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APPLICATION OF THE CHARACTERISTIC POLYNOMIAL OF THE TRANSITION MATRIX TO THE INVESTIGATION OF THE PROPERTIES OF A FINITE GRAPH

Definition 1. We call a finite multigraph the triplet $G = (X,U,R)$, where:

$|X| < +\infty$; X is the set of vertices of the multigraph,

$|U| < +\infty$; U is the set of branches of the multigraph,

$R \subset (X, U, X)$,

the following conditions being satisfied:

1) to any $u \in U$ there exist $x, y \in X$ such that $(x,u,y) \in R$,
2) to any $u \in U$ if there exist $x, y, z, k \in X$, such that $(x,u,z) \in R$ and $(y,u,k) \in R$, then $x = y$ and $z = k$ or $x = k$ and $y = z$.

Definition 2. In the finite multigraph $G = (X,U,R)$ the branch $u \in U$ is said to be

a) a loop, if there exists an $x \in X$ such that $(x,u,x) \in R$,
b) an arc, if there exist $x,y \in X$ such that $(x,u,y) \in R$, $(y,u,x) \notin R$,
c) an edge, if there exist $x,y \in X$ such that $(x,u,y) \in R$ and $(y,u,x) \in R$.

In what follows we shall consider finite multigraphs without loops.

To any finite multigraph we can give a geometrical interpretation in the following way:

1) to each vertex $x \in X$ we assign a point x of the plane, so that to distinct vertices are assignet distinct points,

2) to each arc $u \in U$ such that $(x,u,y) \in R$ we assign in the plane a segment directed from the point x to the point y.
3) to each edge $u \in U$ such that $(x,u,y) \in R$ we assign in the plane an undirected segment joining the points x and y.

A multigraph containing no edges (arcs) is said to be directed (undirected).

If in a multigraph any pair of vertices are joined by no more than one branch, the multigraph is called a graph. In particular, a graph cannot contain arcs joining the same vertices and reversely directed.

We call route of length l with origin x_{i_0} and end x_{i_l} an alternate sequence of vertices and branches of the multigraph of the form $\{x_{i_0}, u_{i_1}, x_{i_1}, u_i, \ldots, u_{i_l}, x_{i_l}\}$, such that for any $k = 1,\ldots,l$ the branch u_{i_k} joins the vertices $x_{i_{k-1}}$ and x_{i_k} (if u_{i_k} is an arc, then it may be directed from $x_{i_{k-1}}$ to x_{i_k} or reversely). A route of length l such that all the branches are distinct is said to be a chain of length l.

A chain whose each arc u_{i_k} is directed from $x_{i_{k-1}}$ to x_{i_k} is said to be a path. A chain (route, path) whose all vertices are distinct is said to be simple. A chain (route, path) which begins and ends at the same vertex is called a cycle (cyclic route, cyclic path). A cycle (cyclic path) of length l, having l distinct vertices is called a simple cycle (simple cyclic path) of lenght l. A multigraph is said to be connected if to any pair of its vertices $x,y \in X$ there exists a chain with origin x and end y.

D e f i n i t i o n 3. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of vertices of a finite multigraph $G = (X,U,R)$. We call a transition matrix $P = [p_{i,j}]$ the matrix such that

$$p_{i,j} = \left|\{u \in U \,/,\, (x_i,u,x_j) \in R\}\right| \quad \text{for} \quad i,j=1,\ldots,n.$$

D e f i n i t i o n 4. a) The metric of the multigraph $G = (X,U,R)$ is defined as the function

$$\rho(x,y) = \begin{cases} 0, & \text{if } x = y, \\ +\infty, & \text{if } x \neq y \text{ and if there exists no chain with origin } x \text{ and end } y, \\ l, & \text{otherwise,} \end{cases}$$

where l denotes the length of the shortest chain with origin x and end y.

b) The vertex $x_0 \in X$ is said to be: central, if for any $x \in X$ $\max_{y\in X} \rho(x_0,y) \leqslant \max_{y\in X} \rho(x,y)$ or peripheral, if for any $x \in X$ $\max_{y\in X} (x_0,y) \geqslant \max_{y\in X} \rho(x,y)$.

