DEMONSTRATIO MATHEMATICA
Vol. XV No 3 1982

Marian Majchrowski

ON CERTAIN NONLOCAL PROBLEM WITH MIXED BOUNDARY CONDITION FOR A PARABOLIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

In the present paper we give the solution of a certain nonlocal problem with mixed boundary condition for a parabolic system of partial differential equations. The method presented in this paper makes possible the reduction of nonlocal problem to a system of Volterra integral equations. We shall make use of the results established by the author in papers [1] and [2].

Consider the problem (N):

Find a vector-function $\vec{u}(x,t)$ satisfying the system of equations

$$\frac{\partial \vec{u}}{\partial t} = 4D\,\frac{\partial^2 \vec{u}}{\partial x^2} + \vec{f}(x,t) \tag{1.1}$$

in the domain $(x,t) \in Q = (0,1)\times(0,T)$, $0 < T < \infty$, where

$$\vec{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{bmatrix},\quad \vec{f}(x,t) = \begin{bmatrix} f_1(x,t) \\ \vdots \\ f_n(x,t) \end{bmatrix},\quad D = \left[d_{ij}\right]\ i,j=1,\dots,n.$$

and Re $\lambda > 0$ for any eigenvalue λ of the matrix D, such that

$$(1.2) \qquad \lim_{t\to 0^+} \vec{u}(x,t) = \vec{\varphi}(x) \quad \text{for} \quad x \in (0,1)$$

$$(1.3) \qquad A(t)\vec{u}_x(0,t) - B(t)\vec{u}(1,t) = \vec{H}(t),$$

where

$$\vec{u}_x(0,t) = \lim_{x\to 0^+} \vec{u}_x(x,t), \quad \vec{u}(1,t) = \lim_{x\to 1^-} \vec{u}(x,t) \quad \text{for} \quad t \in (0,T)$$

$$(1.4) \qquad \vec{u}(1,t) - \sum_{j=1}^{p} \mu_j(t)\cdot\vec{u}(x_j,t) = \vec{h}(t) \quad \text{for} \quad t \in (0,T),$$

$x_1,\dots,x_p$ are fixed points of the interval $(0,1)$.

We assume that

(i) $\vec{\varphi}(x)$ is a continuous vector-function with bounded variation in the interval $(0,1)$,

(ii) $\vec{H}(t)$, $\vec{h}(t)$ are two continuous vector-functions, piecewise of class C^1 in the interval $\langle 0,T\rangle$,

(iii) $A(t)$, $B(t)$, $\mu_j(t)$ $(j=1,\dots,p)$ are n-dimensional matrices piecewise of class $C^1(0,T)$, $\det A(t) \neq 0$ for every $t \in (0,T)$.

We recall that if $D = [d_{ij}]$ is an n-dimensional matrix such that Re $\lambda > 0$ for any eigenvalue λ of the matrix D then by $M(x,t)$ for $(x,t) \in (-\infty,+\infty)\times(0,+\infty)$ we shall understand the sum

$$(1.5) \qquad M(x,t) = I + 2\sum_{k=1}^{+\infty} \exp\left[-k^2\pi^2 t\, D\right] \cos k\pi x.$$

Introduce the following notations

(1.6) $$\vec{u}_1(x,t) = -2D\int_0^t \left[M\left(\frac{x}{2}, t-s\right) - M\left(\frac{x-2}{2}, t-s\right)\right]\vec{f}_1(s)ds$$

(1.7) $$\vec{u}_2(x,t) = D\int_0^t \left[M'_x\left(\frac{x-1}{2}, t-s\right) - M'_x\left(\frac{x+1}{2}, t-s\right)\right]\vec{f}_2(s)ds$$

(1.8) $$\vec{u}_3(x,t) = \frac{1}{4}\int_0^1 \left[M\left(\frac{x-s}{2}, t\right) + M\left(\frac{x+s}{2}, t\right) - M\left(\frac{x-s-2}{2}, t\right) - M\left(\frac{x+s-2}{2}, t\right)\right]\vec{\varphi}(s)ds$$

(1.9) $$\vec{u}_4(x,t) = \frac{1}{4}\int_0^t\int_0^1 \left[M\left(\frac{x-s}{2}, t-\eta\right) + M\left(\frac{x+s}{2}, t-\eta\right) - M\left(\frac{x-s-2}{2}, t-\eta\right) - M\left(\frac{x+s-2}{2}, t-\eta\right)\right]\vec{f}(s,\eta)ds\, d\eta,$$

where $\vec{f}_1(s)$, $\vec{f}_2(s)$, $\vec{\varphi}(s)$, $\vec{f}(s,\eta)$ are some fixed functions.

