DEMONSTRATIO MATHEMATICA
Vol. XV No 3 1982

Jarosław Gela

THE GROUP AND THE EDGE-GROUPS OF A MULTIGRAPH

1. Introduction, definitions

In [2] and [5] the authors have proved that for a nonempty finite graph the group and the induced edge-group are isomorphic if and only if the graph contains neither K_2 as a component nor two or more isolated vertices. From Whitney's results [6] it follows that for a nonempty finite graph the induced edge-group and the edge-group are isomorphic if and only if not both K_3 and $K_{1,3}$ are components of a graph and none of the graphs K_4, K_4-x, and $K_{1,3}+x$ (x is an edge) is a component of a graph. In [4] the author has shoved that, with four exceptions (K_2, K_4, K_4-x, $K_{1,3}+x$), the group and the edge-group of a connected graph are isomorphic.

In this paper we shall study the connection between the group and the edge-groups of a multigraph.

By a multigraphs G we mean an ordered triple $(V(G),X(G),F(G))$ where $V(G)$ is a nonempty set (it may be an infinite set) called a vertex-set, $X(G)$ is an arbitrary set called an edge-set, and $F(G): X(G) \to V(G)\cdot V(G)$ is a mapping from $X(G)$ to the set of unordered pairs of $V(G)$. The mapping $F(G)$ is called an incidence mapping. If $x \in X(G)$ and $F(G)(x) = (u,u)$, x is called a loop; if $x,y \in X(G)$ and $F(G)(x) = F(G)(y)$, then x and y are called multiple edges. For $v \in V(G)$, let $S_G(v) = \{x \in X(G): F(G)(x) = (v,v')$ for some $v' \in V(G)\}$. $S_G(v)$ is called the cluster at v and C is a cluster if $C = S_G(v)$ for some $v \in V(G)$. C is called a star, or star with center v, if $C \subseteq S_G(v)$. A vertex $v \in V(G)$ is a terminal vertex of G if there exists a vertex $v' \neq v$ such that $\emptyset \neq S_G(v) \subsetneq S_G(v')$. If $S_G(v) = \emptyset$, then v is called an isolated vertex of G. If every vertex of G is isolated, then G is called an empty multigraph. The empty multigraph with $|V(G)| = 1$ is a trivial multigraph. The union $\bigcup_{i \in I} G_i$ of multigraphs G_i, $i \in I$, where $V(G_i) \cap V(G_j) = \emptyset$ for $i \neq j$, is defined as the multigraph with $V(\bigcup_{i\in I} G_i) = \bigcup_{i\in I} V(G_i)$, $X(\bigcup_{i\in I} G_i) = \bigcup_{i\in I} X(G_i)$, and $F(\bigcup_{i\in I} G_i)(x) = F(G_i)(x)$ for each $x \in X(G_i)$ and $i \in I$. G is a connected multigraph if it cannot be expressed as the union of two multigraphs; otherwise it is disconnected. Any disconnected multigraph G can be expressed as the union of connected multigraphs; each of these connected multigraphs is called a component of G. A multigraph which has neither loops nor multiple edges is a graph. Another terms and denotions used and not defined in this paper can be found in [1].

2. The isomorphism and the edge-isomorphisms of multigraphs

Let G and H be two multigraphs. An isomorphism φ of G onto H is a one-to-one mapping of $V(G)$ onto $V(H)$ with the property that for each pair of (not necessarily distinct) vertices $v,v' \in V(G)$, $|F(G)^{-1}(v,v')| = |F(H)^{-1}(\varphi(v), \varphi(v'))|$. An edge-isomorphism ψ of G onto H is a one-to-one mapping of $X(G)$ onto $X(H)$ such that for each pair $x,x' \in X(G)$, $x,x' \in S_G(v)$ for some $v \in V(G)$ if and only if $\psi(x), \psi(x') \in S_H(v')$ for some $v' \in V(H)$. An edge-isomorphism ψ is induced by an isomorphism φ if for each $x \in X(G)$ we have, $F(G)(x) = (v,v')$ if and only if $F(H)(\psi(x)) = (\varphi(v),\varphi(v'))$.

We now shall consider simple relationships between the isomorphisms and the induced edge-isomorphisms of multigraphs.

T h e o r e m 2.1. An edge-isomorphism ψ of G onto H is induced by an isomorphism φ if and only if for every $x \in X(G)$,

$$\varphi(F(G)(x)) = F(H)(\psi(x)).$$

P r o o f . First, let ψ be an edge-isomorphism of G onto H induced by φ and let $F(G)(x) = (u,v)$. Then $F(H)(\psi(x)) = (\varphi(u),\varphi(v)) = \varphi(u,v) = \varphi(F(G)(x))$.

Now let $\varphi(F(G)(x)) = F(H)(\psi(x))$. If $F(G)(x) = (u,v)$, then $(\varphi(u),\varphi(v)) = \varphi(u,v) = \varphi(F(G)(x)) = F(H)(\psi(x))$. On the other hand, if $F(H)(\psi(x)) = (\varphi(u),\varphi(v))$, then $(u,v) = \varphi^{-1}\varphi(u,v) = \varphi^{-1}(\varphi(u),\varphi(v)) = \varphi^{-1}(F(H)(\psi(x))) = \varphi^{-1}(\varphi(F(G)(x))) = F(G)(x)$.

T h e o r e m 2.2. If an edge-isomorphism ψ_1 of G onto H is induced by φ_1 and an edge-isomorphism ψ_2 of H onto S is induced by φ_2, then $\psi_2\psi_1$ is an edge-isomarphism of G onto S induced by $\varphi_2\varphi_1$.

P r o o f . It is clear that $\psi_2\psi_1$ is an edge-isomorphism of G onto S. Let $F(G)(x) = (u,v)$. Since ψ_1 is induced by φ_1, $F(H)(\psi_1(x)) = (\varphi_1(u),\varphi_1(v))$ and since ψ_2 is induced by φ_2, $F(S)(\psi_2\psi_1(x)) = (\varphi_2\varphi_1(u),\varphi_2\varphi_1(v))$. Therefore $\psi_2\psi_1$ is induced by $\varphi_2\varphi_1$.

T h e o r e m 2.3. If an edge-isomorphism ψ of G onto H is induced by φ, then ψ^{-1} is an edge-isomorphism of H onto G induced by φ^{-1}.

P r o o f . Evidently, ψ^{-1} is an edge-isomorphism of H onto G. Let ψ be an edge-isomorphism of G onto H induced by φ. Then, by Theorem 2.1, for every $x \in X(G)$, $\varphi^{-1}(F(H)(\psi(x))) = F(G)(x) = F(G)(\psi^{-1}\psi(x))$. Since for each $y \in X(H)$ there exists $x \in X(G)$ such that $y = \psi(x)$, therefore, for every $y \in X(H)$, $\varphi^{-1}(F(H)(y)) = F(G)(\psi^{-1}(y))$, whence, by Theorem 2.1, ψ^{-1} is induced by φ^{-1}.

The edge-isomorphisms not induced by any isomorphism was described by Hemminger in [3]. For the purpose of presenting these results we need a few additional definitions.

A multigraph G' is a multiversion of G if G is a submultigraph of G', with the same vertex-set as G', such that there is a partition $\{M(x): x \in X(G)\}$ of $X(G')$ with $M(x)$ a set of multiple edges containing x for each $x \in X(G)$.

Let ψ be a one-to-one mapping of $X(G)$ onto $X(H)$ and let ψ preserves multiple edges. A pair of multigraphs, G' and H', is a ψ-multiversion of the pair G and H if G' and H' are multiversions of G and H with partitions $\{M(x): x \in X(G)\}$ and $\{N(y): y \in X(H)\}$ respectively, such that there is a one-to-one and onto mapping $\psi' : X(G') \longrightarrow X(H')$ with $\psi'(M(x)) = N(\psi(x))$ for each $x \in X(G)$.

We say that G_1 is a terminal piece of G based at u if $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{u\}$.

Now we shall describe three pairs of multigraphs G and H for which there exists an edge-isomorphism ψ not induced by any isomorphism.

(1) G and H are multigraphs with the property that their edge-sets are clusters or triangles (with multiple edges) and ψ is arbitrary except that ψ or ψ^{-1} does not preserve stars, loops, or multiple edges.

(2) The pair, G and H, is a σ-multiversion of the pair of graphs K_4 and K_4 or K_4-x and K_4-x (x is an edge) where σ is an edge-isomorphism not induced by any isomorphism in either cases.

(3) Let the pair, G' and H', be σ-multiversion (with partitions $\{M(x)\}$ and $\{N(x)\}$ respectively) of the pair of graphs $K_{1,3}+x_0$ and $K_{1,3}+y_0$ where σ is an edge-isomorphism not induced by any isomorphism. Let $V(K_{1,3}+x_0) = \{v_0,v_1,v_2,v_3\}$, $X(K_{1,3}+x_0) = \{x_0,x_1,x_2,x_3\}$, $F(K_{1,3}+x_0)(x_0) = (v_1,v_2)$, $F(K_{1,3}+x_0)(x_i) = (v_0,v_i)$, $i=1,2,3$, and $\sigma(x_j) = x_j'$, $j = 0,1,2,3$. Then $M(x_3)$ is a terminal star (with center v_0) of G' based at v_0. Similarly let the star $N(x_0')$ of H' be based at v_0'. Now the pair G and H we obtain from the G' and H' respectively by replacing $M(x_3)$ and $N(x_0')$ by terminal stars A based at v_0 and B based at v_0' with $|A| = |M(x_3)|$ and $|B| = |N(x_0')|$. Let ψ be any one-to-one mapping from $X(G)$ onto $X(H)$ with the property that $\psi(A) = N(x_3')$, $\psi(M(x_1) \cup M(x_2)) = N(x_1') \cup N(x_2')$, and $\psi(M(x_0)) = B$. Then ψ is an edge-isomorphism of G onto H that is not induced by any isomorphism.

T h e o r e m 2.4 (Hemminger 1971). Let ψ be an edge-isomorphism of G onto H, G and H connected multigraphs, where ψ is not induced by an isomorphism and where G and H are not as in (1), (2), or (3) above. Then G has a terminal piece G_1 based at u and H has a terminal piece H_1 based at v such that $\psi(X(G_1)) = X(H_1)$, $\psi(S_G(u)) = S_H(v)$, ψ restricted to G_1 is not induced by an isomorphism, and where G_1 and H_1 are one of the following: (a) terminal stars with center u or v, (b) terminal triangles, or (c) a terminal triangle and a terminal cluster with center $w \neq u,v$.

3. The group and the induced edge-group of a multigraph

An isomorphism of G onto itself is called an automorphism of G. The set of all automorphisms of G is a group $\Gamma(G)$ called a group of G. An edge-isomorphism of G onto itself is called an edge-automorphism of G. The set of all edge-automorphisms of G is a group $\Gamma'(G)$ and from Theorems 2.2 and 2.3 it follows that the set of edge-automorhpisms induced by all automorphisms of G is its subgroup $\Gamma^*(G)$. The first group is called an edge-group of G an the second one is called an induced edge-group of G. For the study of a relationship between the induced edge-group and the group of a given multigraph we shall define additional two groups. Namely, let us denote by $\Gamma_1(G)$ the set of all automorphisms of G which induce the identity mapping on $X(G)$, and let $\Gamma_1^*(G)$ be the set of all edge-automorhpisms of G induced by the identity mapping on $V(G)$. It is clear that $\Gamma_1(G)$ is a subgroup of $\Gamma(G)$ and $\Gamma_1^*(G)$ is a subgroup of $\Gamma^*(G)$.

T h e o r e m 3.1. Let ψ be an edge-automorphism of G induced by φ. Then $\psi \in \Gamma_1^*(G)$ if and only if $\varphi \in \Gamma_1(G)$.

P r o o f . Let $\psi \in \Gamma_1^*(G)$. Then $\psi^{-1} \in \Gamma_1^*(G)$ and so ψ^{-1} is induced by the identity mapping on $V(G)$. Let φ be any automorphism that induces ψ. Then, by Theorem 2.2, $\psi^{-1}\psi = id_{X(G)}$ is induced by $id_{V(G)}\varphi = \varphi$ and therefore $\varphi \in \Gamma_1(G)$.

Conversely, let $\varphi \in \Gamma_1(G)$. Then $\varphi^{-1} \in \Gamma_1(G)$ and so φ^{-1} induces the identity mapping on $X(G)$. Let ψ be any edge-automorphism induced by φ. Then, by Theorem 2.2, $\varphi^{-1}\varphi = id_{V(G)}$ induces $id_{X(G)}\psi = \psi$. Thus $\psi \in \Gamma_1^*(G)$.

