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R. Doliński, K. Zima

ON WEAK DIFFERENTIAL-FUNCTIONAL INEQUALITIES

The purpose of this paper is to transfer some J.Szarski's theorem on weak differential inequalities [1] to differential - functional inequalities of Volterra's type. Owing to the modification of the method of proofs in J.Szarski's theorems, which is introduced in this paper, we can obtain theorems on weak inequalities also in the cases, where the so-called comparison function σ does not belong to the class of comparison functions in [1].

1. Introduction

Let $f_i(t,x_1,\dots,x_n)$, $i = 1,\dots,n$, be a system of functions defined in the set $D = \langle 0,T\rangle \times R^n$. Let the functions $\Phi(t) = \{\varphi_1(t),\dots,\varphi_n(t)\}$ be continuous in the interval $\langle\tau,T\rangle$, where $-\infty\leqslant\tau\leqslant 0$. We will consider the operator $H^t(\Phi) = \{H_1^t(\Phi),\dots,H_n^t(\Phi)\}$, where $H_i^t(\Phi) = \{h_{i1}^t(\varphi_1),h_{i2}^t(\varphi_2),\dots$ $\dots, h_{in}^t(\varphi_n)\}$ for $i = 1,\dots,n$, having the properties:

1* If $\Phi(\theta) = \Psi(\theta)$ for any functions Φ,Ψ and for all $\theta\in\langle\tau,t\rangle$, $t\in\langle 0,T\rangle$, then $H^t(\Phi) = H^t(\Psi)$.

2* If $\Phi(t)\leqslant\Psi(t)$ for $t\in\langle\tau,T\rangle$, then $H^t(\Phi)\leqslant H^t(\Psi)$.

We will apply the notation: $\{x_1,\dots,x_n\} = X$, $\{f_1,\dots,f_n\} = F$, $X\leqslant(<)\tilde X$ if and only if $x_i\leqslant(<)\tilde x_i$ for $i = 1,\dots,n$. We will write that a function F satisfies the condition $\tilde V_+$ in the set D, if $X\leqslant\tilde X$ implies $f_i(t,X)\leqslant f_i(t,\tilde X)$ for $i = 1,\dots,n$ and for any points (t,X), $(t,\tilde X)\in D$.

2. A theorem on the strong inequality for the system of differential-functional equations of Volterra's type

T h e o r e m 1. Suppose that:

(i) The function F satisfies the condition $\tilde{V}_+$ in the set D.

(ii)
$$D_-\Phi(t) \leqslant F(t,H^t(\Phi)), \quad t \in (0,T\rangle,$$
$$D_-\Psi(t) > F(t,H^t(\Psi)), \quad t \in (0,T\rangle,$$
$$\Phi(t) < \Psi(t), \quad t \in \langle\tau,0\rangle,$$

where $D_-\Phi(t) = \{D_-\varphi_1(t),\dots,D_-\varphi_n(t)\}$, $D_-\varphi_i(t) = \liminf_{h\to 0^-} \frac{\varphi_i(t+h)-\varphi_i(t)}{h}$ for $i=1,\dots,n$ and $F(t,H^t(\Phi)) = \{f_1(t,H_1^t(\Phi)),\dots,f_n(t,H_n^t(\Phi))\}$ (analogously for the function Ψ. Then

(1)
$$\Phi(t) < \Psi(t), \quad t \in \langle\tau,T\rangle.$$

P r o o f . By the continuity of Φ i Ψ and by (ii), the inequality (1) is true in a certain interval $\langle 0,\alpha)$. Let us consider the values $\alpha > 0$ which satisfy the inequality (1). Let α^* be the least upper bound of the set of these values. Suppose that $\alpha^* < T$. Then, for at least one k we have

$$\left.\begin{aligned} \varphi_k(t) &< \psi_k(t) \\ \varphi_k(\alpha^*) &= \psi_k(\alpha^*) \end{aligned}\right. \qquad t \in \langle 0,\alpha^*).$$

Hence

(2)
$$D_-\varphi_k(\alpha^*) \geqslant D_-\Psi_k(\alpha^*).$$

But, by (ii) and by properties 1^* and 2^* of the operator $H^t(\Phi)$, we have

$$D_-\Phi(\alpha^*) \leqslant F(\alpha^*,H^{\alpha^*}(\Phi)) \leqslant F(\alpha^*,H^{\alpha^*}(\Psi)) < D_-\Psi(\alpha^*)$$

which contradicts the inequality (2). Therefore $\alpha^* = T$. This completes the proof.

R e m a r k . If $h^t_{ii}(\varphi_i) \equiv \varphi_i(t)$, then we can replace the condition $\tilde{V}_+$ in the assumption (i) by the condition W_+ (see [1]).