2. The auxiliary problem

Let us consider the problem (F):

Find vector-function $\vec{u}(x,t)$ such that (1.1) holds in Q subject to the following boundary conditions:

(2.1) $$\lim_{t\to 0^+} \vec{u}(x,t) = \vec{\varphi}(x) \quad \text{for} \quad x \in (0,1)$$

(2.2) $$\lim_{t\to 0^+} \vec{u}'_x(x,t) = \vec{f}_1(t) \quad \text{for} \quad t \in (0,T)$$

(2.3) $$\lim_{x\to 1^-} \vec{u}(x,t) = \vec{f}_2(t) \quad \text{for} \quad t \in (0,T).$$

We shall construct solution of (F) problem explicitely, without referring to integral equations. To this aim, let us recall the main results of the paper [1].

T h e o r e m 1. If for every eigenvalue λ of the matrix D the condition $\mathrm{Re}\,\lambda > 0$ is satisfied then:

a) $M(x,t)$ is of the class C^{∞} in the domain (x,t) $(-\infty,+\infty)\times(0,+\infty)$ and the respective derivatives can be obtained by term-by-term differentiation of the respective series.

b) $\lim\limits_{t\to 0^+} \frac{\partial^s M(x,t)}{\partial x^s} = 0$ for $s = 0,1,2,\dots,\ x \neq 0,\pm 2,\pm 4,\dots$

c) For $(x,t) \in (-\infty,+\infty)\times(0,+\infty)$ each column of the matrix $M(x,t)$ is a solution of the homogeneous system (1.1) i.e.

$$\frac{\partial M(x,t)}{\partial t} = D\,\frac{\partial^2 M(x,t)}{\partial x^2}\,.$$

Now, we can prove the following theorem.

T h e o r e m 2. If the vector-functions $\vec{f}_1(t),\vec{f}_2(t)$ are continuous in the interval $\langle 0,T\rangle$ and possess bounded and piecewise continuous derivatives $\vec{f}_1',\ \vec{f}_2'$ for $t \in (0,T)$ then:

a) the functions $\vec{u}_1,\ \vec{u}_2$ defined by (1.6), (1.7) satisfy the homogeneous system of equations (1.1) $(\vec{f}(x,t) = 0)$

b) $\lim\limits_{t\to 0^+} \vec{u}_i(x,t) = 0$ $(i=1,2)$ for every fixed $x \in (0,1)$

c) $\lim\limits_{x\to 0^+} \vec{u}_{1x}'(x,t) = f_1(t)$, $\lim\limits_{x\to 1^-} \vec{u}_1(x,t) = 0$ for every $t \in (0,T)$

d) $\lim\limits_{x\to 0^+} \vec{u}_{2x}'(x,t) = 0$, $\lim\limits_{x\to 1^-} \vec{u}_2(x,t) = \vec{f}_2(t)$ for every $t \in (0,T)$.

P r o o f . The validity of part a) and b) of Theorem 2 results from Theorem 1 and from elementary analytical calculations (cf. [1]). Now, we establish the validity of part c) of the theorem. To this aim let us note that $\lim\limits_{x\to 1^-} \vec{u}_1(x,t) = 0$ because $M\left(\frac{1}{2},\ t-s\right) = M\left(\frac{-1}{2},\ t-s\right)$ and

$$\vec{u}'_{1x}(x,t) = -2D\int_0^t \left[\frac{1}{2} M'_x\left(\frac{x}{2}, t-s\right) - \frac{1}{2} M'_x\left(\frac{x-2}{2}, t-s\right)\right] \vec{f}_1(s)ds =$$

$$= -D\int_0^t M'_x\left(\frac{x}{2}, t-s\right) \vec{f}_1(s)ds + D\int_0^t M'_x\left(\frac{x-2}{2}, t-s\right) \vec{f}_1(s)ds.$$