Theorem 3.2. An automorphism φ is an element of $\Gamma_1(G)$ if and only if φ fixes

(1) each nonterminal vertex of G which is not an isolated vertex, and

(2) every terminal vertex that is a vertex of a component having also at least one nonterminal vertex, and

(3) each set of vertices of a component having only terminal vertices.

Proof. Let $\varphi \in \Gamma_1(G)$. Since the identity mapping on $V(G)$ fixes each vertex of G, we may assume that $\varphi \neq id_{V(G)}$. Let u be such vertex of G for which $\varphi(u) = v \neq u$ and let us consider two cases.

Case 1. Assume that there exists $x \in X(G)$ such that $F(G)(x) = (u,v)$. In this case both u and v are either nonterminal nonisolated vertices or terminal ones. If u and v are nonterminal and nonisolated vertices, then there exists (loop or non-loop) $y \in X(G)$ such that $F(G)(y) = (u,w)$, where $w \neq v$. But then for each edge-automorphism ψ of G induced by φ, $\psi(y) \neq y$. Thus ψ is not the identity mapping on $X(G)$ and so $\varphi \notin \Gamma_1(G)$; a contradiction with the assumption. Therefore, in this case, for each nonterminal vertex u of G, $\varphi(u) = u$. Now if u and v are terminal vertices, then they are vertices of some component having as a vertex-set $\{u,v\}$. If $\varphi(v) = w \neq u$, then for each edge-automorphism ψ induced by φ, $\psi(x) \neq x$. Hence $\varphi \notin \Gamma_1(G)$ and we obtain again a contradiction. Therefore $\varphi(v) = u$, whence $\varphi(\{u,v\}) = \{u,v\}$.

Case 2. Assume that for every $x \in X(G)$, $F(G)(x) \neq (u,v)$. Furthermore, let u be a nonisolated vertex of G. Then there exists a vertex $w \neq v$ such that $F(G)(x) = (u,w)$. But, in this case, for each edge-automorphism ψ of G induced by φ, $\psi(x) \neq x$. Similarly as in the Case 1 we obtain a contradiction and therefore $\varphi(u) = u$.

Conversely, let us assume that an automorphism φ of G fixes each vertex described in (1), (2), and each set described in (3) of our theorem. Then obviously the identity mapping on $X(G)$ is induced by φ, i.e. $\varphi \in \Gamma_1(G)$.

Let us denote by $\prod_{i\in I} \Gamma_i$ the direct product of groups Γ_i for $i \in I$, i.e. the group of all mappings $f: I \to \bigcup_{i\in I} \Gamma_i$ such that for each $i \in I$, $f(i) \in \Gamma_i$ with $(fg)(i) = f(i)g(i)$.

For a multigraph G we consider the set $A \subseteq V(G)$ such that $u \in A$ if and only if u is either an isolated vertex of G or u is a vertex of a component of G having only terminal vertices. Let $\sim$ be the equivalence relation on A defined by $u \sim v$ if and only if either both u and v are isolated vertices or belong to the same component. Then, if we introduce the notation $A/\sim = \{A_i : i \in I\}$, we obtain the following

T h e o r e m 3.3. $\Gamma_1(G) \simeq \prod_{i\in I} S(A_i)$ where $S(A_i)$ is a group of all one-to-one mappings of A_i onto itself for $i \in I$.

P r o o f . This isomorphism of groups follows from the preceding theorem. Namely, let us consider a mapping $\Phi: \Gamma_1(G) \to \prod_{i\in I} S(A_i)$ such that for $\varphi \in \Gamma_1(G)$, $\Phi(\varphi) = f$ where $f(i) \in S(A_i)$ for each $i \in I$ and furthermore $f(i)(u) = \varphi(u)$ for each $u \in A_i$ and for each $i \in I$. Since for every $u \in V(G) \setminus \bigcup_{i\in I} A_i$ φ fixes u, Φ is well defined. From the definition of Φ it is obvious that Φ is one-to-one and onto. We show that Φ is a homomorphism. Let us take $\varphi_1, \varphi_2 \in \Gamma_1(G)$ and let $\Phi(\varphi_1) = f_1$, $\Phi(\varphi_2) = f_2$, and $\Phi(\varphi_2\varphi_1) = f$. Then for each $u \in A_i$ and for each $i \in I$, $f(i)(u) = \varphi_2\varphi_1(u) = \varphi_2 f_1(i)(u) = f_2(i) f_1(i)(u) = f_2 f_1(i)(u)$. Hence $f = f_2 f_1$ and therefore $\Phi(\varphi_2\varphi_1) = f = f_2 f_1 = \Phi(\varphi_2)\Phi(\varphi_1)$.

For a multigraph G, let $\sim^*$ be the equivalence relation on $X(G)$ defined by $x \sim^* y$ if and only if x and y are multiple edges. If we shall denote by $\{B_i : i \in I\}$ the set $X(G)/\sim^*$, we obtain the

T h e o r e m 3.4. $\Gamma_1^*(G) \simeq \prod_{j\in J} S(B_j)$ where $S(B_j)$ is a group of all one-to-one mappings of B_j onto itself for $j \in J$.

P r o o f . Let us consider a mapping $\Phi : \Gamma_1^*(G) \to \prod_{j\in J} S(B_j)$ such that for $\psi \in \Gamma_1^*(G)$, $\Phi(\psi) = f$ where $f(j) \in S(B_j)$ for each $j \in J$ and $f(j)(x) = \psi(x)$ for each $j \in J$ and for each $x \in B_j$. Similarly as in the previous theorem it is easy to observe that Φ is one-to-one, onto, and Φ is operation-preserving.

R e m a r k . The sets A_i for $i \in I$ and the sets $\{u\}$ for $u \in V(G)\setminus\bigcup_{i\in I} A_i$ are orbits of $\Gamma_1(G)$. Similarly the sets B_j for $j \in J$ are orbits of $\Gamma_1^*(G)$.

T h e o r e m 3.5. The group $\Gamma_1(G)$ is a normal subgroup of $\Gamma(G)$.

P r o o f . Let $\varphi' \in \Gamma_1(G)$ and $\varphi \in \Gamma(G)$. Then φ' induces the identity mapping on $X(G)$. If furthermore φ induces ψ, then, by Theorems 2.2 and 2.3, $\varphi^{-1}\varphi'\varphi$ induces $\psi^{-1}\mathrm{id}_{X(G)}\psi = \mathrm{id}_{X(G)}$. Thus $\varphi^{-1}\varphi'\varphi \in \Gamma_1(G)$.

T h e o r e m 3.6. The group $\Gamma_1^*(G)$ is a normal subgroup of $\Gamma^*(G)$.

P r o o f . Let $\psi' \in \Gamma_1^*(G)$ and $\psi \in \Gamma^*(G)$. Then ψ' is induced by the identity mapping on $V(G)$. If ψ is induced by φ, then, by Theorems 2.2, 2.3, $\psi^{-1}\psi'\psi$ is induced by $\varphi^{-1}\mathrm{id}_{V(G)}\varphi = \mathrm{id}_{V(G)}$. Thus $\psi^{-1}\psi'\psi \in \Gamma_1(G)$.

T h e o r e m 3.7. For a nonempty multigraph G,

$$\Gamma^*(G)/\Gamma_1^*(G) \simeq \Gamma(G)/\Gamma_1(G).$$

P r o o f . Let us define a mapping $\Phi : \Gamma^*(G)/\Gamma_1^*(G) \to \Gamma(G)/\Gamma_1(G)$ by $\Phi(\psi\Gamma_1^*(G)) = \varphi\Gamma_1(G)$ where $\psi \in \Gamma^*(G)$ and ψ is induced by $\varphi \in \Gamma(G)$. First, we shall show that Φ is well defined. For that purpose, let $\psi\Gamma_1^*(G) = \psi'\Gamma_1^*(G)$, whence $\psi^{-1}\psi' \in \Gamma_1^*(G)$. Now we choose, arbitrarily φ and φ' such that φ induces ψ and φ' induces ψ'. Then by Theorems 2.2 and 2.3, $\varphi^{-1}\varphi'$ induces $\psi^{-1}\psi'$. Since

$\psi^{-1}\psi' \in \Gamma_1^*(G)$, Theorem 3.1 implies that $\varphi^{-1}\varphi' \in \Gamma_1(G)$. But then $\varphi\Gamma_1(G) = \varphi'\Gamma_1(G)$ and Φ is well defined. Analogously we use Theorems 2.2, 2.3 and 3.1 to show that Φ is one--to-one. It remains to verify that Φ is a homomorphism. Let ψ and ψ' be any edge-automorphism induced by φ and φ', respectively. Then, by Theorem 2.3, $\psi'\psi$ is an edge-automorphism induced by $\varphi'\varphi$ and $\Phi(\psi'\Gamma_1^*(G)\psi\Gamma_1^*(G)) = \Phi(\psi'\psi\Gamma_1^*(G)) = \varphi'\varphi\Gamma_1(G) = \varphi'\Gamma_1(G)\varphi\Gamma_1(G) = \Phi(\psi'\Gamma_1^*(G))\Phi(\psi\Gamma_1^*(G))$. This completes the proof.

Theorem 3.8. If a nontrivial multigraph G has at most one isolated vertex and any component of G, different from the isolated vertex, has at least one nonterminal vertex, then

$$\Gamma(G) \simeq \Gamma^*(G)/\Gamma_1^*(G).$$

Proof. If G satisfies the hypotheses of the theorem, then, by Theorem 3.2, any $\varphi \in \Gamma_1(G)$ fixes every vertex of G. Thus $\Gamma_1(G) = \{id_{V(G)}\}$ and, by Theorem 3.7, $\Gamma(G) \simeq \Gamma^*(G)/\Gamma_1^*(G)$.

Corollary 3.1. For a nontrivial connected multigraph G,

$$\Gamma(G) \simeq \Gamma^*(G)/\Gamma_1^*(G)$$

if and only if G has at least one nonterminal vertex.

Proof. It suffices to show that if a nontrivial connected multigraph G has only terminal vertices, then $\Gamma(G) \not\simeq \Gamma^*(G)/\Gamma_1^*(G)$. But under this assumption $V(G) = \{u,v\}$, $X(G)$ is an arbitrary set, and $F(G)(x) = (u,v)$ for each $x \in X(G)$. For such multigraph G, $\Gamma(G) = S_2$, $\Gamma_1^*(G) = \Gamma^*(G)$ and, therefore, $\Gamma^*(G)/\Gamma_1^*(G) \simeq S_1$.

C o r o l l a r y 3.2. Let G be a nontrivial multigraph and $|\Gamma(G)| < \infty$. Then

$$\Gamma(G) \simeq \Gamma^*(G)/\Gamma_1^*(G)$$

if and only if G has at most one isolated vertex and each component of G, different from the isolated vertex, has at least one nonterminal vertex.

P r o o f . If a nontrivial multigraph G has more than one isolated vertex or there exists a component of G which has only terminal vertices, then, by Theorem 3.3, $\Gamma_1(G) \neq \{id_{V(G)}\}$. Since $|\Gamma(G)| < \infty$, by Theorem 3.7, $\Gamma(G) \not\simeq \Gamma^*(G)/\Gamma_1^*(G)$. (The converse arises from Theorem 3.8).

T h e o r e m 3.9. If a nonempty multigraph G has no distinct multiple edges (single loops are allowed), then

$$\Gamma^*(G) \simeq \Gamma(G)/\Gamma_1(G).$$

P r o o f . By Theorem 3.4, for a multigraph G without distinct multiple edges, $\Gamma_1(G) = \{id_{X(G)}\}$. The result now follows immediately from Theorem 3.7.

C o r o l l a r y 3.3. Let G be a nonempty multigraph with $|\Gamma^*(G)| < \infty$. Then

$$\Gamma^*(G) \simeq \Gamma(G)/\Gamma_1(G)$$

if and only if G has no distinct multiple edges.

P r o o f . This result follows immediately from Theorems 3.7, 3.9, and from the fact that if G has distinct multiple edges, then $\Gamma_1^*(G) \neq \{id_{X(G)}\}$.