3. A theorem on the weak inequality for the system of differential-functional equations of Volterra's type

We will prove the main theorem of this paper.

T h e o r e m 2. We suppose that

1^0 The function F satisfies the condition $\tilde{V}_+$ in the set D (the condition W_+, if $h^t_{ii}(\varphi_i) \equiv \varphi_i(t)$),

2^0
$$D_-\Phi(t) \leqslant F(t,H^t(\Phi)), \quad t \in (0,T\rangle,$$
$$D_-\Psi(t) \geqslant F(t,H^t(\Psi)), \quad t \in (0,T\rangle,$$
$$\Phi(t) \leqslant \Psi(t), \quad t \in \langle\tau,0\rangle.$$

There exists a sequence $U^\nu(t) = \{u^\nu_1(t), u^\nu_2(t), \dots, u^\nu_n(t)\}$, $\nu = 1,2,\dots$, which satisfies the conditions:

3^0 The functions $u^\nu_j(t)$ are continuous, positive and non-decreasing in the interval $\langle\tau,T\rangle$,

4^0 $\lim_{\nu\to\infty} u^\nu_j(t) = 0$ for $t \in \langle\tau,T\rangle$ and $j=1,2,\dots,n$,

5^0 $D_-[\Psi(t) + U^\nu(t)] > F(t,H^t(\psi) + U^\nu(t))$, $t \in (0,T\rangle$,

6^0 $H^t(\psi) + U^\nu(t) \geqslant H^t(\psi + U^\nu)$, $t \in (0,T\rangle$, $\nu = 1,2,\dots$

Then $\Phi(t) \leqslant \Psi(t)$ for $t \in \langle\tau,T\rangle$.

P r o o f . Let $\tilde{\Psi}(t) = \Psi(t) + U^\nu(t)$. From the properties 1^0 and 6^0 we obtain $F(t,H^t(\Psi)+U^\nu(t)) \geqslant F(t,H^t(\Psi+U^\nu)) = F(t,H^t(\tilde{\Psi}))$, $t \in (0,T\rangle$.

By the inequality 5^0, we have

$$D_-\tilde{\Psi}(t) > F(t,H^t(\tilde{\Psi})), \quad t \in (0,T\rangle. \tag{3}$$

Taking into consideration the assumptions 2^0 and 3^0 and the inequality (3), by the theorem on the strong inequality for the system of differential-functional equations, we obtain

$$\Phi(t) < \tilde{\Psi}(t) = \Psi(t) + U^\nu(t), \quad t \in \langle\tau,T\rangle, \ \nu=1,2,\dots \tag{4}$$

Passing in the inequality (4) to the limit as $\nu \to \infty$, we get the thesis of our theorem.

Now we will give examples of the operator $H^t(\Phi)$ which

(a) satisfies the conditions 1^* and 2^* for any continuous function Φ,

(b) satisfies the condition 6^0 for any continuous function Φ and any function $U(t) = \{u_1(t),\dots,u_n(t)\}$ such that $u_i(t)$ for $i=1,2,\dots,n$ is continuous, positive and non-decreasing in the interval $\langle \tau, T\rangle$.

E x a m p l e 1. Consider the operator $H^t(\Phi) = \Phi(\Omega(t))$, where $\Omega(t) = \{\Omega_1(t),\dots,\Omega_n(t)\}$, $\Omega_i(t) = \{\omega_{1i}(t),\omega_{2i}(t),\dots$ $\dots,\omega_{ni}(t)\}$, $-\infty \leqslant \tau \leqslant \omega_{ij}(t) \leqslant t$, $\tau = \min\limits_{ij}\,(\inf\limits_{t\in\langle 0,T\rangle} \omega_{ij}(t))$.
Let us notice that in this case

$$F(t,H^t(\Phi)) = \Big\{f_1(t,\varphi_1(\omega_{11}(t)),\dots,\varphi_n(\omega_{n1}(t))),\dots$$

$$\dots,f_n(t,\varphi_1(\omega_{1n}(t)),\dots,\varphi_n(\omega_{nn}(t)))\Big\}.$$

By definition, this operator satisfies the conditions (a), (b), because

$$\Phi(\Omega(t)) + U(t) \geqslant \Phi(\Omega(t)) + U(\Omega(t)) = (\Phi+U)(\Omega(t)).$$

E x a m p l e 2. Let

$$H^{\tilde t}(\Phi) = \max_{\langle 0,\tilde t\rangle} \Phi(t) = \Big\{\max_{\langle 0,\tilde t\rangle} \varphi_1(t), \max_{\langle 0,\tilde t\rangle} \varphi_2(t),\dots, \max_{\langle 0,\tilde t\rangle} \varphi_n(t)\Big\}.$$

