(y).$$

In the same way we obtain the category $\mathcal{C}$ with convex processes as the morphisms.

Denoting by $\mathcal{L}$ the category of linear transformations we then have

$$\mathcal{L} \subsetneq \mathcal{P} \subsetneq \mathcal{C}.$$

First, let us observe that the class $\bar{\mathcal{C}}$ of all closed convex processes, with ordinary composition of multifunctions, is not a category.

E x a m p l e 1. Let G be closed convex cone in the space R^3 defined as follows

$$G = \left\{ t(x_1, x_2, 1) \;\middle|\; t \geqslant 0,\ x_1^2 + x_2^2 \leqslant 2x_1 \right\}.$$

It is easy to see that for every linear transformation $f : X \to R$, convex process $T : X \to R^3$ defined as

$$T(x) = f(x) + G \quad \text{for} \quad x \in X$$

is a closed convex process.

Let $\pi : R^3 \to R^2$ be a projection on the space R^2, e.g.

$$\pi(x_1, x_2, x_3) = (x_1, x_2) \quad \text{for} \quad (x_1, x_2, x_3) \in R^2,$$

then the composition $\pi T : X \to R^2$ of these closed convex processes is not closed, because the set

$$(\{0\} \times R^2) \cap G(\pi T) = \pi T(0) = \pi(G) = \{(x_1,x_2) | x_1 > 0\} \cup \{(0,0)\}$$

is not closed.

We also do not obtain good results, if we change the definition of composition in the class $\overline{\mathcal{C}}$, such that in the cases of the categories $\mathcal{L}$ and $\mathcal{P}$ it is ordinary composition.

E x a m p l e 2. For $T_1 : X \to Y$, $T_2 : Y \to Z$, $T_1, T_2 \in \overline{\mathcal{C}}$ we define composition $T_2 * T_1 : X \to Z$ as follows

$$G(T_2 * T_1) = \overline{G(T_2 T_1)},$$

where $\overline{A}$ denotes the closure of A in the usual topology of a linear space. Of course composition $T_2 * T_1$ is closed convex process and in the case when both processes T_2, T_1 are polyhedral then $T_2 * T_1 = T_2 T_1$. Unfortunately the class $\overline{\mathcal{C}}$ with such defined composition does not form a category, because the associative law of composition of morphisms is not fulfilled.

Indeed, let $G \subseteq R^3$ and $\pi : R^3 \to R^2$ be such as in Example 1. We consider closed convex processes

$$\{0\} \xrightarrow{T} R^3 \xrightarrow{\pi} R^2 \xrightarrow{S} R^3$$

such that $T(0) = G$ and

$$S(x,y) = \begin{cases} (y,y,y) + G, & x = 0 \\ \emptyset, & x \neq 0. \end{cases}$$

Then

$$(-1,-1,-1) \in (S * (\pi * T))(0) = \overline{\mathrm{lin}(1,1,1) + G},$$

but $(-1,-1,-1) \notin ((S * \pi) * T)(0) = G$. This means that $S * (\pi * T) \neq (S * \pi) * T$.

Let us observe that the second way of composition in the class $\bar{\mathcal{C}}$, namely for $T_1 : X \to Y$, $T_2 : X \to Y$ the composition $T_2 \cdot T_1 : X \to Z$ defined as

$$(T_2 \cdot T_1)(x) = \overline{\bigcup_{y \in T_1(x)} T_2(y)},$$

is a convex process, but leads out of the class $\bar{\mathcal{C}}$. Moreover, this composition is not associative which is easy to see if we compose the same processes from Example 2.

E x a m p l e 3. Now we show that the composition $T_2 \cdot T_1$ leads out of the class $\bar{\mathcal{C}}$.

Let $G \subseteq R^3$ be such as in Example 1. Let us consider two closed convex processes

$$R^2 \xrightarrow{T} R \xrightarrow{S} R$$

defined as follows

$$G(T) = G, \quad G(S) = \{(x,y) \mid x \geq 0,\ y \geq 0\}.$$

Then

$$G(S \cdot T) = \{(x,y,z) \in R^3 \mid x > 0,\ z \geq 0\} \cup \{(x,y,z) \in R^3 \mid x=y=0,\ z \geq 0\}$$

is not a closed cone, in spite of this, that one of these processes, S, is polyhedral.