By Theorem 5 of paper [2] it follows that

$$\lim_{x\to 0^+} D\int_0^t M'_x\left(\frac{x}{2}, t-s\right) \vec{f}_1(s)ds = -\vec{f}_1(t) \quad \text{for} \quad t \in (0,T)$$

and

$$\lim_{x\to 0^+} D\int_0^t M'_x\left(\frac{x-2}{2}, t-s\right) \vec{f}_1(s)ds = 0 \quad \text{for} \quad t \in (0,T)$$

which completes the proof of the part c). Analogously part d) can be proved.

T h e o r e m 3. If the vector-function $\vec{\varphi}(x)$ possesses a bounded variation in the interval $x \in (0,1)$ and it is a sum of its trigonometric Fourier series for every $x \in (0,1)$, then the vector-function $\vec{u}_3(x,t)$ defined by (1.8) is a solution of the homogeneous system (1.1) ($\vec{f}(x,t) = 0$)) and satisfies the conditions

$$\lim_{t\to 0^+} \vec{u}_3(x,t) = \vec{\varphi}(x) \quad \text{for} \quad x \in (0,1),$$

$$\lim_{x\to 0^+} \vec{u}'_{3x}(x,t) = 0,$$

$$\lim_{x\to 1^-} \vec{u}_3(x,t) = 0 \qquad \text{for} \quad t \in (0,T).$$

P r o o f . From the definition (1.5) we obtain

$$\frac{1}{4}\left[M\left(\frac{x-s}{2}, t\right) + M\left(\frac{x+s}{2}, t\right) - M\left(\frac{x-s-2}{2}, t\right) - M\left(\frac{x+s-2}{2}, t\right)\right] =$$

$$= 2\sum_{k=0}^{+\infty} \exp\left[-(2k+1)^2\pi^2 t\ D\right] \cos\frac{2k+1}{2}\pi x \cos\frac{2k+1}{2}\pi s.$$

This series is uniformly convergent with respect to x and s (t being fixed). Thus, we have

$$(2.4)\qquad \vec{u}_3(x,t) = \sum_{k=1}^{+\infty} \exp\left[-(2k+1)^2\pi^2 t\ D\right] \cos\frac{2k+1}{2}\pi x\cdot\vec{a}_k,$$

where

$$\vec{a}_k = 2\int_0^1 \vec{\varphi}(s)\cos\frac{2k+1}{2}\pi s\, ds \quad \text{for} \quad k = 0,1,2,\ldots$$

The series (2.4) converges uniformly for $t \in \langle 0,T\rangle$ because the series $\sum_{k=0}^{+\infty} \vec{a}_k$ converges (cf. lemma in [2]). Then

$$\lim_{t\to 0^+} \vec{u}_3(x,t) = \sum_{k=0}^{+\infty} \cos\frac{2k+1}{2}\pi x\cdot\vec{a}_k = \vec{\varphi}(x)$$

by the assumption.

The formula (2.4) implies that $\lim_{x\to 0^+} \vec{u}\,'_{3x}(x,t) = = \lim_{x\to 1^-} \vec{u}_3(x,t) = 0$. The series (2.4) is almost uniformly convergent in Q, so it can be differentiated term-by-term and it is clear that $\vec{u}_3(x,t)$ satisfies the homogeneous system of equations (1.1).

Now, we can prove the following theorem.