Then

$$F(\tilde t,H^{\tilde t}(\Phi)) = \Big\{f_1\Big(\tilde t, \max_{\langle 0,\tilde t\rangle} \varphi_1(t), \max_{\langle 0,\tilde t\rangle} \varphi_2(t),\dots, \max_{\langle 0,\tilde t\rangle} \varphi_n(t)\Big),\dots$$

$$\dots, f_n\Big(\tilde t, \max_{\langle 0,\tilde t\rangle} \varphi_1(t), \max_{\langle 0,\tilde t\rangle} \varphi_2(t),\dots, \max_{\langle 0,\tilde t\rangle} \varphi_n(t)\Big)\Big\}.$$

The operator $H^{\tilde{t}}(\Phi)$ satisfies the conditions (a), (b) and the following inequality is true

$$\max_{\langle 0,\tilde{t}\rangle} \Phi(t) + U(\tilde{t}) = \max_{\langle 0,\tilde{t}\rangle} \Phi(t) + \max_{\langle 0,\tilde{t}\rangle} U(t) \geq$$

$$\geq \max_{\langle 0,\tilde{t}\rangle} (\Phi(t) + U(t)).$$

Example 3. Let

$$H^t(\Phi) = \int_{t-h}^{t} K(s)\Phi(s)ds, \quad t \in \langle \tau+h,T\rangle ,$$

where $K(s) = \{k_1(s),\dots,k_n(s)\}$, $k_i(s) > 0$ for $i=1,2,\dots,n$, $s \in \langle \tau,T\rangle$, and h is chosen in such a way that $u(t) \geq \int_{t-h}^{t} k(s)u(s)ds$ for any continuous, positive and non-decreasing function $u(t)$. This operator fulfils the conditions (a), (b).

4. Properties of the function F implying the existence of the sequence $\{U^{\nu}(t)\}$

Now we will give certain properties of a function F such that there exists a sequence $\{U^{\nu}(t)\}$ fulfilling the conditions 3°, 4°, 5°, 6° if the operator $H^t(\Phi)$ fulfils the conditions (a), (b).

Property A. Let $\sigma(t,u)$ be a comparison function of the first type (see [1]) and let for $a > 0$ and $(t,X) \in D$

$$(5) \qquad f_i(t,X+A) \leq f_i(t,X) + \sigma(t,a),$$

where $A = (a,a,\dots,a) \in R^n$. Let $w^{\nu}(t)$ be the solution of the problem

(6) $$w'(t) = \sigma(t,w) + \frac{1}{\nu}, \quad t \in \langle 0,T^*\rangle, \quad T^* \leq T, \ w(0) = \frac{1}{\nu}.$$

We define the sequence $\{U^\nu(t)\} = \{u_1^\nu(t),\dots,u_n^\nu(t)\}$, $\nu=1,2,\dots$ in the following way:

(7) $$u_i^\nu(t) = \begin{cases} w^\nu(t) & \text{for } t \in \langle 0,T^*\rangle \\ \frac{1}{\nu} & \text{for } t \in \langle \tau,0\rangle \end{cases}, \quad i=1,2,\dots,n.$$

By the properties of a comparison function σ and by (5), (6), the sequence (4) fulfils the conditions 3°, 4°, 5°, 6°.

P r o p e r t y B. Let $\sigma(t,u)$ be a comparison function of II type (see [1]) and let for $a > 0$, $(t,X) \in D$

(8) $$f_i(t,X+A) < f_i(t,X) + \sigma(t,a),$$

where $A = (a,a,\dots,a) \in R^n$. If $w_\nu(t)$ is the solution of the problem

(9) $$w' = \sigma(t,w), \quad w(T) = \frac{1}{\nu}, \quad t \in (0,T\rangle, \quad \nu=1,2,\dots,$$

then the sequence $\{U^\nu(t)\} = \{u_1^\nu(t),\dots,u_n^\nu(t)\}$, $\nu=1,2,\dots$ defined in the following way

$$u_i^\nu(t) = \begin{cases} w_\nu(t) & \text{for } t \in (0,T\rangle \\ \alpha_\nu & \text{for } t \in \langle \tau,0\rangle \end{cases} \quad i=1,2,\dots,n,$$

where $\alpha_\nu = \lim_{t\to 0^+} w_\nu(t)$, fulfils the conditions $3^\circ - 6^\circ$.

Really, a solution $w_\nu(t)$ of the equation (9) is defined and non-decreasing in the whole interval $(0,T\rangle$ and $\alpha_\nu > 0$ for any $\nu=1,2,\dots$. Therefore $w_\nu(t) > 0$ for $t \in (0,T\rangle$. With regard to the fact that the function $w_\nu(t)$ is non-decreasing, we have

$$0 < w_\nu(t) \leqslant w_\nu(T), \quad t \in (0,T\rangle.$$

Moreover $w_\nu(T) = \frac{1}{\nu}$. Therefore $\lim_{\nu\to\infty} w_\nu(t) = 0$ for $t \in (0,T\rangle$. The conditions $3^0, 4^0, 6^0$ are satisfied. With regard to the inequality (8) the condition 5^0 is satisfied, too.