For every cone $G \subseteq X$ we can define a polar cone G^o in X^* as

$$G^o = \{q \in X^* \mid q(x) \leq 0 \text{ for all } x \in G\}.$$

The polar cone G^o is always closed, if G is a closed; if G is a closed convex cone, then $G^{oo} = G$ [10]; if G is a polyhedral convex cone, then G^o is also polyhedral [10], [11].

For every $T : X \to Y$ of class $\mathcal{C}$ we define two adjoint convex processes $T^* : Y^* \to X^*$, $T^{\#} : Y^* \to X^*$ as follows

$$G(T^*) = \{(p,q) \mid (-q,p) \in G(T)^o\}$$

and

$$G(T^{\#}) = \{(p,q) \mid (q,-p) \in G(T)^o\}.$$

Let us note that for $f \in \mathcal{L}$ the adjoint of f as a convex process is the adjoint linear transformation in the usual sense and $f^* = f^{\#}$.

It is easy to see that $T^*, T^{\#} \in \bar{\mathcal{C}}$ for all $T \in \mathcal{C}$, and for closed $T \in \mathcal{C}$ we have $(T^{\#})^* = T$ and $(T^*)^{\#} = T$. We have also the following theorem.

Theorem 1 [11]. Let $T_1, T_2 \in \mathcal{P}$, then

$$(T_2T_1)^* = T_1^*T_2^*, \quad (T_2T_1)^{\#} = T_1^{\#}T_2^{\#}.$$

By Theorem 1, we obtain immediately the following one which is similar to that in the category $\mathcal{L}$.

Theorem 2. The above mentioned adjoint operations are contravariant functors [8] from the category $\mathcal{P}$ into itself.

Exactly:

a) The mapping $\mathcal{F}_1 : \mathcal{P} \to \mathcal{P}$ such that $\mathcal{F}_1(X) = X^*$ for every space X, $\mathcal{F}_1(T) = T^*$ for every $T \in \mathcal{P}$ is a contravariant functor.

b) The mapping $\mathcal{F}_2 : \mathcal{P} \to \mathcal{P}$ such that $\mathcal{F}_2(X) = X^*$ for every space X, $\mathcal{F}_2(T) = T^{\#}$ for every $T \in \mathcal{P}$ is a contravariant functor. Moreover $\mathcal{F}_1 \circ \mathcal{F}_2 = \mathcal{F}_2 \circ \mathcal{F}_1 = id_{\mathcal{P}}$.

Now we show that the adjoint operations are not functors in the category C.

Example 4. Let $T : R \to R^2$, $S : R^2 \to R$ be convex processes defined as follows

$$T(x) = (x,0) + \{\{(y,z) \mid y < 0 < z\} \cup \{(0,0)\}\} \text{ for } x \in R$$

and

$$S(y,z) = \begin{cases} y & \text{for } y \leq 0,\ z \leq 0 \\ \emptyset & \text{in the other cases.} \end{cases}$$

Then

$$G(ST) = \{(x,x) \in R^2 \mid x \leq 0\},$$

hence

$$G((ST)^*) = \{(p,q) \mid -q + p \geq 0\}.$$

Let $T_1 : R \longrightarrow R^2$, $T_1 \in \mathcal{P}$, be such that $G(T_1) = \overline{G(T)}$. Then $G(T_1)^0 = G(T)^0$, thus $T_1^* = T^*$.

Since both processes S and T_1 are polyhedral we obtain that

$$(ST_1)^* = T_1^* S^*.$$

If $(ST)^* = T^*S^*$, then $(ST)^* = T^*S^* = T_1^*S^* = (ST_1)^*$ and

$$ST = (ST)^{*\#} = (ST_1)^{*\#} = ST_1, \quad \text{because } ST, ST_1 \in \bar{\mathcal{C}},$$

but

$$G(ST_1) = \{(x,y) \in R^2 \mid y \leq 0,\ y \leq x\}.$$

This contradiction proves that in spite of this we have $S \in \mathcal{P}$, $\overline{G(T)}$ is polyhedral convex cone, both ST and T^*S^* are closed convex processes, but $(ST)^* \neq T^*S^*$.

In Example 4 one of these processes are not closed. Now we show that $(ST)^*$ is not necessarily equal to $T^* S^*$ even if we assume that both S and T are closed.