T h e o r e m 4. If:

a) $\vec{f}(x,t)$ is continuous in the rectangle $\bar{Q}$,

b) $\lim_{x\to 0^+} \vec{f}(x,t) = \lim_{x\to 1^-} \vec{f}(x,t) = 0$,

c) $f(x,t)$ is piecewise of the class C^2 with respect to x (t being fixed),

d) $\sup_{t\in(0,T)} \int_0^1 \|\vec{f}''_{xx}(x,t)\| dx < +\infty$,

e) the number of points of discontinuity $\vec{f}''_{xx}(x,t)$ as a function of x is not greater than n_0 (n_0 is independent on t),

then the vector-function $\vec{u}_4(x,t)$ defined by (1.9) is a solution of the system (1.1) and satisfies the homogeneous boundary and initial conditions

$$\lim_{t\to 0^+} \vec{u}_4(x,t) = \lim_{x\to 0^+} \vec{u}'_{4x}(x,t) = \lim_{x\to 1^-} \vec{u}_4(x,t) = 0.$$

P r o o f . According to the theory of Fourier series, the function $\vec{f}(x,t)$ which satisfies given assumptions can be written in the form

$$\vec{f}(x,t) = \sum_{k=0}^{+\infty} \vec{b}_k(t) \cos \frac{2k+1}{2} \pi x,$$

where

$$\vec{b}_k(t) = 2 \int_0^1 \vec{f}(s,t) \cos \frac{2k+1}{2} \pi s \, ds \quad \text{for} \quad k=0,1,2,\dots$$

are continuous functions for $t \in (0,T)$ satisfying the inequality

$$\left\| \vec{b}_k(t) \right\| \leq \frac{A}{k^2}, \quad k = 0,1,2,\dots .$$

A is a positive constant.

From the uniform convergence of the series

$$\frac{1}{4}\int_0^1 \left[M\left(\frac{x-s}{2}, t-\eta\right) + M\left(\frac{x+s}{2}, t-\eta\right) - M\left(\frac{x-s-2}{2}, t-\eta\right) - \right.$$

$$\left. - M\left(\frac{x+s-2}{2}, t-\eta\right)\right] \vec{f}(s,\eta)ds =$$

$$= \sum_{k=0}^{+\infty} \exp\left[-(2k+1)^2\pi^2(t-\eta)D\right] \cos\frac{2k+1}{2}\pi x\, \vec{b}_k(\eta)$$

it follows that

$$(2.5) \qquad \vec{u}_4(x,t) =$$

$$= \sum_{k=0}^{+\infty} \cos\frac{2k+1}{2}\pi x \int_0^t \exp\left[-(2k+1)^2\pi^2(t-\eta)D\right] b_k(\eta)d\eta.$$

Hence

$$\vec{u}'_{4x}(x,t) = \sum_{k=0}^{+\infty} \vec{b}_k(t) \cos\frac{2k+1}{2}\pi x +$$

$$- D\sum_{k=0}^{+\infty} \cos\frac{2k+1}{2}\pi x \int_0^t (2k+1)^2\pi^2 \exp\left[-(2k+1)^2\pi^2(t-\eta)D\right]\vec{b}_k(\eta)d\eta =$$

$$= \vec{f}(x,t) + 4D\frac{\partial^2\vec{u}_4}{\partial x^2}.$$

The proof of the second part of the theorem follows from (2.5)

Theorems 2, 3 and 4 imply the following corollary.

C o r o l l a r y . If the functions $\vec{f}_1$, $\vec{f}_2$, $\vec{\varphi}$, $\vec{f}$ satisfy the assumptions of Theorems 2, 3 and 4 respectively, then the problem (F) possesses the solution. This solution can be written in the form

$$(2.6)\qquad \vec{u}(x,t) = \vec{u}_1(x,t) + \vec{u}_2(x,t) + \vec{u}_3(x,t) + \vec{u}_4(x,t)$$

where $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, $\vec{u}_4$ are defined by (1.6), (1.7), (1.8), (1.9).

3. Solution of the problem (N)

We can prove the following theorem.

T h e o r e m 5. If $\vec{h}$, $\vec{\varphi}$, $\vec{H}$, A, B, μ_j (j=1,...,p) satisfy conditions (i), (ii), (iii), and $\vec{f}(x,t)$ satisfies the assumptions of Theorem 4 then there exists a solution of the problem (N).

P r o o f . Let us denote

$$\vec{f}_1(t) = \lim_{x\to 0^+} \vec{u}_x'(x,t), \qquad \vec{f}_2(t) = \lim_{x\to 1^-} \vec{u}(x,t)$$

where a solution $\vec{u}(x,t)$ of the problem (N) is sought in the form (2.6).