Combining Theorems 3.8 and 3.9 we obtain the following

T h e o r e m 3.10. If a nontrivial multigraph G without distinct multiple edges has at most one isolated vertex and each component of G, different from the isolated vertex, has at least one nonterminal vertex, then

$$\Gamma(G) \simeq \Gamma^*(G).$$

C o r o l l a r y 3.4. If a nontrivial connected multigraph G has no distinct multiple edges, then

$$\Gamma(G) \simeq \Gamma^*(G)$$

if and only if $G \neq K_2$ (complete graph with two vertices).

By Q we denote the multigraph with $V(Q) = \{u,v\}$, $X(Q) = \{x,y\}$, and $F(Q)(x) = F(Q)(y) = (u,v)$.

C o r o l l a r y 3.5. Let $G \neq K_2$ be a nontrivial connected multigraph with $|\Gamma^*(G)| < \infty$. Then

$$\Gamma(G) \simeq \Gamma^*(G)$$

if and only if either G has no distinct multiple edges or G is isomorphic to Q.

4. The induced edge-group and the edge-group of a multigraph

In this section we shall use Hemminger's theorem to describe of multigraphs for which the induced edge-group and the group are not identical.

In Fig.4.1 we have the following multigraphs:

G_i, i=1,2,4,5 are multiversions of multigraphs (with additional conditions) whose edges being represented by solid lines. G_3 was obtained from a multiversion of the multigraph whose edges being represented by solid lines by replacing a star with distinct multiple edges by a star with single edges. G_i, $i=6,7$ are arbitrary multigraphs with terminal stars with center v_1 based at v_1; each of these stars is a multiversion of the star whose edges being represented by solid lines. G_8 is an arbitrary multigraph with a terminal triangle; this triangle is a multiversion of the triangle whose edges being represented by solid lines. The last multigraph, G_9, is an arbitrary multigraph which has terminal pieces H_i, $i \in I_t$, $t \in T$, $|I| \geq 2$, $|T| \geq 1$, based at u_i, $u_i \neq u_j$ for $i \neq j$ with $|X(H_i)| = |X(H_j)|$ for each $i,j \in I_t$ where H_i is either one of A_1, A_2, A_3 or B_1 and one of B_2, B_3, B_4; moreover there exists an automorphism φ of H such that $\varphi(\{u_i : i \in I_t\}) = \{u_i : i \in I_t\}$ for each $t \in T$ and there exist $t \in T$ and $i,j \in I_t$, $i \neq j$ such that $H_i \neq H_j$ but $\varphi(u_i) = u_j$.

T h e o r e m 4.1. Let G be a nontrivial connected multigraph. If $G \neq G_i$, $i = 1,\dots,9$ (of Fig.4.1), then

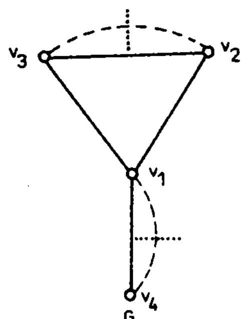
$$\Gamma^*(G) \simeq \Gamma'(G).$$

P r o o f . Suppose $\Gamma^*(G) \not\simeq \Gamma'(G)$. Since $\Gamma^*(G)$ is a subgroup of $\Gamma'(G)$, there exists an edge-automorphism ψ not induced by any automorphism. Then, by Theorem 2.4, G has at least one of the following properties:

(1) The edge-set of G is a cluster or a triangle and ψ or ψ^{-1} does not preserve stars, loops, or multiple edges. Thus G must be isomorphic to G_6, G_7, or G_8; H in Fig.4.1 is the trivial or, in the case of multigraphs G_6 and G_7, the edge-set of H may be a star with center v_1.

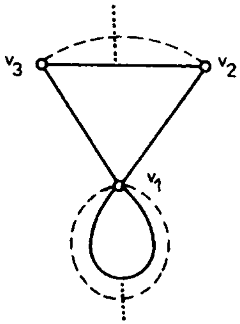
(2) The pair G and G is a σ-multiversion of the pair K_4 and K_4 or K_4-x and K_4-x where σ is an edge-isomorphism not induced by any isomorphism. Hence G must be isomorphic to G_4 or G_5.

(3) The edge-set of G consists of a terminal triangle with vertices, say v_1, v_2, v_3 and a terminal star A with



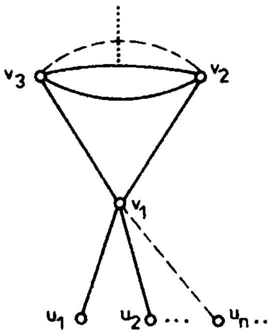
G_1

$$|F(G_1)^{-1}(v_2,v_3)| = |F(G_1)^{-1}(v_1,v_4)| \geq 1,$$
$$|F(G_1)^{-1}(v_1,v_2)| = |F(G_1)^{-1}(v_1,v_3)| = 1.$$



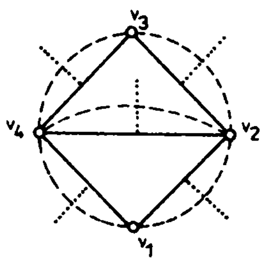
G_2

$$|F(G_2)^{-1}(v_1,v_1)| = |F(G_2)^{-1}(v_2,v_3)| \geq 1,$$
$$|F(G_2)^{-1}(v_1,v_2)| = |F(G_2)^{-1}(v_1,v_3)| = 1.$$

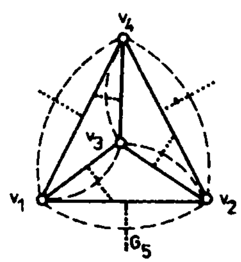


G_3

$$|F(G_3)^{-1}(v_2,v_3)| = |F(G_3)^{-1}(u_1,v_1) \cup F(G_3)^{-1}(u_2,v_1) \cup \ldots \cup F(G_3)^{-1}(u_n,v_1) \cup \ldots| \geq 2,$$
$$|F(G_3)^{-1}(u_1,v_1)| = |F(G_3)^{-1}(u_2,v_1)| = \ldots = |F(G_3)^{-1}(u_n,v_1)| = \ldots = 1,$$
$$|F(G_3)^{-1}(v_1,v_2)| = |F(G_3)^{-1}(v_1,v_3)| = 1.$$



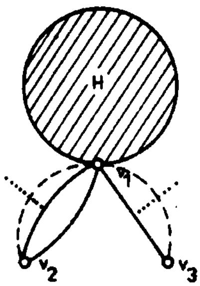
G_4



$$|F(G_4)^{-1}(v_1,v_2)| = |F(G_4)^{-1}(v_3,v_4)| \geq 1, \quad |F(G_5)^{-1}(v_1,v_2)| = |F(G_5)^{-1}(v_3,v_4)| \geq 1,$$

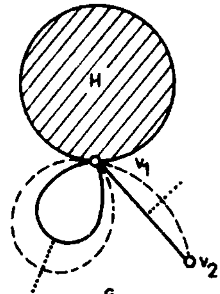
$$\left|F(G_4)^{-1}(v_1,v_4)\right| \geq 1,$$
$$\left|F(G_4)^{-1}(v_2,v_3)\right| \geq 1,$$
$$\left|F(G_4)^{-1}(v_2,v_4)\right| \geq 1.$$

$$\left|F(G_5)^{-1}(v_1,v_3)\right| \geq 1,$$
$$\left|F(G_5)^{-1}(v_1,v_4)\right| \geq 1,$$
$$\left|F(G_5)^{-1}(v_2,v_3)\right| \geq 1,$$
$$\left|F(G_5)^{-1}(v_2,v_4)\right| \geq 1.$$



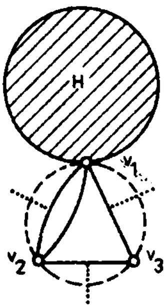
G_6

$$\left|F(G_6)^{-1}(v_1,v_2)\right| \geq 2,$$
$$\left|F(G_6)^{-1}(v_1,v_3)\right| \geq 1.$$



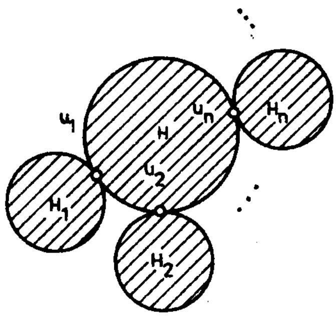
G_7

$$\left|F(G_7)^{-1}(v_1,v_1)\right| \geq 1,$$
$$\left|F(G_7)^{-1}(v_1,v_2)\right| \geq 1.$$

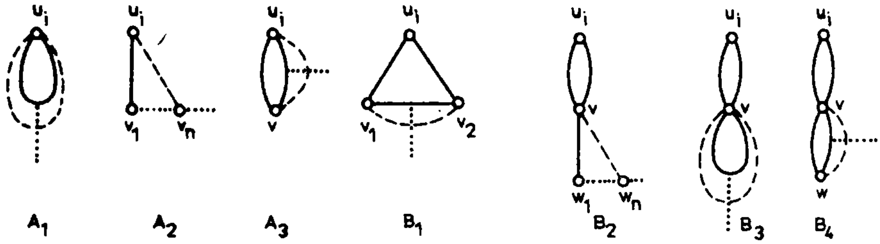


G_8

$$\left|F(G_8)^{-1}(v_1,v_2)\right| \geq 2,$$
$$\left|F(G_8)^{-1}(v_2,v_3)\right| \geq 1,$$
$$\left|F(G_8)^{-1}(v_1,v_3)\right| \geq 1.$$



G_9



$$\left|F(A_1)^{-1}(u_i,u_i)\right| \geqslant 1.$$

$$\left|F(A_2)^{-1}(u_i,v_1) \cup \ldots \cup F(A_2)^{-1}(u_i,v_n) \cup \ldots\right| \geqslant 1,$$

$$\left|F(A_2)^{-1}(u_i,v_1)\right| = \ldots = \left|F(A_2)^{-1}(u_i,v_n)\right| = \ldots = 1.$$

$$\left|F(A_3)^{-1}(u_i,v)\right| \geqslant 2.$$

$$\left|F(B_1)^{-1}(u_i,v_1)\right| = \left|F(B_1)^{-1}(u_i,v_2)\right| = 1,$$

$$\left|F(B_1)^{-1}(v_1,v_2)\right| \geqslant 1.$$

$$\left|F(B_2)^{-1}(u_i,v)\right| = 2,$$

$$\left|F(B_2)^{-1}(v,w_1) \cup \ldots \cup F(B_2)^{-1}(v,w_n) \cup \ldots\right| \geqslant 1,$$

$$\left|F(B_2)^{-1}(v,w_1)\right| = \ldots = \left|F(B_2)^{-1}(v,w_n)\right| = \ldots = 1.$$

$$\left|F(B_3)^{-1}(u_i,v)\right| = 2,$$

$$\left|F(B_3)^{-1}(v,v)\right| \geqslant 1,$$

$$\left|F(B_4)^{-1}(u_i,v)\right| = 2,$$

$$\left|F(B_4)^{-1}(v,w)\right| \geqslant 2.$$

Figure 4.1

center, say v_1 based at v_1. Then $|F(G)^{-1}(v_2,v_3)| = |A|$. Let $B = F(G)^{-1}(v_1,v_2) \cup F(G)^{-1}(v_1,v_3)$. We distinguish three cases.

(a) B contains distinct multiple edges. Then G is isomorphic to G_8; the edge-set of H is a star A.

(b) B contains no distinct multiple edges and A is a set of edges of the same kind. The G is isomorphic to G_1, G_2 or G_3.

(c) B contains no distinct multiple edges and A is a set of edges of the distinct kind. Then G is isomorphic to G_6 or G_7; H is then triangle with an eventual star with center v_1 based at v_1.

(4) G has a terminal pieces H' and H'' based at u' and u'' respectively, such that $\psi(X(H')) = \psi(X(H''))$, $\psi(S_G(u')) = \psi(S_G(u''))$, ψ restricted to H' is not induced by an automorphism and H' and H'' are one of the following: terminal stars, terminal triangles, or a terminal triangle and a terminal cluster. Hence, if $u' = u''$ and H' is the same as H'', then G is isomorphic to G_6, G_7, or G_8. Conversely, G is isomorphic to G_9.

Corollary 4.1. Let G be a nontrivial connected multigraph with $|\Gamma^*(G)| < \infty$. Then

$$\Gamma^*(G) \simeq \Gamma'(G)$$

if and only if $G \neq G_i$, $i = 1,\dots,9$ (of Fig.4.1).

Proof. It suffices to observe that if $G = G_i$, $i = 1,\dots,9$ (of Fig.4.1), then there exists an edge-automorphism of G not induced by any automorphism.