c) The radius of the multigraph is defined as the number

$$r(G) = \min_{x\in X} \max_{y\in X} \rho(x,y).$$

The diameter of the multigraph is defined as the number

$$d(G) = \max_{x,y\in X} \rho(x,y).$$

d) Let $A, B \subset X$, $A \neq \emptyset$, $B \neq \emptyset$. The distance between the sets A, B is defined as the number

$$\rho(A,B) = \min_{\substack{x\in A \\ y\in B}} \rho(x,y).$$

The distance of the vertex x_0 to the set A is defined as the number

$$\rho(x_0,A) = \min_{y\in A} \rho(x_0,y).$$

In what follows we shall assume that

$G = (X,U,R)$ is a finite multigraph without loops,

$X = \{x_1, x_2, \dots, x_n\}$;

Let us now denote by:

a) $D^*_{i,k}$: $k = 2,\dots,n$; $i = 1,\dots,\left[\frac{k}{2}\right]$, the set whose elements are i sets of cyclic simple paths without common vertices, such that the sum of their lengths equals k. In special cases, for some i, k the set $D^*_{i,k}$ may be empty.

b) $P = P(G)$ the transition matrix for G, with the assumpted numbering of vertices.

c) $D_{i,j}(\lambda)$ the co-factor of the element in the i-th row and j-th column of the matrix $P - \lambda.E$; $D_{i,j}(\lambda)$ is a polynomial of degree $k_{i,j}$ in the variable λ, such that the highest power of λ has the coefficient $H_{i,j}$.

R e m a r k 1. Given a multigraph G it is possible to build:

a) a directed multigraph $G_1 = (X,U_1,R_1)$ in such a way that each edge of G joining the vertices x and y is replaced by the pair of reversely directed arcs in G_1 which join the same vertices,

b) a directed multigraph $G_2 = (X,U_2,R_2)$ with a symmetrical transition matrix, in such a way that to each branch of G joining the vertices x,y there corresponds in G_2 exactly one pair of reversely directed arcs joining these vertices. It is easy to see that to each path in G there corresponds in G_1 exactly one path passing through the same vertices. The transition matrices for G and G_1 are identical and the transition matrix of G_2 is symmetrical.

Let us now recall the following theorem (see [1]):

T h e o r e m 1. The characteristic polynomial of the matrix P is

$$\det(P - \lambda E) = (-1)^n(\lambda^n + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \dots + a_n),$$

where, for any $k = 2,\dots,n$:

$$a_k = -\left|D^*_{1,k}\right| + \left|D^*_{2,k}\right| - \left|D^*_{3,k}\right| + \dots + (-1)^{\left[\frac{k}{2}\right]}\left|D^*_{\left[\frac{k}{2}\right],k}\right|.$$

T h e o r e m 2. Let G_1 be a directed multigraph corresponding to G in accordance with Remark 1.

Let be given the sets of vertices:

$$X_1 = \left\{x_{j_1}, x_{j_2}, \dots, x_{j_r}\right\} \subset X; \quad r \geq 1,$$
$$X_2 = \left\{x_{i_1}, x_{i_2}, \dots, x_{i_s}\right\} \subset X; \quad s \geq 1,$$
$$X_1 \cap X_2 = \emptyset.$$

Denote:

a) by $G_3 = (X, U_3, R_3)$ the multigraph obtained from G_1 in the following way: for each pair of vertices $(x_{i_k}, x_{j_w}) \in X_2 \times X_1$ we remove in G_1 all the arcs joining x_{i_k} with x_{j_w}, then we join them by exactly one arc,

b) by $G_4 = (X, U_4, R_4)$ the directed multigraph obtained from G_1 by removing from U_1 the arcs joining vertices which belong to the set X_2 with the vertices of set X_1,

c) by P_3, P_4 the transition matrices for G_3 and G_4 respectively,

d) by $W(\lambda) = \det(P_3 - \lambda E) - \det(P_4 - \lambda E)$ a polynomial in the variable λ of degree $k_1(X_1, X_2)$ such that the coefficient of the highest power of λ equals $H(X_1, X_2)$.

A s s e r t i o n . 1) A necessary and sufficient condition in order that there exist in G a path whose origin belongs to X_1 and end belongs to X_2 is that $W(\lambda) \neq 0$.

2) If $W(\lambda) \neq 0$, then the shortest of these paths has the length $n-1-k(X_1, X_2)$. The number of these shortest paths is $|H(X_1, X_2)|$.

