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Czesław Bagiński

ON THE PROPERTY W IN FINITE p-GROUPS

In this paper we shall study the property W considered in [1,2,3,4] concerning sets of elements of this same order. In our considerations we restrict ourselves to the case of finite p-groups ($p > 2$). In [1] it was shown that for two groups G and H with the property W and of relatively prime orders, $G \times H$ has the property W. In [4] an analogous result for p-groups of exponent p^2 was obtained. The aim of the paper is to prove that the above results cannot be generalized to the class of p-groups of exponent grater than p^2. We also give the proof that the property W is not hereditary under taking subgroups and homomorphism images. These results are based on a characterization of p-groups of maximal class.

Most of the notation used is standard. The basic results concerning regular p-groups and groups of maximal class can be found in [5].

We recall the basic definition.

A p-group G has the property W if for each natural n $K_n(G) \neq \emptyset$ implies $K_n(G)K_n(G) \leqslant G$, where $K_n(G) = \{x \in G : o(x) = p^n\}$ and $AB = \{ab : a \in A,\ b \in B\}$.

We need the following simple but useful lemma which was proved in [1] in a weaker form.

L e m m a 1. If H is a subgroup of G and $G \neq H$ then

$$(G - H)(G - H) = \begin{cases} H & \text{if } |G:H| = 2 \\ G & \text{otherwise.} \end{cases}$$

P r o o f . First let us assume that $|G:H| = 2$. Then there exists an element $g \in G$ such that $G - H = gH$. Therefore the normality of H in G implies $(G - H)(G - H) = (gH)(gH) = g^2H = H$. Suppose now that $|G:H| > 2$. Thus $H = g^{-1}(gH) \subset (G - H)(G - H)$. Moreover, there exists an element g_1 in $G - (H \cup gH)$. Because $g_1^{-1}gH \neq H$, we have $gH = g_1(g_1^{-1}gH) \subset (G - H)(G - H)$, which ends the proof.

T h e o r e m 2 ([2]). Each regular p-group has the property W.

This can be proved by observing that in a regular p-group G the set of elements of order less than p^n (n fixed) constitutes a subgroup of G ([5]).

Before going further in Lemma 3 we recall some results on p-groups of maximal class. By G_1 we denote the subgroup of G

$$C_G(\gamma_2G/\gamma_4G) = \left\{ g \in G: \bigwedge_{x \in \gamma_2G} [g,x] \in \gamma_4G \right\}$$

and we call this subgroup fundamental.

L e m m a 3 ([5]). If a p-group G is of maximal class and $|G| \geqslant p^{p+2}$ then:

a) G_1 is the unique maximal subgroup being regular;

b) Each maximal subgroup of G not equal to G_1 is a group of maximal class;

c) For each element $x \in G - G_1$ we have $o(x) \leqslant p^2$, $\{x^g : g \in G\} = x\gamma_2G$ and $x^p \in Z(G)$.

By III.14.6 ([5]) these are special cases of III.14.14 and III.14.22.

L e m m a 4. If G is of maximal class, $|G| \geqslant p^{p+2}$, then for each maximal subgroup $M \neq G_1$ of G the set $M - \gamma_2G$ is contained in $K_1(G)$ or $K_2(G)$.

P r o o f . Let M be a maximal subgroup of G, $M \neq G_1$. By III.14.2 $|M:\mho_2 G| = p$ and because of that $M = \langle x, \mho_2 G\rangle$ for $x \notin G_1$. Thus by Lemma 3c we obtain

$$M - \mho_2 G = x\mho_2 G \cup x^2\mho_2 G \cup \dots \cup x^{p-1}\mho_2 G =$$

$$= \{x^g : g \in G\} \cup \{(x^2)^g : g \in G\} \cup \dots \cup \{(x^{p-1})^g : g \in G\}.$$

But by Lemma 3c the elements $x, x^2, \dots, x^{p-1}$ are of this same order equal to p or p^2. This yields $M - \mho_2 G \subset K_1(G)$ or $M - \mho_2 G \subset K_2(G)$.

T h e o r e m 5. If G is a p-group of maximal class, then G has the property W.

P r o o f . If $|G| \leqslant p^{p+1}$ the result is known ([4] Theorem 1), so we may assume that $|G| \geqslant p^{p+2}$. Since G_1 is regular, by Theorem 2 we infer that the inclusion $K_i(G) \subset G_1$ yields $K_i(G)K_i(G) \leqslant G$. By Lemma 3c for $x \notin G_1$ we have $o(x) = p$ or $o(x) = p^2$, thus for $i > 2$ we obtain $K_i(G)K_i(G) \leqslant G_1$ in virtue of the above. So we shall consider the sets $K_i(G)$ for $i = 1,2$ only. To prove our theorem it is sufficient to consider the case where $K_i(G)$ is not included in G_1. If $K_i(G)$ is included in a maximal subgroup M, $M \neq G_1$, then by Lemma 4 $M - \mho_2 G \subset K_i(G)$ and

$$K_i(G)K_i(G) \supset (M - \mho_2 G)(M - \mho_2 G) = M \supset K_i(G)K_i(G).$$

Now let $\langle K_i(G)\rangle = G$. Then there exist at least two distinct maximal subgroups M_1, M_2 such that $M_1 - \mho_2 G \subset K_i(G)$ and $M_2 - \mho_2 G \subset K_i(G)$. This implies $M_j = (M_j - \mho_2 G)(M_j - \mho_2 G) \subset$ $\subset K_i(G)K_i(G)$. Hence

$$G = M_1 M_2 = [(M_1 - \mho_2 G) \cup \mho_2 G][(M_2 - \mho_2 G) \cup \mho_2 G] =$$

$$= (M_1 - \mho_2 G)(M_2 - \mho_2 G) \cup (M_1 - \mho_2 G)\mho_2 G \cup \mho_2 G(M_2 - \mho_2 G) \cup$$

$$\cup\ \mho_2 G \mho_2 G \subset K_i(G)K_i(G).$$

Thus the theorem is proved.