Corollary 4.2. Let G be a nontrivial connected multigraph. Then

$$\Gamma^*(G) = \Gamma'(G)$$

if and only if $G \neq G_i$, $i = 1,\dots,9$ (of Fig.4.1).

Theorem 4.1 may be generalized to arbitrary multigraphs. In Fig.4.2 we have shown multigraphs "similar" to the multigraphs of Fig.4.1 but not isomorphic to them. G_i' and G_i'', $i = 1,2,4,5$ are multiversions of multigraphs with additional conditions whose edges being represented by solid lines and G_3' and G_3'' was obtained in similar way as G_3 (of Fig.4.1). G_6' and G_6'' are multigraphs having terminal pieces H_i' and H_i'' based at u_i' and u_i'', $i \in I$, $|I| \geq 1$, respectively, and furthermore:

(a) none of the multigraphs G_6' and G_6'' is isomorphic to G_i, $i = 1,2,3,6,7,8,9$ (of Fig.4.1),

(b) for every pair H_i' and H_i'', $i \in I$, $|X(H_i')| = |X(H_i'')|$ and $H_i' \neq H_i''$,

(c) H' and H'' are isomorphic and there is an isomorphism φ of H' onto H'' such that $\varphi(u_i') = u_i''$ for each $i \in I$,

(d) H_i' and H_i'' are either the terminal pieces A_i, $i = 1,2,3$ or B_1 and one of B_j, $j = 2,3,4$ (of Fig.4.1) based at u_i.

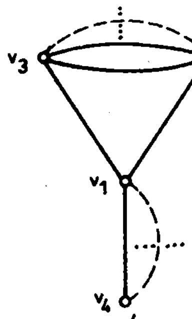
T h e o r e m 4.2. Let G be a nonempty multigraph. If neither

1) G_i, $i = 1,\dots,9$ (of Fig.4.1) is a component of G nor

2) both a) K_3 and $K_{1,3}$, b) G_i' and G_j'', $i,j=1,2,3$ (of Fig.4.2), and c) G_i' and G_i'', $i=4,5,6$ (of Fig.4.2) are components of G, then

$$\Gamma^*(G) \simeq \Gamma'(G).$$

P r o o f . Assume G to be a multigraph satisfying 1) and 2). Since $\Gamma^*(G)$ is a subgroup of $\Gamma'(G)$, it suffices to show that any edge-automorphism ψ of G is induced by an automorphism. For every component K of G, the subgraph $\langle\psi(X(K))\rangle$ is also a component of G. If K is isomorphic to one of multigraphs K_3, $K_{1,3}$, G_i', G_i'', $i=1,\dots,6$ (of Fig.4.2), then, by Theorem 2.4, since G satisfies 2), we have $\langle\psi(X(K))\rangle = K$. Therefore, if ψ is restricted to K, then ψ is induced by an automorphism of K. If K



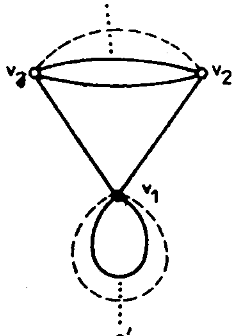
G$'_1$

$F(G'_1)^{-1}(v_1,v_4) = A'_1$

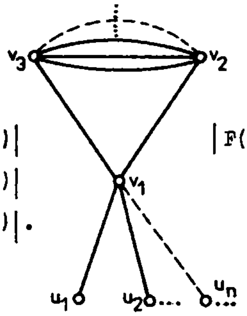
$F(G'_2)^{-1}(v_1,v_1) = A'_2$

$F(G'_3)^{-1}(u_1,v_1) \cup \ldots \cup$

$\cup F(G'_3)^{-1}(u_n,v_1) \cup \ldots = A'_3$



G$'_2$



G$'_3$

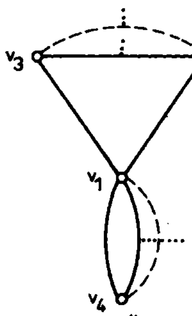
$1 \leqslant |A'_1| < |F(G'_1)^{-1}(v_2,v_3)|$

$1 \leqslant |A'_2| < |F(G'_2)^{-1}(v_2,v_3)|$

$2 \leqslant |A'_3| < |F(G'_3)^{-1}(v_2,v_3)|.$

$|F(G'_i)^{-1}(v_1,v_2)| = |F(G'_i)^{-1}(v_1,v_3)| = 1$

for $i=1,2,3$.



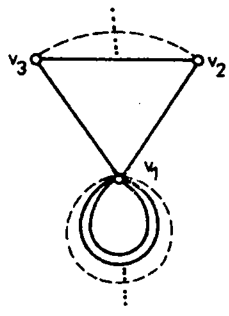
G$''_1$

$F(G''_1)^{-1}(v_1,v_4) = A''_1$

$F(G''_2)^{-1}(v_1,v_1) = A''_2$

$F(G''_3)^{-1}(u_1,v_1) \cup \ldots \cup F(G''_3)^{-1}(u_n,v_1)$

$\ldots = A''_3$



G$''_2$

$1 \leqslant |F(G''_1)^{-1}(v_2,v_3)| < |A''_1|$

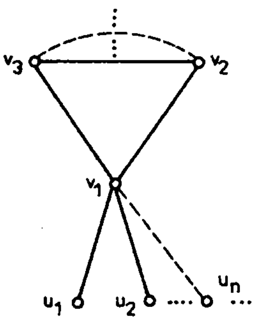
$1 \leqslant |F(G''_2)^{-1}(v_2,v_3)| < |A''_2|$

$1 \leqslant |F(G''_3)^{-1}(v_2,v_3)| < |A''_3|$

$|F(G''_i)^{-1}(v_2,v_3)| = |A''_j|$

$|F(G''_j)^{-1}(v_2,v_3)| = |A'_i|$

for $i,j=1,2,3$.

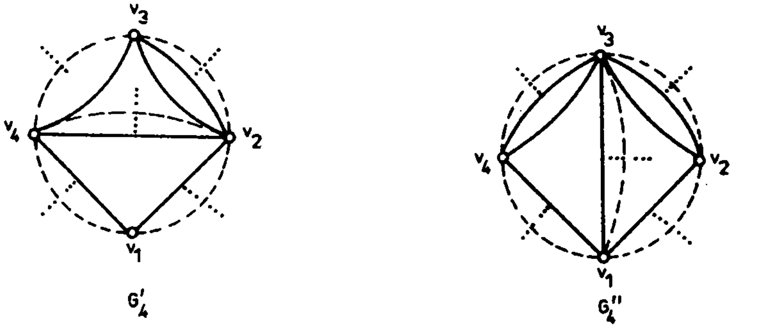


G$''_3$

$|F(G''_i)^{-1}(v_1,v_2)| =$

$= |F(G''_i)^{-1}(v_1,v_3)| = 1$

for $i=1,2,3$.



$$1 \leqslant \left| F(G'_4)^{-1}(v_2,v_4) \right| = \left| F(G''_4)^{-1}(v_1,v_3) \right|,$$

$$1 \leqslant \left| F(G'_i)^{-1}(v_1,v_2) \right| = \left| F(G''_i)^{-1}(v_1,v_2) \right| < \left| F(G'_i)^{-1}(v_3,v_4) \right| = \left| F(G''_i)^{-1}(v_3,v_4) \right|,$$

$$1 \leqslant \left| F(G'_i)^{-1}(v_1,v_4) \right| = \left| F(G''_i)^{-1}(v_1,v_4) \right| < \left| F(G'_i)^{-1}(v_2,v_3) \right| = \left| F(G''_i)^{-1}(v_2,v_3) \right|, \quad i = 4,5,$$

$$1 \leqslant \left| F(G'_5)^{-1}(v_2,v_4) \right| = \left| F(G''_5)^{-1}(v_1,v_3) \right| < \left| F(G'_5)^{-1}(v_1,v_3) \right| = \left| F(G''_5)^{-1}(v_2,v_4) \right|.$$

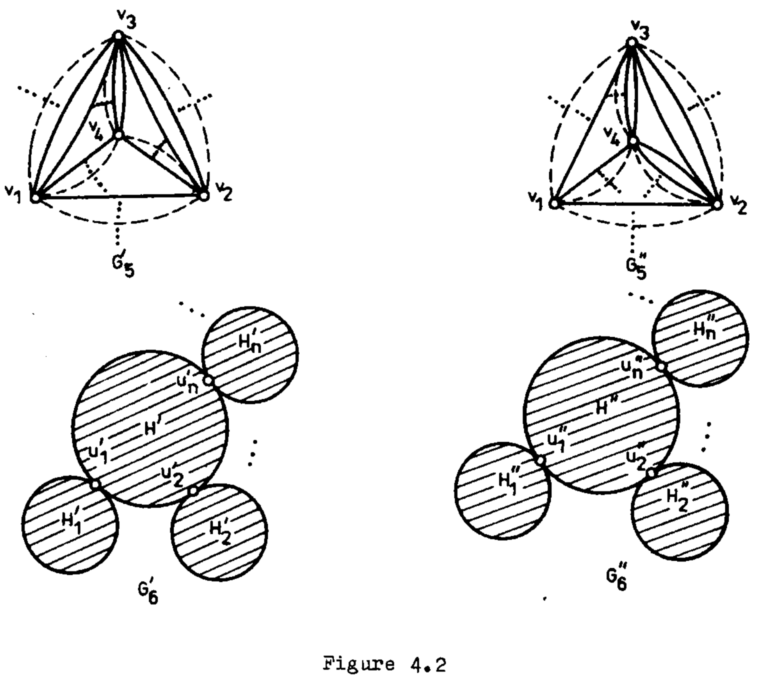


Figure 4.2

is not isomorphic to K_3, $K_{1,3}$, G_i', and G_i'', $i=1,\dots,6$, then, by 1), Theorem 2.4 implies that if ψ is restricted to K, then ψ is induced by an automorphism of K. Hence by applying the above argument to every component of G, we obtain that ψ is induced by an automorphism of G.

Corollary 4.3. Let G be a nonempty multigraph with $|\Gamma^*(G)| < \infty$. Then

$$\Gamma^*(G) \simeq \Gamma'(G)$$

if and only if neither G_i, $i=1,\dots,9$ (of Fig.4.1) is a component of G nor both K_3 and $K_{1,3}$, G_i' and G_j'', $i,j=1,2,3$ (of Fig.4.2), and G_i' and G_i'', $i=4,5,6$ (of Fig.4.2) are components of G.

Corollary 4.4. Let G be a nonempty multigraph. Then

$$\Gamma^*(G) = \Gamma'(G)$$

if and only if neither G_i, $i=1,\dots,9$ (of Fig.4.1) is a component of G nor both K_3 and $K_{1,3}$, G_i' and G_j'', $i,j=1,2,3$ (of Fig.4.2), and G_i' and G_i'', $i=4,5,6$ (of Fig.4.2) are components of G.

REFERENCES

[1] M. Behzad, G. Chartrand: Introduction to the theory of graphs. Boston 1971.

[2] F. Harary, E.M. Palmer: On the point--group and line-group of a graph, Acta Math. Acad. Sci. Hugn. 19 (1968) 263-269.

[3] R.L. Hemminger: Isomorphism-induced line isomorphism on pseudographs, Czechoslovak Math. J. 21 (96) (1971) 672-679.

[4] G. S a b i d u s s i : Graph derivatives, Math. Z. 76 (1961) 385-401.

[5] S a b i d u s s i G.: Loewy-grupoids related to linear graphs, Amer. J. Math. 76 (1954) 477-487.

[6] H. W h i t n e y : Congruent graphs and the connectivity of graphs, Amer. J. Math. 54 (1932) 150-168.

INSTITUTE OF MATHEMATICS AND PHYSICS, HIGHER ENGINEERING SCHOOL, ZIELONA GÓRA

Received December 11, 1980.

DEMONSTRATIO MATHEMATICA
Vol. XV No 3 1982

Adam Kucharz

ON THE DETERMINATION OF HALL POLYNOMIALS

1. Introduction

The aim of this paper is to deduce formulas for the Hall polynomials for the product ($n = 3,4$; r arbitrary) and for the exponent ($n = 3,4$; r arbitrary, $m = -1$). These formulas essentially accelerate the determination of Hall polynomials, since they require neither the application of the processus of selection, nor the knowledge of the computation rules. By means of them a method is proposed for determining Hall polynomials for the exponent ($n = 3,4$, r and m arbitrary).