P r o o f . By Theorem 1 we have

$$\text{(A)} \quad \begin{cases} \det(P_3 - \lambda E) = (-1)^n(\lambda^n + (a_2)_3\lambda^{n-2} + (a_3)_3\lambda^{n-3} + \dots + (a_n)_3) \\ \det(P_4 - \lambda E) = (-1)^n(\lambda^n + (a_2)_4\lambda^{n-2} + (a_3)_4\lambda^{n-3} + \dots + (a_n)_4), \end{cases}$$

where for each $r = 2,\dots,n$

$$(B)\quad \begin{cases} (a_r)_3 = -\left|D^*_{1,r}\right|_3 + \left|D^*_{2,r}\right|_3 - \left|D^*_{3,r}\right|_3 + \dots + (-1)^{\left[\frac{r}{2}\right]}\left|D^*_{\left[\frac{r}{2}\right],r}\right|_3 \\ (a_r)_4 = -\left|D^*_{1,r}\right|_4 + \left|D^*_{2,r}\right|_4 - \left|D^*_{3,r}\right|_4 + \dots + (-1)^{\left[\frac{r}{2}\right]}\left|D^*_{\left[\frac{r}{2}\right],r}\right|_4 . \end{cases}$$

We shall restrict ourselves to simple paths, since the shortest paths are always simple ones. G_4 is a partial multigraph of G_3, obtained by removing from the set U_3 the subset

$$U_{1,3} = \left\{u \in U_3/(x_{i_k}, u, x_{j_w}) \in R_3;\ x_{i_k} \in X_2;\ x_{j_w} \in X_1\right\}.$$

All the simple cyclic paths belonging to G_3 and not belonging to G_4 are obtained by composing the simple paths joining the vertices of X_1 with the vertices of X_2 with the corresponding arcs of the set $U_{1,3}$. The multigraph G does not contain paths joining the vertices of X_1 and X_2 if and only if all the sets of simple cyclic paths in G_3 and G_4 are identical. Hence

$$\left|D^*_{i,r}\right|_3 = \left|D^*_{i,r}\right|_4 \quad \text{for} \quad r = 2,\dots,n;\ i = 1,\dots,\left[\frac{r}{2}\right].$$

By (A) and (B) this is equivalent to the equality $W(\lambda) = 0$.

Suppose that $W(\lambda) \neq 0$ and that k denotes the least length of a cyclic path belonging to G_3 and not belonging to G_4 we have

$$\left|D^*_{i,r}\right|_3 = \left|D^*_{i,r}\right|_4 \quad \text{for} \quad 2 \leqslant r < k,$$

$$(a_r)_3 = (a_r)_4 \quad \text{for} \quad 2 \leqslant r < k.$$

Moreover,

$$\left| (a_k)_4 - (a_k)_3 \right| = \left| \left| D^*_{1,k} \right|_4 - \left| D^*_{1,k} \right|_3 \right|,$$

since

$$\left| D^*_{i,k} \right|_3 = \left| D^*_{i,k} \right|_4 \quad \text{for} \quad i \geqslant 2.$$

The length of the shortest path whose origin belongs to X_1 and end belongs to X_2 is $k-1$ and the number of such paths equals

$$\left| \left| D^*_{1,k} \right|_4 - \left| D^*_{1,k} \right|_3 \right| = \left| (a_k)_4 - (a_k)_3 \right| = \left| H(X_1,X_2) \right|$$

$$k-1 = n-1 - k(X_1,X_2).$$

T h e o r e m . If G is an undirected graph, then for all $x,y \in X$ there exists in G exactly one shortest path joining these vertices if and only if

$$D_{i,j}(\lambda) \neq 0, \quad \left| H_{i,j} \right| = 1 \quad \text{for} \quad i,j=1,\dots,n; \quad i > j.$$

P r o o f . When building the directed multigraph G_1 corresponding to G by Remark 1, it should be noted that the matrix P_1 for G_1 is identical with the matrix P for G. Since the matrix P is symmetrical, we have merely to consider the case $i > j$. The above theorem follows directly from Theorem 2 for $r = 1$ and $s = 1$, if me make use of the true equality

$$W(\lambda) = D_{i_1,j_1}(\lambda) = D_{j_1,i_1}(\lambda).$$

The theorem yields in algebraic form the solution of a problem stated by Ore (see [3], p.119): "Describe the class of undirected graphs such that for any pair of vertices x,y there exists exactly one shortest path joining them". This problem has been till now solved only for planar graphs (see [2]).