From the first nonlocal condition (1.3) we obtain

$$(3.1)\qquad \vec{f}_1(t) = A^{-1}(t)\vec{H}(t) + A^{-1}(t)B(t)\vec{f}_2(t).$$

Combining formulas (2.6) and (3.1) we can write

$$\vec{u}(x,t) = \int_0^t K(x,t,s)\ \vec{f}_2(s)ds -$$

$$- 2D \int_0^t \left[M\left(\frac{x}{2},\ t-s\right) - M\left(\frac{x-2}{2},\ t-s\right)\right] A^{-1}(s)H(s)ds +$$

$$+ \vec{u}_3(x,t) + \vec{u}_4(x,t),$$

where

$$K(x,t,s) = D\left[M'_x\left(\frac{x-1}{2}, t-s\right)\right] - M'_x\left(\frac{x+1}{2}, t-s\right) -$$

$$- 2D\left[M\left(\frac{x}{2}, t-s\right) - M\left(\frac{x-2}{2}, t-s\right)\right] A^{-1}(s)B(s),$$

is continuous with respect to $t \in \langle 0,T\rangle$, $s \in \langle 0,t\rangle$ for every fixed $x \in (0,1)$.

From the second nonlocal condition (1.4) we infer that the function $\vec{f}_2(t)$ must satisfy the following system of Volterra integral equations of the second kind

$$(3.2) \qquad \vec{f}_2(t) - \int_0^t \sum_{j=1}^{p} \mu_j(t) K(x_j,t,s)\vec{f}_2(s)ds = \vec{F}(t),$$

where

$$(3.3) \qquad \vec{F}(t) = -2\sum_{j=1}^{p} \mu_j(t) D\int_0^t \left[M\left(\frac{x_j}{2}, t-s\right) - \right.$$

$$\left. - M\left(\frac{x_j}{2}, t-s\right)\right] A^{-1}(s)\vec{H}(s)ds +$$

$$+ \vec{h}(t) + \sum_{j=1}^{p} \mu_j(t)\left[\vec{u}_3(x_j,t) + \vec{u}_4(x_j,t)\right].$$

Hence we conclude that there exists exactly one solution of the system (3.2). This solution is continuous and piecewise of class C^1. The formulas (3.1) and (2.6) define the solution of the problem (N).

R e m a r k . If the condition $\lim_{x\to 0^+} \vec{f}(x,t) = \lim_{x\to 1^-} f(x,t) = 0$ is not satisfied but $\vec{f}(x,t)$ is continuous in $\langle 0,1\rangle \times \langle 0,T\rangle$, then we can use the substitution

$$\vec{v}(x,t) = \vec{u}(x,t) - x\int_0^t \left[\vec{f}(1,s) - \vec{f}(0,s)\right] ds - \int_0^t \vec{f}(0,s)ds.$$

This substitution reduces the problem (N) to the problem (N*)

$$\vec{v}_t' = 4D\,\vec{v}_{xx}'' + \vec{g},$$

where

$$\vec{g}(x,t) = \vec{f}(x,t) - x\left[\vec{f}(1,t) - \vec{f}(0,t)\right] - \vec{f}(0,t)$$

$$\vec{g}(0,t) = \vec{g}(1,t) = 0$$

$$\vec{v}(x,0) = \vec{\varphi}(x)$$

with conditions

$$A(t)\vec{v}_x'(0,t) - B(t)\vec{v}(1,t) =$$

$$= \vec{H}(t) - A(t)\int_0^t \left[\vec{f}(1,s) - \vec{f}(0,s)\right] ds + B(t)\int_0^t \vec{f}(1,s)\, ds,$$

$$\vec{v}(1,t) - \sum_{j=1}^{p} \mu_j(t)\vec{v}(x_j,t) =$$

$$= \vec{h}(t) - \int_0^t \vec{f}(1,s)ds + \sum_{j=1}^{p} \mu_j(t)\left\{x_j\int_0^t \left[\vec{f}(1,s) - f(0,s)\right] ds + \int_0^t \vec{f}(0,s)ds\right\}.$$

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
Received December 5, 1980.