The notion of Hall polynomials appears for the first time in Hall's paper on nilpotent groups [1], where the author gives their definition and proves that the defined functions have in fact the form of polynomials. In papers [2], [3] the Hall polynomials for the product and the inverse were determined for $n \leqslant 4$, $r = 4$, $m = -1$ and a method was presented for determining them for $n \leqslant 4$ and for an arbitrary number r of generators. The present paper rests on the results of papers [1], [2], [3] and on the classification of basic commutators given in [4].

2. Hall polynomials

Let $G(X)$ be a free nilpotent group of a given nil n, generated by the set $X = \{x_1, x_2, \ldots, x_r\}$. Let t be the number of basic commutators. Any element $g \in G(X)$ can be written in the form

$$g = u_1^{g_1} u_2^{g_2} \dots u_t^{g_t},$$

where $u_1,\dots,u_t$ are basic commutators of weight $1,2,\dots,n$; $u_1 \leqslant u_2 \leqslant \dots \leqslant u_t$, and $g_1,g_2,\dots,g_t$ are integers. For the definition of basic commutators we refer to paper [7], chapter 11.

Let a and b be arbitrary elements of the group $G(X)$ and

(1.1) $$a = u_1^{a_1} \dots u_t^{a_t}, \quad b = u_1^{b_1} \dots u_t^{b_t};$$

then, according to [1], for the elements $p = ab$ and $q = a^m$ (m integer) we have the relation

(1.2) $$p = u_1^{p_1} \dots u_t^{p_t}; \quad q = u_1^{q_1} \dots u_t^{q_t},$$

where

$$p_i = a_i + b_i + P_i(a_1,\dots,a_{i-1};b_1,\dots,b_{i-1})$$

$$q_i = ma_i + E_i(a_1,\dots,a_{i-1};m),$$

where P_i, E_i are Hall polynomials for the product and the exponent, respectively ($i = 1,\dots,t$).

3. Formulas for the Hall polynomials

The relations given in tables 1, 2, 3, 4 are based on the forms, determined in [3], of the Hall polynomials and on the classification of the basic commutators given in [4]. It has been observed that to a determined form of the commutator corresponds uniquely a form of the Hall polynomial.

3.1. Let us now introduce the following classification of the basic commutators ($n = 3,4$; we assume the notation: $(u_i,u_j,\dots) = ij\dots$)):

basic commutators of weight 3

1. ijj; 2. jij; 3. kji, $k < i$; 4. kji, $k > i$

basic commutators of weight 4

1. ijjj, $i > j$; 2. ijji; 3. ijii; 4.1 ikjj, $i > j$; 4.2 ikjj, $i < j$; 5. ikji; 6.1 ikij, $i > j$; 6.2. ikij, $i < j$; 7. ijkj - complex commutator; 8.1 ijjk, $i < k$; 8.2 ijjk, $i > k$; 8.3 ijjk - complex commutator; 9.1 ijkt, $i < k$, $i < t$; 9.2 ijkt, $i > k$, $i < t$; 9.3 ijkt, $i > k$, $i > t$; 9.4 ijkt, $j < k$, $j < t$ - complex commutator; 9.5 ijkt, $j > k$, $j > t$ - complex commutator.

3.2. The commutator of the structure ijj is represented by the commutator 211 and the corresponding Hall polynomial has the form

$$p_{211} = a_2 \binom{b_1}{2} + a_{21} b_1.$$

Hence we can write the general relation for the Hall polynomials for the product for $n = 3$, based on the basic commutators of the structure ijj:

$$p_{ijj} = a_i \binom{b_j}{2} + a_{ij} b_j.$$

Proceeding as above we can obtain general formulas for the Hall polynomials for the product and the exponent ($n = 3,4$; $m = -1$; r arbitrary). They are given in tables 1-4.

4. Method of determining Hall polynomials for any m (for $n = 3,4$)

4.1. Let

$$p = (u_1^{a_1} \ldots u_t^{a_t})^m. \tag{4.1}$$

It is easy to reduce the formula (4.1) to the form (1.2) for $m = 2$; to do this we have merely to modify formulas for the

Hall polynomials for the product (tables 1 and 3) by setting $b_i = a_i$. Then

$$I_{k,2} = P_k \Big|_{b_i=a_i},$$

where $I_{k,2}$ is the Hall polynomial for the exponent, with number k and $m = 2$, P_k is the Hall polynomial for the product with number k. Moreover, we have for $m < 0$

$$p = (u_1^{a_1} \dots u_t^{a_t})^{-m} = ((u_1^{a_1} \dots u_t^{a_t})^m)^{-1}.$$

4.2. Let $m > 0$. For $m = 2$ we have

$$p = (\dots u_i^{a_i} \dots)^2 = (\dots u_i^{a_i} \dots)(\dots u_i^{a_i} \dots) = \dots u_i^{a_i+a_i+P_{i_1}} \dots$$

where P_{i_1} is obtained from P_i by substituting $b_j = a_j$, $j = (1,\dots,i-1)$.

For $m = 3$ we have

$$p = (\dots u_i^{a_i} \dots)^3 = (\dots u_i^{a_i} \dots)^2 (\dots u_i^{a_i} \dots) =$$
$$= \dots i_i^{3a_i+P_{i_1}+P_{i_2}} \dots$$

where P_{i_2} is obtained from P_i by substituting $a_j = 2a_j + P_{j_1}$, $b_j = a_j$, $j = (1,\dots,i-1)$.

For arbitrary m

$$p = (\dots u_i^{a_i} \dots)^m = \dots u_i^{ma_i+P_{i_1}+P_{i_2}+\dots+P_{i_{m-1}}} \dots$$

where

$$P_{i_1} = P_i \Big|_{a_j=a_j,\, b_j=a_j}$$

$$P_{i_2} = P_i \Big|_{a_j=2a_j+P_{j_1},\, b_j=a_j}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_{i_m} = P_i \Big|_{a_j=ma_j+P_{j_1}+\ldots+P_{j_{n-1}},\, b_j=a_j}.$$

Finally, for arbitrary $m > 0$ the i-th Hall polynomial for the exponent takes the form

$$I_{i,m} = P_{i_{m-1}} \quad (m > 0). \tag{4.2}$$

4.3. Let now $m < 0$; then, taking into account §§4.1,4.2 and the definition of the Hall polynomials we have:

$$p = (\ldots\, u_i^{a_i} \ldots)^{-m} = \left((\ldots\, u_i^{a_i} \ldots)^{m}\right)^{-1} =$$

$$= \left(\ldots\, u_i^{ma_i+\sum_{k=1}^{m-1} P_{i_k}} \ldots\right)^{-1} =$$

$$= \ldots\, u_i^{-\left. \left(ma_i+\sum_{k=1}^{m-1} P_{i_k} + I_i(a_1,\ldots,a_{i-1}) \right)\right|_{a_j}} =$$

$$= ma_j + \sum_{k=1}^{m-1} P_{j_k}.$$

Finally, for arbitrary $m < 0$ the i-th Hall polynomial for the exponent has the form

$$(4.3) \qquad I_{i,m} = I_{i,-1}\Big|_{a_j = ma_j + \sum_{k=1}^{m-1} P_{j_k}} \qquad (m < 0).$$

E x a m p l e . We shall determine the Hall polynomial for the exponent for $n = 3$, $r = 4$, $m = -3$. From relation (4.3) we get

$$I_{i,-3} = I_{i,-1}\Big|_{a_j = 3a_j + P_{j_1} + P_{j_2}}$$

(it should be observed that i may be considered not only as the number of the commutator, but also as the form of the basic commutator). We shall find the form of the Hall polynomial based on the commutator jii; according to table 2 we have

$$I_{ijj} = -\binom{-a_j}{2} a_i + a_j a_{ij}\Big|_{a_j = 3a_j + P_{j_1} + P_{j_2}} .$$

We next calculate P_{j_1} and P_{j_2}, where $j \in (i, j, ij)$, substitute and get

$$I_{ijj} = -\binom{-3a_j}{2} 3a_i + 3a_j 3a_{ij} + 3a_i a_j .$$

5. Remarks on the determination of the Hall polynomials for $n > 4$

It is probable that this problem may be solved by determining, by means of an arbitrary method, formulas for the Hall polynomials for several initial values of n, for polynomials based on basid commutators of similar structure.

Table 1
Hall polynomials formulae for the product, nil = 3

commutator	formula
1. ijj	$P_{ijj} = a_i \binom{b_j}{2} + a_{ij}b_j$
2. iji	$P_{iji} = b_j \binom{a_i}{2} + a_{ij}b_i + a_i b_j b_i$
3. kji $k < i$	$P_{kji} = a_k b_j b_i + a_k a_i b_j + a_{kj} b_i - a_{ik} b_j$
4. kji $k > i$	$P_{kji} = a_k b_j b_i + a_{kj} b_i + a_{ki} b_j$

Table 2
Hall polynomials formulae for the exponent, nil = 3

commutator	formula
1. ijj	$I_{ijj} = - \binom{-a_j}{2} a_i + a_j a_{ij}$
2. iji	$I_{iji} = - \binom{-a_i}{2} a_j + a_i a_{ij}$
3. kji $k > i$	$I_{kji} = -a_j a_k a_i + a_j a_{ki} + a_i a_{kj}$
4. kji $k < i$	$I_{kji} = -a_j a_{ik} + a_i a_{kj}$

Table 3
Hall polynomials formulae for the product, nil = 4

commutator	formula
1. ijjj	$P_{ijjj} = a_i \binom{b_j}{2} + a_{ij} \binom{b_j}{2} + a_{ijj} b_j$
2. ijji	$P_{ijji} = \binom{a_i}{2} \binom{b_j}{2} + a_i b_i \binom{b_j}{2} + a_{ij} b_i b_j + a_{ijj} b_i + a_{iji} b_j$

3. ijii	$P_{ijii} = \binom{a_i}{3} b_j + \binom{a_i}{2} b_i b_j + a_i b_j \binom{b_i}{2} +$ $+ a_{ij} \binom{b_i}{2} + a_{iji} b_i$
4.1 ikjj $i < j$	$P_{ikjj} = a_i b_k \binom{a_j}{2} + a_i b_k \binom{b_j}{2} + a_{ik} \binom{b_j}{2} +$ $+ a_i a_j b_k b_j - a_{ji} b_k b_j + a_{ikj} b_j - a_{jij} b_k$
4.2 ikjj $i > j$	$P_{ikjj} = a_i b_k \binom{b_j}{2} + a_{ik} \binom{b_j}{2} + a_{ij} b_k b_j +$ $+ a_{ikj} b_j + a_{ijj} b_k$
5. ikji	$P_{ikji} = \binom{a_i}{2} b_k b_j + a_i b_k b_j b_i + a_{ik} b_j b_i +$ $+ a_{ij} b_k b_i + a_{ikj} b_i + a_{iki} b_j + a_{iji} b_k$
6.1 ikij $j < i$	$P_{ikij} = \binom{a_i + 1}{2} b_j b_k + a_i a_{ij} b_k + a_i b_k b_{ij} +$ $+ a_{ik} b_{ij} - a_{iji} b_k$
6.2 ikij $j > i$	$P_{ikij} = \binom{a_i}{2} a_j b_k + \binom{a_i}{2} b_j b_k + a_i a_j b_k b_i +$ $+ a_i b_k b_i b_j + a_{ik} b_i b_j - a_{ji} b_k b_i + a_{iki} b_j +$ $+ a_{ikj} b_i - a_{jii} b_k$
7. ijkj	$P_{ijkj} = a_i a_{kj} b_j + a_i b_j b_{kj} + a_{ij} b_{kj} +$ $+ a_{ik} b_j b_j - a_{kji} b_j + a_{ijk} b_j$
8.1 ijjk $k > i$	$P_{ijjk} = a_i a_k \binom{b_j}{2} + a_i b_k \binom{b_j}{2} + a_{ij} b_j b_k +$ $- a_{ki} \binom{b_j}{2} + a_{ijj} b_k + a_{ijk} b_j$
8.2 ijjk $k < i$	$P_{ijjk} = a_i b_k \binom{b_j}{2} + a_{ik} \binom{b_j}{2} + a_{ij} b_j b_k +$ $+ a_{ijj} b_k + a_{ijk} b_j$