T h e o r e m 6. If A and B are p-groups of maximal class and of exponent grater than p^3, then the direct product G of these groups has not the property W.

P r o o f . Let A_1, B_1 be fundamental subgroups of groups A and B respectively. If $x \in K_3(A)$ is a fixed element, then for each element $y \in B - B_1$ we have $xy \in K_3(G)$. So $B = (B - B_1)(B - B_1) = x^{-1}(B - B_1)x(B - B_1) = [x^{-1}(B-B_1)][x(B - B_1)] \subset K_3(G)K_3(G)$. Similarly $A \subset K_3(G)K_3(G)$. Therefore $G = \langle K_3(G)\rangle$. Now we show that $K_3(G)K_3(G) \neq G$. Really, each element from $K_3(G)$ is of the form xy, $x \in A$, $y \in P$, where both of multiplicators are of the order less than p^4 and at least one of them is of the order p^3. Thus for $xy, uv \in K_3(G)$ $(x,u \in A,\ y,v \in B)$ the element $(xy)(uv) = (xu)(yv)$ does not belong to $K_4(A)K_4(B)$. Otherwise $xu \in K_4(A) \subset A_1$, $yv \in K_4(B) \subset B_1$ and then we have the following cases:

i) $x,u \in A_1;\ y,v \in B_1$
ii) $x,u \in A_1;\ y,v \in B - B_1$
iii) $x,u \in A - A_1;\ y,v \in B_1$
iV) $x,u \in A - A_1;\ y,v \in B - B_1$.

By regularity of A_1 and B_1 we obtain in two first cases $xu \notin K_4(A)$ and in the third - $yv \notin K_4(B)$. In the fourth case by Lemma 3c all elements x,y,u,v are of the order less than p^3 and then $xy, uv \notin K_3(G)$.

C o r o l l a r y 7. If p-groups A and B have the property W and are of exponent grater than p^2 then their direct product need not have this property.

P r o o f . For p-groups of exponent grater than p^3 this is an immediate consequence of two last theorems. For groups of exponent p^3 this can be easily proved using the method from the proof of Theorem 6 and by observing that if G is a p-group of maximal class, then $G/Z(G)$ is of maximal class too, moreover, each element from $G/Z(G) - G_1/Z(G)$ is of order p.

Now we show that there exist p-groups with the property W containing subgroups which have not this property.

First we prove the following lemma.

L e m m a 8. If A is a p-group such that $K_1(A)K_1(A) = A$, then in the direct product G of A and a cyclic group B of the order p^n we have $K_n(G)K_n(G) = G$.

P r o o f . Let $B = \langle b: b^{p^n} = 1\rangle$ and let G be the direct product of A and B. By assumption, for each $a \in A$ there exist elements $a_1, a_2 \in A$ of order p such that $a_1a_2 = a$. Let xy $(x \in A,\ y \in B)$ be any element of G. Then $x = x_1x_2$ $/o(x_1) = o(x_2) = p/$ and $y = b^m$. Hence $xy = (x_1b^{m+k})(x_2b^{-k}) \in K_n(G)K_n(G)$, where k is a natural number such that $k, m+k \not\equiv 0(\mathrm{mod}\,p)$. Such a number exists by the assumption $p > 2$.

T h e o r e m 9 . The class of all finite p-groups with the property W is not closed under taking subgroups and homomorphism images.

P r o o f . Let A and B be p-groups of maximal class and of exponent grater than p^2 such that $(A - A_1) \subset K_1(A)$, $(B - B_1) \subset K_1(B)$, where A_1 and B_1 are fundamental subgroups of A and B respectively. By Lemma 1 $A = K_1(A)K_1(A)$, $B = K_1(B)K_1(B)$ and it is easy to see that $K_1(H)K_1(H) = H$, where H denotes the direct product of A and B. As it was shown in Theorem 6, H does not have the property W and by Lemma 8 $G = H \times C$ has the property W, where C is a cyclic p-group of an order not smaller than the exponents of A and B. It is obvious that $H \leqslant G$ and $G/C \simeq H$.

REFERENCES

[1] E. A m b r o s i e w i c z : O kwadratach zbiorów elementów tego samego rzędu w grupach, Doctoral dissertation, Technical University of Warsaw, Warsaw 1978.

[2] E. A m b r o s i e w i c z : The property W for regular p-groups and for nilpotent groups of degree 2, Demonstratio Math. 13 (1980) 613-617.

[3] J. Ambrosiewicz: O pewnych podgrupach danej grupy, Zeszyty Nauk.-Dydakt. Filii UW w Białymstoku, z.7, t.II, Nauki Mat.-Przyr. Białystok 1974.

[4] Cz. Bagiński: Some remarks on finite p-groups, Demonstratio Math. 15 (1981) 279-285.

[5] B. Huppert: Endliche Gruppen, Berlin 1967.

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, BIAŁYSTOK BRANCH, BIAŁYSTOK

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