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R. P. Agarwal, Rajiv K. Srivastava

A REMARK ON A CERTAIN SPACE OF SEQUENCES

1. Introduction and motivation

Agarwal and Bose [1] have recently studied a class S_1 of integral functions defined as

$$S_1 = \left\{ f : f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n ; \sum_{n=0}^{\infty} |a_n|^{1/2n+1} < \infty \right\}.$$

They have provided S_1 with a Banach algebraic structure by defining binary operations and norm suitably.

There are several misprints and mistakes in their paper. In section 2 of this paper, we have pointed them out and have corrected the mistakes in section 3 and 4 of [1].

In section 3, we have defined a class $\#$ of sequences closely related with the class S_1 of [1]. Suitably defining binary operations and norm on $\#$, we have provided it with a separable, commutative Banach algebraic structure. Section 4 is devoted to characterizing continuous linear functionals on $\#$ and studying weak and weak - $*$ convergences in $\#$. Topological zero divisors have been characterized in the same section. In section 5, an inner product has been defined on $\#$ which equips it with an inner product space structure. A superset $\#_1$ of $\#$ is shown to be a Hilbert space with the same structure as $\#$.

2. The arguments in Section 3 of [1] do not hold, in general. The fact that only real arithmetic roots of the quantities like $a_n^{1/2n+1}$ are considered, in order to make the ordering well defined, should have been specifically mentioned.

In section 4 of [1], there are several mistakes in the proof of Proposition 5. In the first part, (p.68, third line from below), c_n's should be defined as

$$|c_n|^{1/2n+3} = \max\left\{|a_n|^{1/2n+3},\ |b_n|^{1/2n+3}\right\}.$$

In the second part, the closure of S_3 (considered as a subset of S_1) with respect to multiplication and not scalar multiplication should be asserted, the proof being obvious. Moreover, the last sentence of the same proposition should read:

Thus $h(z) \times f(z) \in S_3$, which shows that S_3 is an ideal in S_1.

3. Let us consider a class $\#$ of sequences, defined by

$$\# = \left[f = \{a_n\}:\ a_n \in \mathbf{C}\ ,\ \sum_{n=1}^{\infty} |a_n|^{1/2n+1} < \infty\right].$$

It may be noted that every member of $\#$ forms the sequence of coefficients of an entire Taylor series.

We now define binary algebraic operations on $\#$ as follows:

$$f + g = \{a_n\} + \{b_n\} = \left\{(a_n^{1/2n+1} + b_n^{1/2n+1})^{2n+1}\right\};$$

$$\alpha \cdot f = \alpha \cdot \{a_n\} = \left\{(\alpha \cdot a_n^{1/2n+1})^{2n+1}\right\},\quad \alpha \in \mathbf{C};$$

and

$$f \times g = \{a_n\} \times \{b_n\} = \{a_n \cdot b_n\},$$

where $f = \{a_n\}$, $g = \{b_n\}$ are arbitrary members of $\#$.

The norm on $\#$ is defined as

$$\| f \| = \sum_{n=1}^{\infty} |a_n|^{1/2n+1}, \quad f = \{a_n\} \in \# .$$

It is easy to verify that $\#$, equipped with these operations and norm, becomes a commutative Banach algebra. The only possible identity element is $\{1,1...\}$ which does not belong to $\#$.

In $\#$, consider elements

$$f_n = \{a_i^{(n)}\}$$

with

$$a_i^{(n)} = \begin{cases} 1 & \text{if } i=n \\ 0 & \text{if } i \neq n . \end{cases} \tag{3.1}$$

If $f = \{a_i\}$ is any arbitrary element of $\#$, then we put

$$g_n = f - \sum_{k=1}^{n} a_n^{1/2k+1} \cdot f_k .$$

Obviously, g_n can be rewritten as $\{0,0,\ldots,0,a_{n+1},a_{n+2},\ldots\}$. Hence, $\| g_n \| = \sum_{n+1}^{\infty} |a_k|^{1/2k+1} \to 0$ as $n \to \infty$, because the series $\sum_{k=1}^{\infty} |a_k|^{1/2k+1}$ converges. Consequently, f can be uniquely represented as

$$f = \sum_{k=1}^{\infty} a_k^{1/2k+1} \cdot f_k . \tag{3.2}$$

Consider the set of all polynomials of the type

$$p = \sum_{k=1}^{n} b_k^{2k+1} \cdot f_k,$$

where b_k is a rational complex number. This is a countable set and can be shown to be dense in $\#$, considered as the set of elements of the type (3.2). Hence, we have

T h e o r e m 1. $\#$ is a separable, commutative Banach algebra without identity.