We can replace the inequality (8) by the weak inequality, but then we ought to modify the equation adequately. In C and D we give examples of such a case, where (8) is replaced by a weak inequality.

P r o p e r t y C. Let $\sigma(u)$ be a continuous function for $u > 0$, $\sigma(u) > 0$ and

$$\int_0^{u_0} \frac{du}{\sigma(u)} = +\infty, \qquad u_0 > 0.$$

Let for $a > 0$, $(t,X) \in D$

$f_i(t,X+A) \leqslant f_i(t,X) + \sigma(a)$, where $A = (a,a,\dots,a) \in R^n$.

Let $w^\nu(t)$ be a solution of the problem

$$w' = 2\sigma(w), \quad w(0) = \frac{1}{\nu}.$$

Let us notice that $\frac{1}{\nu} \leqslant w^\nu(t) \leqslant w^\nu(T)$ for $t \in \langle 0,T\rangle$ and

$$\int_{\frac{1}{\nu}}^{w^\nu(T)} \frac{du}{\sigma(u)} = 2T. \tag{10}$$

Since the integral $\int_0^{u_0} \frac{du}{\sigma(u)}$ is divergent, by the equality (10), we have $w^\nu(T) \to 0$, as $\nu \to +\infty$.

We construct the sequence $\{U^\nu(t)\} = \{u_1^\nu(t),\dots,u_n^\nu(t)\}$ like in A.

P r o p e r t y D. Let for $t \in (0,T\rangle$, $a > 0$:

$$f_i(t,X+A) \leqslant f_i(t,X) + |\ln t|\,a, \quad A = (a,a,\dots,a) \in R^n.$$

Let $w^{\nu}(t)$ be a solution of the problem

$$w'(t) = (|\ln t| + \gamma)w, \quad w(T) = \frac{1}{\nu}, \quad \gamma > 0, \quad t \in (0,T\rangle.$$

If for example $T \leqslant 1$, then $w^{\nu}(t)$ has the form

$$w^{\nu}(t) = \frac{1}{\nu}\, e^{T(1-\gamma-\ln T)}\, e^{t(\gamma-1-\ln t)}.$$

In this case we construct the sequence $\{U^{\nu}(t)\} = \{u_1^{\nu}(t),\dots,u_n^{\nu}(t)\}$ in the following way

$$u^{\nu}(t) = \begin{cases} w^{\nu}(t) & \text{for} \quad t \in (0,T\rangle \\ \frac{1}{\nu}\, e^{T(1-\gamma-\ln T)} & \text{for} \quad t \in \langle \tau, 0\rangle . \end{cases}$$

It is easy to verify that this sequence has the properties $3^0 - 6^0$. There exist examples of the comparison function of III type (see [1]) that there exists the sequence $\{U^{\nu}(t)\}$ for them. The existence of the sequence $\{U^{\nu}(t)\}$ is possible in the case, when $\sigma(t) = \frac{1}{t}\, a$.

P r o p e r t y E. Let for $t \in (0,T\rangle$, $a \geqslant 0$, $i = 1,\dots,n$,

$$f_i(t,X+A) \leqslant f_i(t,X) + \frac{1}{t}\, a, \quad \text{where} \quad A = (a,a,\dots,a) \in R^n.$$

Let $w^{\nu}(t)$ be a solution of the problem

$$(11) \qquad w' = \left(\frac{1}{t} + 1\right)w, \quad w(T) = \frac{1}{\nu}, \quad t \in (0,T\rangle,$$

namely $w^{\nu}(t) = \frac{1}{\nu T}\, e^{-T}\, e^{t}$. We see that the sequence $\{U^{\nu}(t)\} = \{u_1^{\nu}(t),\dots,u_n^{\nu}(t)\}$, $\nu = 1,2,\dots,$ defined in the following way

$$u_i^{\nu}(t) = \begin{cases} w^{\nu}(t) & \text{for} \quad t \in (0,T\rangle \\ 0 & \text{for} \quad t \in \langle \tau,0\rangle \end{cases} \qquad i = 1,2,\dots,n$$

satisfies the conditions 3^{o} - 6^{o}.

C o r o l l a r y . If the assumptions 1^{o} and 2^{o} are satisfied and the function F has one of the properties A, B, C, D, E, then the theorem on weak differential inequalities is true.

REFERENCE

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INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY, RZESZÓW
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