8.3	ijjk	$P_{ijjk} = a_i a_{jk} b_j + a_i b_j b_{jk} + a_{ij} b_{jk} + a_{ij} b_k b_j + a_{ij} b_{jk} - a_{jki} b_j + 2a_{ijj} b_k$
9.1	ijkt i < k i < t	$P_{ijkt} = a_i a_k b_j b_t + a_i a_k a_t b_j + a_i a_t b_j b_k + a_i b_j b_k b_t + a_{ij} b_k b_t - a_{ki} b_j b_t - a_{ti} b_j b_k + a_{ijk} b_t + a_{ijt} b_k - a_{kit} b_j - a_{tik} b_j$
9.2	ijkt i > k i < t	$P_{ijkt} = a_i a_t b_j b_k + a_i b_j b_k b_t + a_{ij} b_k b_t + a_{ik} b_j b_t - a_{ti} b_j b_k + a_{ijk} b_t + a_{ijt} b_k + a_{ikt} b_j$
9.3	ijkt i > k i > t	$P_{ijkt} = a_i a_j b_k b_t + a_{ij} b_k b_t + a_{ik} b_j b_t + a_{it} b_j b_k + a_{ijk} b_t + a_{ijt} b_k + a_{ikt} b_j$
9.4	ijkt j < k j < t	$P_{ijkt} = a_i a_{kt} b_j + a_i b_j b_{kt} + a_{ij} b_{kt} + a_{ik} b_j b_t - a_{kti} b_j + a_{ijk} b_t$
9.5	ijkt j > k j > t	$P_{ijkt} = a_i a_{kt} b_j + a_i b_j b_{kt} + a_{ik} b_t b_j + - a_{kti} b_j + a_{ijk} b_t$

Table 4
Hall polynomials formulae for the exponent, nil = 4

commutator	formula
1. ijjj	$I_{ijjj} = -\binom{-a_j}{3}a_i - \binom{-a_j}{2}a_{ij} + a_j a_{ijj}$
2. ijji	$I_{ijji} = \binom{-a_j}{2}\binom{-a_j}{2} - a_j a_i a_{ij} + a_j a_{iji} + a_i a_{ijj}$
3. ijii	$I_{ijii} = -\binom{-a_i}{3} a_j - \binom{-a_i}{2} a_{ij} + a_i a_{iji}$
4.1 ikjj $i > j$	$I_{ikjj} = \binom{-a_j}{2} a_k a_i - a_k a_j a_{ij} + a_k a_{ijj} +$ $- \binom{-a_j}{2} a_{ik} + a_j a_{ikj}$
4.2 ikjj $i < j$	$I_{ikjj} = a_k a_j a_{ji} - a_k a_{jij} + a_j a_{ikj} - \binom{-a_j}{2} a_{ik}$
5. ikji	$I_{ikji} = a_k a_j \binom{-a_i}{2} - a_k a_i a_{ij} + a_k a_{iji} - a_j a_i a_{ik} +$ $+ a_j a_{iki} + a_i a_{ikj}$
6.1 ikij $j < i$	$I_{ikij} = a_j a_k \binom{-a_i + 1}{2} - a_j a_i a_{ik} + a_j a_{ik} +$ $+ a_j a_{iki} - a_k a_{iji} + a_{ij} a_{ik}$
6.2 ikij $j > i$	$I_{ikij} = -a_k a_{jii} + a_k a_i a_{ji} - a_i a_j a_{ik} + a_i a_{ikj} +$ $+ a_j a_{iki}$
7. ijkj	$I_{ijkj} = a_j a_j a_k a_i - a_j a_j a_{ik} - a_j a_k a_{ij} - a_j a_{kji} +$ $+ a_j a_{kji} + a_{kj} a_{ij}$

8.1 ijjk $k > i$	$I_{ijjk} = \binom{-a_j}{2} a_{ki} - a_j a_k a_{ij} + a_j a_{ijk} + a_k a_{ijj}$
8.2 ijjk $k < i$	$I_{ijjk} = \binom{-a_j}{2} a_k a_i - \binom{-a_j}{2} a_{ik} - a_j a_k a_{ij} +$ $+ a_j a_{ijk} + a_k a_{ijj}$
8.3 ijjk	$I_{ijjk} = a_k a_j a_j a_i - 2a_k a_j a_{ij} + a_k a_{ij} + 2a_k a_{ijj} +$ $- a_j a_{jki} + a_{jk} a_{ij}$
9.1 ijkt $i < k$ $i < t$	$I_{ijkt} = a_j a_k a_{ti} + a_j a_t a_{ki} - a_j a_{kit} - a_j a_{tik} +$ $- a_k a_t a_{ij} + a_k a_{ijt} + a_t a_{ijk}$
9.2 ijkt $i < t$ $i > k$	$I_{ijkt} = a_j a_k a_{ti} - a_j a_t a_{ik} - a_j a_{ikt} - a_k a_t a_{ij} +$ $+ a_k a_{ijt} + a_t a_{ijk}$
9.3 ijkt $i > t$ $i > k$	$I_{ijkt} = a_i a_j a_k a_t - a_j a_k a_{it} - a_j a_t a_{ik} + a_j a_{ikt} +$ $- a_k a_t a_{ij} + a_k a_{ijt} + a_t a_{ijk}$
9.4 ijkt $j < k$ $j < t$	$I_{ijkt} = a_i a_j a_k a_t - a_j a_t a_{ik} - a_j a_{kti} - a_t a_k a_{ij} +$ $+ a_t a_{ijk} + a_{kt} a_{ij}$
9.5 ijkt $j > k$ $j > t$	$I_{ijkt} = a_i a_j a_k a_t - a_t a_k a_{ij} - a_j a_{kti} - a_t a_j a_{ik} +$ $+ a_t a_{ijk} + a_{kt} a_{ij}$

On the ground of tables 1 and 3 it is possible to find the general form of the Hall polynomials for the product for any n, based on the commutator of the form ijj ...; thus we obtain a relation of the form

$$P_{ijj\ldots\ \ldots jj} = \sum_{k=2}^{n-2} a_{ij}{}^{k} \binom{b_j}{n-(k+1)}$$

where $j^k = \underbrace{jj\ldots jj}_{k}$.

To establish this general formula we had merely to apply the form of the polynomial for $n = 3, 4,$ owing to the simple structure of the basic commutator. If the commutator has a more complex structure, this does not suffice; the polynomials based, for instance, on the commutators of structures iji, ijii contain monomials of the structure

$$b_j\binom{a_i}{n-1}; \quad a_{ij}\binom{b_i}{n-2}; \quad a_i b_j\binom{b_i}{n-2}.$$

Should we know the polynomials for some further values of n, based on the commutator ijii... ...ii, it should be probably possible to find the general relation.

REFERENCES

[1] P. H a l l : Nilpotent groups, Canadian Mathematical Congress 1957 in Russian , Matematika 12/1 (1968).

[2] A.W. M o s t o w s k i : Computational algorithms for deciding some problems for nilpotent groups, Fund. Math., 59 (1966) 137-152.

[3] E. C z y ż o : Problemy programowania zagadnień teorii grup i automatyzacja rachunku komutatorów [in Polish] Doctoral Thesis, Warsaw University (1974).

[4] K. D a ł e k : Computation of exponents in Hall formula, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 19 (1971) 711-718.

[5] A. K u c h a r z : Wielomiany Halla (in Polish), Master Thesis, Warsaw University (1979).

[6] E. C z y ż o : On the determination of Hall polynomials, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 23 (1975), 739-745.

[7] M. H a l l : The theory of groups. New York 1959.

Address of the Author: ul. 3 Maja 33 m.5, 05-120 Legionowo
Received December 12, 1980.

DEMONSTRATIO MATHEMATICA
Vol. XV No 3 1982

Romuald Małecki, Ewa Zielińska

CONTOUR-INTEGRAL METHOD APPLIED FOR SOLVING A CERTAIN MIXED PROBLEM FOR A PARABOLIC EQUATION OF ORDER FOUR

In this paper, the existence and uniqueness of a certain mixed problem with nonhomogeneous boundary conditions is proved by using the contour-integral method introduced by M.L.Rasulov [1]. In the case of nonhomogeneous boundary conditions it is necessary to use a certain generalized Laplace's transform given by the relation

$$\bar{f}(\lambda) = L_p(f(t))(\lambda) = \int_0^{+\infty} \exp(-\lambda^p t)f(t)dt,$$

where p is a natural number.

For the above transform L_p the following theorem has been proved in [2].

Theorem 1. If

1^0 f is continuously differentiable for $t \geq 0$, except may-be countable number of points in which f and its derivative can possess discontinuity of first kind, but on each bounded interval $\langle 0,t\rangle$ the number of such points is finite,

2^0 there exists a $M > 0$ and a $s \geq 0$ such that

$$|f(t)| \leq M \exp(st),$$

3^0 there exists an $\alpha \in (0, \frac{\pi}{2p})$ such that for an analitic continuation $\bar{f}(\lambda)$ the following formula holds: $\lim_{\lambda\to\infty} \bar{f}(\lambda) = 0$ uniformly with respect to arg λ for $|\arg\lambda| \leq \frac{\pi}{2p} + \alpha$,

4^{0} S is an infinite curve lying in a region $|\arg\lambda| \leqslant \frac{\pi}{2p} + \alpha$ and coincides, for sufficiently large $|\lambda|$, with half-lines $|\arg\lambda| = \frac{\pi}{2p} + \alpha$,
then in each point of continuity of function f we have

$$f(t) = \frac{p}{2\lambda\sqrt{-1}} \int_S \lambda^{p-1} \exp(\lambda^p t)\bar{f}(\lambda)d\lambda.$$

The L_2 transform will be applied to solve the following problem

$$(1) \quad \frac{\partial^2 v(x,t)}{\partial t^2} = \sum_{k=0}^{1} \sum_{l=0}^{4-2k} A_{kl}(x) \frac{\partial^{k+l} v(x,t)}{\partial t^k \partial x^l} \quad \text{for } (x,t)\in(a,b)\times(0,T\rangle,$$

$$(2) \quad \sum_{k=0}^{2} \sum_{l=0}^{3} \left\{ \alpha_{skl} \left.\frac{\partial^{k+l} v(x,t)}{\partial t^k \partial x^l}\right|_{x=a} + \beta_{skl} \left.\frac{\partial^{k+l} v(x,t)}{\partial t^k \partial x^l}\right|_{x=b} \right\} =$$

$$= g_s(t) \quad \text{for } s = 1,2,3,4, \quad t \in (0,T\rangle,$$

$$(3) \quad \left.\frac{\partial^k v(x,t)}{\partial t^k}\right|_{t=0} = 0 \quad \text{for } k = 0,1, \quad x \in (a,b),$$

where A_{kl} are given functions and the constants $T > 0$, α_{skl}, β_{skl} are real numbers.

Let us suppose that the following conditions are satisfied:

$$(I) \quad A_{kl} \in \begin{cases} C^i(\langle a,b\rangle), & 2k+l = 2+i \quad \text{for } i = 1,2, \\ C(\langle a,b\rangle), & 2k+l \leqslant 2 \quad \text{for } k=0,1;\ l=0,\dots,4-2k, \end{cases}$$

$$A_{04}(x) \neq 0 \quad \text{for} \quad x \in \langle a,b\rangle,$$

(II) g_s, $s = 1,2,3,4$, satisfy conditions 1^{0}, 2^{0} and 3^{0} of Theorem 1 for $p = 2$,

(III) there exists $\delta \in (0, \frac{\pi}{2})$ such that $\frac{\pi}{2} + \delta \leq \arg \gamma \leq$ $\leq \frac{3\pi}{2} - \delta$, where γ satisfies the characteristic equation

$$A_{04}(x) - A_{12}(x)\gamma - \gamma^2 = 0. \tag{4}$$

This means that equation (1) is parabolic in the Petrovsky sense.