4. Continuous linear functionals on $\#$

The dual space of $\#$ will be denoted by $\#^*$ and members of $\#^*$ will be denoted by f^*. Our main result of this section is

T h e o r e m 2. The general form of any continuous linear functional on $\#$ is given by the formula

$$f^*(f) = \sum_{n=1}^{\infty} a_n^{1/2n+1} d_n, \tag{4.1}$$

where $f = \{a_n\} \in \#$ and $\{d_n\}$ is a bounded sequence uniquely defined by f^*.

Moreover, $\|f^*\| = \sup\limits_n |d_n|$.

P r o o f. Let $f^* \in \#^*$. Consider the elements $f_n = \{a_i^{(n)}\}$ in $\#$ defined by (3.1). If f is any arbitrary element of $\#$ it can be expressed as (3.2), i.e.,

$$f = \sum_{n=1}^{\infty} a_n^{1/2n+1} \cdot f_n.$$

Linearity and continuity of f^* implies that

$$f^*(f) = f^*\left[\sum_{n=1}^{\infty} a_n^{1/2n+1}\cdot f_n\right] = \sum_{n=1}^{\infty} a_n^{1/2n+1}\ f^*(f_n) =$$

$$= \sum_{n=1}^{\infty} a_n^{1/2n+1}\cdot d_n,$$

where $f^*(f_n) = d_n$ are uniquely determined by f^*.

Since $\|f_n\| = 1$ and $|d_n| = |f^*(f_n)| \leq \|f^*\|\ \|f_n\| = \|f^*\|$ for $n = 1,2,\ldots$ it follows that the sequence $\{d_n\}$ is bounded.

Conversely, let $\{d_n\}$ be a bounded sequence. The series (4.1) converges, since $f = \{a_n\} \in ⧺$, and thus f^* defines a linear functional on ⧺. For continuity of f^*, we note that

$$|f^*(f)| = \left|\sum a_n^{1/2n+1}\ d_n\right|,$$

i.e.,

$$|f^*(f)| \leq \sum |a_n|^{1/2n+1} |d_n| < \infty. \tag{4.2}$$

We further note that, (4.2) gives

$$|f^*(f)| \leq (\sup_n |d_n|)\ \|f\|.$$

Hence $\|f^*\| = \sup\limits_{\|f\|\leq 1} \dfrac{|f^*(f)|}{\|f\|} \leq \sup\limits_n |d_n|$.

On the other hand, if

$$f_n = \{\text{sgn.}\ (\overline{d_n})\}^{2n+1},$$

then $f_n \in ⧺$ and $\|f_n\| = 1$ for every $n \geq 1$.

Also,

$$|f^*(f_n)| = |d_n \cdot \text{sgn.} (\bar{d}_n)| = |d_n|,$$

i.e.,

$$|d_n| = |f^*(f_n)| \leqslant \|f^*\| \|f_n\| = \|f^*\|,$$

so that

$$\sup_n |d_n| \leqslant \|f^*\|.$$

Hence the theorem follows.

Theorem 3. The set of all topological zero divisors in $\#$ is $\#$ itself.

Proof. For the definition of topological zero divisors, we refer to Larsen [2].

Let $f = \{a_n\}$ be any arbitrary element of $\#$. Let $f_n = \{a_i^{(n)}\}$ be defined by (3.1). Obviously, $\|f_n\| = 1$ for every $n \geqslant 1$. Also,

$$f \times f_n = a_n = f_n \times f,$$

so that

$$\|f \times f_n\| = \|f_n \times f\| = |a_n|^{1/2n+1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty .$$

Hence the theorem.

It is easy to see that weak and strong convergences are equivalent in $\#$. We now prove a result on weak convergence in $\#^*$.