In what follows we shall assume that:

a) $G_2 = (X,U_2,R_2)$ is a directed multigraph constructed for G in accordance with Remark 1.

b) P_2 is the transition matrix for G_2,

c) for any $i,j = 1,\dots,n$: $D^2_{i,j}(\lambda)$ denotes the co-factor of the element in the i-th row and j-th column of the matrix $P_2 - \lambda E'$,

d) $k^2_{i,j}$ denotes the degree of the polynomial $D^2_{i,j}(\lambda)$ in the variable λ.

T h e o r e m. If G is a connected multigraph, $|X| \geqslant 2$, then:

1) The radius of the multigraph is

$$r(G) = n - 1 - \max_{i\neq j} \min_{n\geqslant i\geqslant 1} k^2_{i,j}.$$

2) The diameter of the multigraph is

$$d(G) = n - 1 - \min_{n\geqslant i>j\geqslant 1} k^2_{i,j}.$$

3) If $x_i \in X$, then

3a) x_i is a central vertex if and only if for any $r = 1,\dots,n$

$$\min_{\substack{r\neq j\\ j=1,\dots,n}} k^2_{r,j} \leqslant \min_{\substack{i\neq j\\ j=1,\dots,n}} k^2_{i,j}$$

3b) x_i is a peripheral vertex if and only if the inequality under 3a) is replaced either by the reverse inequality or by the equality.

P r o o f . 1) Since $|X| \geqslant 2$, there exists for each i; $j \neq i$ such that $\rho(x_i,x_j) > 0$. We have merely to consider the case $i \neq j$.

Since G is connected, so is G_2 and

$$D_{i,j}^{2}(\lambda) \neq 0 \quad \text{for} \quad i,j = 1,\dots,n, \quad i \neq j.$$

$$r(G) = \min_{i \neq j} \max_{i=1,\dots,n} \rho(x_i,x_j) = n - 1 - \max_{n \geq i \neq j} \min_{n \geq j \geq 1} k_{i,j}^{2}.$$

2) The proof is similar to that of case 1).

3) x_i is a central vertex if and only if for each $x_r \in X$ the following inequality holds

$$\text{(A)} \qquad \max_{j=1,\dots,n} \rho(x_i,x_j) \leqslant \max_{h=1,\dots,n} \rho(x_r,x_h).$$

Similarly to case 1) we can restrict ourselves to the case $i \neq j$, $h \neq r$ we have

$$\rho(x_i,x_j) = n - 1 - k_{i,j}^{2}$$

$$\rho(x_r,x_h) = n - 1 - k_{r,h}^{2}.$$

Hence the inequality (A) is equivalent to the inequality

$$\min_{\substack{i \neq j \\ j=1,\dots,n}} k_{i,j}^{2} \geqslant \min_{\substack{r \neq j \\ j=1,\dots,n}} k_{r,j}^{2}.$$

3b) The proof is analogous to that of case 3a).

Theorem. Let $r \geqslant 2$ and $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ be a sequence of vertices of the multigraph G such that $x_{i_{l+1}} \neq x_{i_l}$ for $l = 1,\dots,r-1$.

Assertion. 1) A directed route from x_{i_1} to x_{i_r} passing in turn through the vertices $x_{i_2}, x_{i_3}, \dots, x_{i_{r-1}}$ exists if and only if

$$\left| \prod_{l=2}^{r} D_{i_l,i_{l-1}}(\lambda) \right| \neq 0.$$

2) If condition 1) is satisfied, then there exist in G exactly $\prod_{l=2}^{r} \left| H_{i_l,i_{l-1}} \right|$ directed routes with origin x_{i_1} and end x_{i_r} passing through the vertices $x_{i_2}, x_{i_3}, \ldots, x_{i_{r-1}}$ and having the least length; the length of each of them is

$$(r - 1)(n - 1) - \sum_{l=2}^{r} k_{i_l,i_{l-1}}.$$

P r o o f . To obtain the shortest route wanted in the theorem we compose the shortest paths joining the vertices x_{i_1} with x_{i_2}, x_{i_2} with $x_{i_3}, \ldots, x_{i_{r-1}}$ with x_{i_r}. Applying Theorem 2 to the paths that form this route and making use of the equality

$$W_l(\lambda) = D_{i_l,i_{l-1}}(\lambda) \quad \text{for} \quad l=1,\ldots,r-1$$

we get the assertion.

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