Applying formally L_2 to (1), (2) and making use of (3) we arrive at the following spectral problem

$$\sum_{k=0}^{1} \sum_{l=0}^{4-2k} \lambda^{2k} A_{kl}(x) \frac{d^l u(x,\lambda)}{dx^l} - \lambda^4 u(x,\lambda) = 0, \tag{5}$$

$$\sum_{k=0}^{2} \sum_{l=0}^{3} \left\{ \alpha_{skl} \lambda^{2k} \frac{d^l u(x,\lambda)}{dx^l}\bigg|_{x=a} + \beta_{skl} \lambda^{2k} \frac{d^l u(x,\lambda)}{dx^l}\bigg|_{x=b} \right\} = \bar{g}_s(\lambda), \tag{6}$$

$s = 1,2,3,4,$ where $\bar{g}_s(\lambda) = L_2(g_s(t))(\lambda) = \int_0^{+\infty} \exp(-\lambda^2 t) \times$ $\times g_s(t)dt.$

The solution of the problem (5) - (6) has the form

$$u(x,\lambda) = \frac{1}{\Delta(\lambda)} \sum_{k=1}^{4} \bar{g}_k(\lambda) \sum_{i=1}^{4} y_i(x,\lambda) \cdot \Delta_{ki}(\lambda), \tag{7}$$

where $y_i(x,\lambda)$, $i = 1,2,3,4$, is a fundamental system of particular solutions of (5),

$$\Delta(\lambda) = \det \left[u_{ki}(\lambda)\right]_{4\times4}, \tag{8}$$

$$(9)\qquad u_{ki}(\lambda) = \sum_{l=0}^{3}\sum_{r=0}^{2}\lambda^{2r}\left\{\alpha_{krl}\left.\frac{d^l y_i}{dx^l}\right|_{x=a} + \beta_{krl}\left.\frac{d^l y_i}{dx^l}\right|_{x=b}\right\}$$

and $\Delta_{ki}(\lambda)$ denotes the cofactor of $u_{ki}(\lambda)$ in $\Delta(\lambda)$. Substituting

$$(10)\qquad \begin{cases} \alpha_{kl}(\lambda) = \sum_{r=0}^{2}\lambda^{2r}\alpha_{krl} \\ \beta_{kl}(\lambda) = \sum_{r=0}^{2}\lambda^{2r}\beta_{krl} \end{cases} \qquad s=1,2,3,4,\; l=0,1,2,3$$

into (9) we get

$$(11)\qquad u_{ki}(\lambda) = \sum_{l=0}^{3}\left\{\alpha_{kl}\left.\frac{d^l y_i}{dx^l}\right|_{x=a} + \beta_{kl}\left.\frac{d^l y_i}{dx^l}\right|_{x=b}\right\}.$$

To obtain an asymptotic representation of the solution (7) and of its derivatives, we use the Tamarkin theorem [1]. Consider the differential equation

$$(12)\qquad \frac{d^n y}{dx^n} + \sum_{i=0}^{n} P_i(x,\lambda)\,\frac{d^{n-i} y}{dx^{n-i}} = 0$$

with coefficients having expansions of the form

$$P_i(x,\lambda) = \sum_{r=0}^{\infty}\lambda^{i-r}P_{ir}(x),$$

for $x \in \langle a,b\rangle$ and $|\lambda| \geq R$, where $R > 0$ is sufficiently large.

Tamarkin's theorem. If

1^0 the functions P_{ir}, $i=1,\dots,n$, $r=0,1,\dots$, are continuous and uniformly bounded on interval $\langle a,b\rangle$,

2^0 at least one of the functions P_{io}, $i=1,\dots,n$, is not identically zero on the interval $\langle a,b\rangle$,

3^0 the solutions $\varphi_k(x)$, $k=1,\dots,n$, of the quasi-characteristic equation of (12) i.e. of

$$(13)\qquad \theta^n + \sum_{i=1}^{n} P_{io}(x)\,\theta^{n-i} = 0$$

are distinct for all values of $x \in \langle a,b\rangle$,

4^0 $\dfrac{d^m P_{io}}{dx^m}, \dfrac{d^{m-1}P_{i1}}{dx^{m-1}}, \dots, P_{im} \in C^q(\langle a,b\rangle)$, $i=1,\dots,n$,

where $q \geq 1$ and $m \geq 1$ are natural numbers,

5^0 there exists an unbounded portion Ω_1 of the region $\Omega_R := \{\lambda;\ |\lambda| \geq R\}$ in which the inequalities
$\operatorname{Re}\lambda\varphi_1(x) \leq \operatorname{Re}\lambda\varphi_2(x) \leq \dots \leq \operatorname{Re}\lambda\varphi_n(x)$
hold for all $x \in \langle a,b\rangle$ with suitable numbering of the solutions φ_k,

then (12) has fundamental system of particular solutions $y_s(x,\lambda)$, $s=1,\dots,n$, which together with their first $n-1$ derivatives have asymptotic representations of the form

$$(14)\qquad \frac{d^k y_s(x,\lambda)}{dx^k} = \lambda^k \exp\left(\lambda\int_a^x \varphi_s(\xi)d\xi\right)\left[\sum_{r=0}^{m-1}\frac{\eta_{ks}^{(r)}(x)}{\lambda^r} + \frac{E_{ks}(x,\lambda)}{\lambda^m}\right],$$

$k = 0,\dots,n-1$, $s = 1,\dots,n$, where the functions E_{ks} are continuous with respect to $x \in \langle a,b\rangle$ and bounded for $\lambda \in \Omega_1$, and $\eta_{ks}^{(r)} \in C^{q-1}(\langle a,b\rangle)$.

R e m a r k 1. The coefficients $\eta_{ks}^{(r)}(x)$ can be obtained by differentiating the expression $\exp\left(\lambda\int_a^x \varphi_s(\xi)d\xi\right) \times \sum_{r=0}^{m-1} \lambda^{-r}\eta_{ks}^{(r)}(x)$ and expanding the result in decreasing powers of λ. It can be shown as well that $\eta_{soo}(x) :=$ $:= \frac{\eta_{ks}^{(o)}(x)}{(\varphi_s(x))^k}$ is independent of the index k.

Asymptotic representation of $y^{(n)}$ is obtained by substituting into (12) the representations (14) for k=0,...,n-1.

In our case the quasi-characteristic equation of (5) has the form

(15) $$A_{04}(x)\ \theta^4 + A_{12}(x)\ \theta^2 - 1 = 0.$$

From (4) and (15) we have

(16) $$\theta = \sqrt{-\frac{1}{\gamma}}$$

and $\arg\theta = \frac{\pi}{2} + \frac{\arg\left(\frac{1}{\gamma}\right) + 2k\pi}{2}$, k = 0,1. Hence, under Condition (III), we get

(17) $$\frac{3\pi}{4} + k\pi + \frac{\delta}{2} \leqslant \arg\theta \leqslant \frac{5\pi}{4} + k\pi - \frac{\delta}{2}, \quad k=0,1.$$

Further we assume the condition

(IV) the solutions $\theta(x)$ of equation (15) are distinct for all $x \in \langle a,b\rangle$ and their arguments and the arguments of their differences are independent of x i.e. $\arg(\theta_k(x)-\theta_s(x))=:$ $=: \psi_{ks} = \text{const}$, $k \neq s$. Hence

$$\mathrm{Re}(\lambda\theta_k(x)) - \mathrm{Re}(\lambda\theta_s(x)) = |\lambda|\,|\theta_k(x)-\theta_s(x)|\cos(\arg\lambda + \psi_{ks})$$

Consequently, by condition (IV), we have

(18) $$\mathrm{Re}\,\lambda\,\theta_k(x) = \mathrm{Re}\,\lambda\,\theta_s(x) \Longleftrightarrow \arg\lambda = \pm\frac{\pi}{2} - \psi_{ks}.$$

The last equalities determine the straight lines passing through the coordinate origin and dividing the λ-plane into a finite number of sectors $\sum_j$. In each of these sectors, for suitable numbering of the solutions $\varphi_k^{\,j}(x)$, the inequalities

(19) $$\mathrm{Re}\ \lambda\varphi_1^{\,j}(x) \leqslant \mathrm{Re}\ \lambda\varphi_2^{\,j}(x) \leqslant \mathrm{Re}\ \lambda\varphi_3^{\,j}(x) \leqslant \mathrm{Re}\ \lambda\varphi_4^{\,j}(x)$$

are satisfied for all $x \in \langle a,b\rangle$.

Let

$$(20) \qquad \Lambda := \left\{\lambda ;\ |\arg\lambda| \leqslant \frac{\pi+\delta}{4}\right\}.$$

Consider the sectors $\Gamma_j := \Lambda \cap \sum_j$. By (17), it is easy to see, that in each sector $\Gamma_j \neq \emptyset$ for suitable numbering of the solutions $\varphi_k^j(x)$ of (15) the following inequalities hold

$$(21) \qquad \operatorname{Re}\lambda\varphi_1^j(x) \leqslant \operatorname{Re}\lambda\varphi_2^j(x) \leqslant -|\lambda||\varphi_2^j(x)|\,\varepsilon < 0 <$$

$$< |\lambda||\varphi_3^j(x)|\,\varepsilon \leqslant \operatorname{Re}\lambda\varphi_3^j(x) \leqslant \operatorname{Re}\lambda\varphi_4^j(x),$$

where

$$(22) \qquad \varepsilon := \sin\frac{\delta}{4} > 0.$$

Thus, by condition I, for suitable numbering of the solutions of (15) and for a sufficiently large $|\lambda|$, the fundamental system $y_i(x,\lambda)$, $i=1,2,3,4$, of (5), has asymptotic representations

$$(23) \qquad \frac{d^l y_i(x,\lambda)}{dx^l} = \lambda^l \exp\left(\lambda\int_a^b \varphi_i^j(\xi)\,d\xi\right)\left(\eta_{li}^{(o)}(x) + \frac{E_{li}(x,\lambda)}{\lambda}\right),$$

where E_{li} are bounded functions for $|\lambda| \geqslant R$.

Substituting (23) into (9) we get

$$(24) \qquad u_{ki}(\lambda) = \sum_{l=0}^{3}\left\{\alpha_{kl}(\lambda)\lambda^l\left(\eta_{li}^{(o)}(a) + \frac{E_{li}(a,\lambda)}{\lambda}\right) + \right.$$

$$\left. + \beta_{kl}(\lambda)\lambda^l\left(\eta_{li}^{(o)}(b) + \frac{E_{li}(b,\lambda)}{\lambda}\right)\exp\left(\lambda\int_a^b \varphi_i(\xi)\,d\xi\right)\right\}.$$

Denoting

$$W_i := \int_a^b \varphi_i(\xi)d\xi,$$

$$A_{ki}(\lambda) = \sum_{l=0}^{3} \alpha_{kl}(\lambda)\lambda^l(\varphi_i^j(a))^l,$$

$$B_{ki}(\lambda) = \sum_{l=0}^{3} \beta_{kl}(\lambda)\lambda^l(\varphi_i^j(b))^l$$

and keeping in mind Remark 1, we get

$$(25) \qquad u_{ki}(\lambda) = A_{ki}(\lambda)\left(\eta_{ioo}(a) + \frac{E(\lambda)}{\lambda}\right) + B_{ki}(\lambda)\left(\eta_{ioo}(b) + \frac{E(\lambda)}{\lambda}\right)\exp(\lambda W_i),$$

where E denotes an arbitrary bounded function.

Let

$$(26) \qquad A := \min_{1\leq i\leq 4}\ \inf_{x\in\langle a,b\rangle} |\varphi_i(x)| > 0.$$

From (21) and (26) we get inequalities

$$(27) \qquad \begin{cases} Re(\lambda W_k) \leq -|\lambda|\,\varepsilon A(b-a) & \text{for } k=1,2, \\ Re(\lambda W_k) \geq |\lambda|\,\varepsilon A(b-a) & \text{for } k=3,4. \end{cases}$$

By (27), putting $\Delta_0(\lambda) = \exp(-\lambda(W_3+W_4))\,\Delta(\lambda)$, we can state that the real parts of the exponential functions occurring in the determinant $\Delta_0(\lambda)$ approach to zero as $|\lambda| \to +\infty$ more rapidly than any positive power of $|\lambda|^{-1}$. Thus, $\Delta_0(\lambda)$ has following asymptotic representation

(28) $$\Delta_0(\lambda) = \vartheta \cdot \left(\det M(\lambda) + \frac{E(\lambda)}{\lambda}\right),$$

where

$$\det M(\lambda) = \begin{vmatrix} A_{11}(\lambda), & A_{12}(\lambda), & B_{13}(\lambda), & B_{14}(\lambda) \\ A_{21}(\lambda), & A_{22}(\lambda), & B_{23}(\lambda), & B_{24}(\lambda) \\ A_{31}(\lambda), & A_{32}(\lambda), & B_{33}(\lambda), & B_{34}(\lambda) \\ A_{41}(\lambda), & A_{42}(\lambda), & B_{43}(\lambda), & B_{44}(\lambda) \end{vmatrix}$$

and

$$\vartheta = \eta_{100}(a)\, \eta_{200}(a)\, \eta_{300}(b)\, \eta_{400}(b).$$

Further we assume the condition

(V) $\vartheta \neq 0$, $\det M(\lambda) \neq 0$ for $|\lambda| \geq R$, R being a sufficiently large number. Let d denote a degree of polynomial $\det M(\lambda)$ and μ coefficient of λ^d. Thus, we can put $\det M(\lambda) = \lambda^d \left(\mu + \frac{E(\lambda)}{\lambda}\right)$.
Finally

(29) $$\Delta(\lambda) = \exp(\lambda(W_3+W_4))\, \Delta_0(x) =$$

$$= \exp(\lambda(W_3+W_4))\, \lambda^d \left(\mu\nu + \frac{E(\lambda)}{\lambda}\right).$$

Let us observe that it comes out from the asymptotic representation (29) that zeros of function $\Delta(\lambda)$ for $\lambda \in \Lambda$ lie in certain circle with radius R. Really, if $\left|\frac{E(\lambda)}{\lambda}\right| < \mu\nu$ for $|\lambda| = R$, then by the Rouché theorem, the function $\mu\nu + \frac{E(\lambda)}{\lambda}$ has not zeros outside the circle with radius R.