E x a m p l e 5. Let $T : X \to R^3$ and $\pi : R^3 \to R^2$ be processes defined in Example 1.

We denote $T_1 = \pi^\#$, $T_2 = T^\#$. Then we have

$$(T_2T_1)^* \neq T_1^*T_2^* = (\pi^\#)^*(T^\#)^* = \pi T,$$

since $\pi T \in \overline{\mathcal{C}}$.

Let $\mathcal{K}$ be an arbitrary category. We shall say that morphism $T \in \mathcal{K}$ is a monomorphism (epimorphism) in the category $\mathcal{K}$ if for any morphisms $S_1, S_2 \in \mathcal{K}$ such that $TS_1 = TS_2$, $(S_1T = S_2T)$ implies $S_1 = S_2$ [8].

Applying Theorem 2 we have in the category $\mathfrak{P}$ the following analogous result to that in the category $\mathcal{L}$.

T h e o r e m 3 [13]. Let $T \in \mathfrak{P}$. Then the following conditions are equivalent

(i) T is a monomorphism

(ii) T^* is an epimorphism

(iii) $T^\#$ is an epimorphism.

Theorem 3 is not true in the category $\mathcal{C}$.

E x a m p l e 6. Let $C \subseteq R^2$ be a convex cone defined as

$$C = \{(y,z) \mid y < 0 < z\} \cup \{(0,0)\}.$$

It is easy to see that a convex process $T : R \to R^2$ such that

$$T(x) = (x,0) + C \quad \text{for} \quad x \in R$$

is a monomorphism in the category $\mathcal{C}$, but $T^* : R^2 \to R$ defined by

$$T^*(p_1,p_2) = \begin{cases} p_1 & \text{for} \quad p_2 \leqslant 0 \leqslant p_1, \\ \emptyset & \text{in the other cases} \end{cases}$$

is not an epimorhpism even in the category $\mathfrak{P}$, because for $S_1, S_2 : R \to R$ defined as follows

$$G(S_1) = \{(x,y) \mid y \geqslant 0\},$$
$$G(S_2) = \{(x,y) \mid y \geqslant 0,\ x+y \geqslant 0\}$$

we have $S_1 T^* = S_2 T^*$.

Polyhedral convex processes are continuous in the sense of the classical definition of a convergence of sets in a topological space [7].

Let A_n, $n=1,2,\ldots$, be subsets of a space X. We shall say that $\lim A_n = A_0 \subseteq X$ if $\mathrm{Li}\, A_n = A_0 = \mathrm{Ls}\, A_n$, where $x \in \mathrm{Li}\, A_n$ if any neighbourhood of x has common points with sets A_n for almost every n, and $x \in \mathrm{Ls}\, A_n$ if any neighbourhood of x has common points with an infinite number of sets A_n.

This continuity of convex processes, which fulfill the condition $T(0) = 0$, were studied in [6].

The other types of continuity of multifunction were investigated by different authors e.g. Berge [1], Dolecki, Rolewicz [4, 5, 12]. Basic relations between different types of semicontinuity may be found in [2, 3].

For the category $\mathfrak{P}$ we have an analogous theorem to that in the category $\mathcal{L}$.

T h e o r e m 4 [13]. Let $T \in \mathfrak{P}$, $T : X \to Y$, $T(x_n) \neq \emptyset$, $n = 0,1,2,\ldots$, and $\lim x_n = x_0$. Then $\lim T(x_n) = T(x_0)$.

Theorem 4 is not true in the category $\mathfrak{C}$, even if we assume that $T \in \bar{\mathfrak{C}}$.

E x a m p l e 7. Let $G \subseteq R^3$ be such closed convex cone as in Example 1. We define $T : R^2 \to R$ as a convex process with a graph $G(T) = G$. Then

$$T(x,y) = \left\{ z \mid z \geqslant \frac{x^2 + y^2}{2x} \right\} \quad \text{for} \quad x > 0,$$

$$T(0,0) = \{z \mid z \geqslant 0\},$$

thus

$$\lim T\left(\frac{1}{n}, \frac{1}{n}\right) = \lim\left\{z \mid z \geq \frac{1}{2}\left(1 + \frac{1}{n}\right)\right\} = \left\{z \mid z \geq \frac{1}{2}\right\} \neq T(0,0).$$

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