Theorem 4. A sequence $\{f_p^*\}$ in $\#^*$ converges weakly to $f^* \in \#^*$, if and only if the following conditions hold:

i) $\{\|f_p^*\|\}$ is bounded; and

ii) $d_{pn} \rightarrow d_n$ for every $n \geqslant 1$ as $p \rightarrow \infty$, where $f_p^* = \{d_{pn}\}$ and $f^* = \{d_n\}$.

P r o o f . "Necessity" part is obvious.

Next, assume that (i) and (ii) hold. By (i) and the fact that $f^* \in \#^*$, we can find M, such that

$$|d_{pn}| \leqslant M, \ |d_n| \leqslant M \quad \text{for every} \quad n.$$

By the definition of $\#$, given $f = \{a_n\} \in \#$ and $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon)$ such that

$$\sum_{n_0+1}^{\infty} |a_n|^{1/2n+1} < \varepsilon/4M.$$

Further, by (ii), we can choose $p_0 = p_0(\varepsilon)$, such that

$$|d_{pn}-d_n| < \varepsilon/2N, \quad \text{for} \quad p \geqslant p_0,$$

where $N = \sum_{n=1}^{n_0} |a_n|^{1/2n+1}$.

Now, for $p \geqslant p_0$, we have

$$|(f_p^* - f^*)(f)| = \left| \sum_{n=1}^{\infty} (d_{pn}-d_n)\cdot a_n^{1/2n+1} \right| \leqslant$$

$$\leqslant \sum_{n=1}^{n_0} |d_{pn}-d_n| \ |a_n|^{1/2n+1} + \sum_{n_0+1}^{\infty} |d_{pn}-d_n| \ |a_n|^{1/2n+1} <$$

$$< \frac{\varepsilon}{2N} \cdot N + 2M. \frac{\varepsilon}{4M} = \varepsilon .$$

Hence f_p^* converges to f^* weakly in $\#^*$.

5. Inner product in $\mathcal{H}$

We define, in $\mathcal{H}$, an inner product as follows

$$(5.1) \qquad \langle f,g\rangle = \sum_{n=1}^{\infty} a_n^{1/2n+1}\, \bar{b}_n^{1/2n+1} ,$$

where $f = \{a_n\}$, $g = \{b_n\} \in \mathcal{H}$ and $\bar{b}_n$ is the complex conjugation of b_n.

This evidently satisfies all the axioms of inner product. The norm induced by this inner product is

$$(5.2) \qquad \| f \| = \left\{ \sum_{n=1}^{\infty} |a_n|^{2/2n+1} \right\}^{1/2} .$$

Thus, $\mathcal{H}$ becomes a unitary space as well as a normed algebra. It is neither a Hilbert space nor a Banach algebra. In fact, $\mathcal{H}$ is not complete with respect to the norm (5.2). Let, for instance,

$$f_p = \left\{ 1^{-3},\ 2^{-5},\ 3^{-7}, \dots, p^{-(2p+1)}, 0, 0 \dots \right\} .$$

Obviously, $f_p \in \mathcal{H}$ for every $p \geqslant 1$. Moreover, in the norm (5.2), f_p converges to $f = \{n^{-(2n+1)}\}$. Hence $\{f_p\}$ is a Cauchy sequence. However, the limit function f does not belong to $\mathcal{H}$, since,

$$\sum_{n=1}^{\infty} \left| n^{-(2n+1)} \right|^{1/2n+1} = \sum_{n=1}^{\infty} 1/n \nless \infty .$$

Let us now consider the set of sequences defined as

$$\mathcal{H}_1 = \left[f = \{a_n\};\ a_n \in \mathbf{C} , \left(\sum_{n=1}^{\infty} |a_n|^{2/2n+1} \right)^{1/2} < \infty \right] .$$

The binary operations in $\#_1$ are the same as in $\#$. The norm and inner product in $\#_1$ are defined by (5.2) and (5.1), respectively, thus $\#$, becomes a Hilbert space.

From the preceding discussions regarding the inner product (5.1), we infer that

Theorem 6. $\#_1$ is a Hilbert space. $\#$ is not a closed linear subspace of $\#_1$.

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DEPARTEMENT OF MATHEMATICS AND ASTRONOMY, LUCKNOW UNIVERSITY, LUCKNOW, INDIA

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