Thus, the function described by (7) is analytic with respect to λ on the set $\Omega_R \cap \Lambda$, where $\Omega_R := \{\lambda;\ |\lambda| \geqslant R\}$.

From (29) and above consideration we obtain in $\Omega_R \cap \Lambda$

$$(30)\qquad |\Delta(\lambda)| \geqslant C\ |\lambda|^d\ |\exp \lambda(W_3+W_4)|,$$

where $C > 0$ depends on R.

To get an asymptotic estimation of the function u and of its derivatives, we must find estimations of the cofactors $\Delta_{ki}(\lambda)$. Let c be the maximum of degree in λ for all determinants of 3-rd degree of elements of matrix $M(\lambda)$. Thus, for $\lambda \in \Omega_R \cap \Lambda$ we have

$$(31)\qquad \begin{cases} |\Delta_{ks}(\lambda)| \leqslant D|\lambda|^c\ |\exp(\lambda(W_3+W_4))|, & s=1,2, \\ |\Delta_{k3}(\lambda)| \leqslant D|\lambda|^c\ |\exp(\lambda\ W_4)|, \\ |\Delta_{k4}(\lambda)| \leqslant D|\lambda|^c\ |\exp(\lambda\ W_3)|, \end{cases}$$

for $k=1,2,3,4$, D being positive constant.

By (7), (23), (30) and (31) we get estimations

$$(32)\qquad \left|\frac{d^m u(x,\lambda)}{dx^m}\right| \leqslant \sum_{k=1}^{4} |\bar{g}_k(\lambda)| \sum_{l=1}^{4} |\lambda|^m \left|\exp\left(\lambda \int_a^x \varphi_l^j(\xi)d\xi\right)\right| \times$$

$$\times \left|\eta_{ml}^{(o)}(x) + \frac{E_{ml}(x,\lambda)}{\lambda}\right| \left|\frac{\Delta_{kl}(\lambda)}{\Delta(\lambda)}\right| \leqslant$$

$$\leqslant H \max_{1\leqslant k\leqslant 4} |\bar{g}_k(\lambda)|\,|\lambda|^{m-d+c} \{\exp(-\varepsilon A\,|\lambda|(x-a)) +$$

$$+ \exp(-\varepsilon A\,|\lambda|\,(b-x))\},\qquad m=0,1,2,3,4,$$

where $H := \frac{8D}{C}\left(\max_{\substack{0\leqslant m\leqslant 4\\ 1\leqslant l\leqslant 4}} \sup_{x\in\langle a,b\rangle} |\eta_{ml}^{(o)}(x)| + 1\right)$.

Now for the problem (1) - (3) we shall prove the following theorem.

T h e o r e m 2. If

1° the conditions (I)-(V) are satisfied,

2° S is an infinite curve lying in $\Omega_R \cap \Lambda$ and coinciding with half-lines $|\arg \lambda| = \frac{\pi+\delta}{4}$ for sufficiently large $|\lambda|$,

then the problem (1)-(3) has in the space $C_t^\infty((0,T\rangle) \cap \cap C_x^4((a,b))$ the solution

$$(33) \qquad v(x,t) = \frac{1}{\pi\sqrt{-1}} \int_S \lambda \exp(\lambda^2 t) u(x,\lambda) d\lambda.$$

P r o o f . We shall first prove that the integrals

$$\int_S \lambda^{2k+1} \exp(\lambda^2 t) \frac{d^m u(x,\lambda)}{dx^m} d\lambda \qquad \begin{matrix} m=0,1,2,3,4, \\ k=0,1,\dots, \end{matrix}$$

are uniformly convergent with respect to $x \in \langle a,b \rangle$ for any $t \in (0,T\rangle$. Let Q_n and Q_s denote points on S in the distance respectively r_n and r_s from 0. We assume that $s > n$ and $\lim_{n\to\infty} r_n = +\infty$. We must prove that

$$(34) \qquad \int_{\widetilde{Q_n Q_s}} \lambda^{2k+1} \exp(\lambda^2 t) \frac{d^m u(x,\lambda)}{dx^m} d\lambda \underset{n\to\infty}{\rightrightarrows} 0 \quad \text{for all } s > n.$$

By (32), we obtain estimation

$$(35) \qquad \left| \int_{\widetilde{Q_n Q_s}} \lambda^{2k+1} \exp(\lambda^2 t) \frac{d^m u(x,\lambda)}{dx^m} d\lambda \right| \leqslant$$

$$\leqslant H \int_{r_n}^{r_s} \rho^{2k+1+h} \max_{1\leqslant i\leqslant 4} \bar{m}_i(\rho) \exp(-\varepsilon_1 t \rho^2) \left\{ \exp(-\varepsilon A \rho(x-a)) + \right.$$

$$\left. + \exp(-\varepsilon A \rho (b-x)) \right\} d\rho,$$

where

$$\varepsilon_1 = -\cos\frac{\pi+\delta}{2} > 0, \quad \bar{m}_s(\varrho) = \sup_{|\lambda|=\varrho} |\bar{g}_s(\lambda)|, \quad h = m+c-d.$$

Thus

$$\left| \int_{\widetilde{Q_nQ_s}} \lambda^{2k+1+h} \exp(\lambda^2 t) \frac{d^m u(x,\lambda)}{dx^m} d\lambda \right| \leqslant$$

$$\leqslant H \max_{1\leqslant i\leqslant 4} \bar{m}_i(r_n)\exp(-\varepsilon_1 t r_n^2) \left\{\exp(-\varepsilon A r_n(x-a)) + \right.$$

$$\left. + \exp(-\varepsilon A r_n(b-x))\right\} \frac{r_s^{2k+2+h} - r_n^{2k+2+h}}{2k+2+h}$$

which implies

$$\text{(36)} \quad \lim_{r_n\to\infty} \int_{\widetilde{Q_nQ_s}} \lambda^{2k+1} \exp(\lambda^2 t) \frac{d^m u(x,\lambda)}{dx^m} d\lambda = 0 \qquad s > n,$$

for all $t \in (0,T\rangle$ and $x \in \langle a,b\rangle$ or $t \in \langle 0,T\rangle$ and $x \in (a,b)$.

We prove analogically that

$$\text{(37)} \quad \lim_{r_n\to\infty} \int_{\widetilde{Q'_nQ'_s}} \lambda^{2k+1} \exp(\lambda^2 t) \frac{d^m u(x,\lambda)}{dx^m} d\lambda = 0, \quad s > n,$$

where Q'_n, Q'_s denote the mirror image points of Q_n, Q_s with respect to the real coordinate axis. From (36) and (37) we conclude that $v \in C_t^\infty((0,T\rangle) \cap C_x^4(\langle a,b\rangle)$. By (5), we have

$$\sum_{k=0}^{1}\sum_{l=0}^{4-2k} A_{kl}(x)\,\frac{\partial^{k+l}v(x,t)}{\partial x^l \partial t^k} - \frac{\partial^2 v(x,t)}{\partial t^2} =$$

$$= \frac{1}{\pi\sqrt{-1}}\int_S \exp(\lambda^2 t)\left\{\sum_{k=0}^{1}\sum_{l=0}^{4-2k}\lambda^{2k}A_{kl}(x)\,\frac{d^l u(x,\lambda)}{dx^l} - \lambda^4 u(x,\lambda)\right\}d\lambda = 0.$$

We shall verify the boundary conditions. From boundary conditions (6) and by Theorem 1, we get

$$\sum_{k=0}^{2}\sum_{l=0}^{3}\left\{\alpha_{skl}\,\frac{\partial^{k+l}v(x,t)}{\partial t^k \partial x^l}\bigg|_{x=a} + \beta_{skl}\,\frac{\partial^{k+l}v(x,t)}{\partial t^k \partial x^l}\bigg|_{x=b}\right\} =$$

$$= \frac{1}{\pi\sqrt{-1}}\int_S \lambda \exp(\lambda^2 t)\sum_{k=0}^{2}\sum_{l=0}^{3}\left\{\alpha_{skl}\lambda^{2k}\,\frac{d^l u(x,\lambda)}{dx^l}\bigg|_{x=a} +\right.$$

$$\left. + \beta_{skl}\lambda^{2k}\,\frac{d^l u(x,\lambda)}{dx^l}\bigg|_{x=b}\right\}d\lambda = \frac{1}{\pi\sqrt{-1}}\int_S \exp(\lambda^2 t)\bar{g}_s(\lambda)d\lambda = g_s(t)$$

for $s = 1,2,3,4$.

Finally, for initial condition (3) we must state that

$$\int_S \lambda^{2k+1}u(x,\lambda)d\lambda = 0 \quad \text{for} \quad k=0,1.$$

From the analysis of asymptotic representation of the function u there results its analyticity in $\Omega_R \cap \Lambda$ including contour S. Suppose that S_n is an arc of the curve S lying inside the circle with radius r_n, and O_n - is an arc of this circle lying in $\Omega_R \cap \Lambda$. By the Cauchy theorem we obtain

$$\int_S \lambda^{2k+1}u(x,\lambda)d\lambda = \lim_{n\to\infty} \int_{S_n} \lambda^{2k+1}u(x,\lambda)d\lambda =$$

$$= \lim_{n\to\infty} \int_{O_n} \lambda^{2k+1}u(x,\lambda)d\lambda.$$

In virtue of (32), we obtain

$$\left|\int_{O_n} \lambda^{2k+1}u(x,\lambda)d\lambda\right| \leqslant H \int_{-\frac{\pi+\delta}{4}}^{\frac{\pi+\delta}{4}} r_n^{2k+1+h} \max_{1\leqslant s\leqslant 4} \bar{m}_s(r_n) \times$$

$$\times \left\{\exp(-A\varepsilon r_n(x-a)) + \exp(-A\varepsilon r_n(b-x))\right\} d\varphi \leqslant$$

$$\leqslant H \frac{\pi+\delta}{2} r_n^{2k+1+h} \max_{1\leqslant s\leqslant 4} \bar{m}_s(r_n)\cdot\left\{\exp(-A\varepsilon r_n(x-a)) +\right.$$

$$\left. + \exp(-A\varepsilon r_n(b-x))\right\}.$$

Finally

$$\frac{\partial^k v(x,t)}{\partial t^k} = \int_S \lambda^{2k+1}u(x,\lambda)d\lambda = 0 \quad \text{for} \quad x \in (a,b).$$

Assuming additionally that the function v increases at least exponentially i.e. the application of transform L_2 to (1)-(3) is sensible, we can prove, by Theorem 1, the unicity of (33). Let us suppose that v_1 and v_2 are distinct solutions of the problem (1)-(3). Hence, $v(x,t) = v_1(x,t) - v_2(x,t)$ is the solution of the same problem with homogeneous boundary conditions. The corresponding spectral problem is a problem (5)-(6) with homogeneous boundary conditions. Let $u(x,\lambda)$ be the solution of this spectral problem. But $u(x,\lambda) = 0$, by (32). Thus, by Theorem 1, we get $v(x,t) = 0$ on the set $\langle a,b \rangle \times \langle 0,T \rangle$ which contradicts the assumption $v_1 \neq v_2$.

REFERENCES

[1] M.L. R a s u l o v : Methods of contour integration. Amsterdam 1967.

[2] И.С. З е й н а л о в : Новые интегральные преобразования, Differencial'nye Uravnienija 9 (1970) 1691-1696.

TECHNICAL UNIVERSITY OF WARSAW, PŁOCK BRANCH, PŁOCK

Received December 15, 1980.