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ON SUBSPACES OF SUBSPACES OF A FINSLER SPACE

This note is devoted to a study of the properties of a subspace F_1 of a subspace F_m of a Finsler space F_n with the help of δ-derivative given by Rund ([2], pp.166-169). The conditions under which F_1 is minimal in F_m or it is minimal in F_n have been investigated.

1. Introduction

Let F_m be a differential subspace of dimension of m of an n-dimensional Finsler space F_n. The coordinates x^i (referred to F_n) of a point of F_m are given by

$$x^i = x^i(u^\alpha), \quad i = 1,\dots,n; \quad \alpha = 1,\dots,m \quad (m \leqslant n). \tag{1.1}$$

Suppose that a vector $\dot{x}^i$ (or $\dot{u}^\alpha$) tangent to F_m is an element of support. We may then write

$$\dot{x}^i = B^i_\alpha \dot{u}^\alpha, \tag{1.2}$$

where

$$B^i_\alpha = \partial x^i / \partial u^\alpha.$$

The (n-m) vectors $N^i_{\mu(x,\dot{x})}$ $(\mu = m+1,\dots,n)$ satisfying the conditions

$$(1.3)\quad \begin{cases} \text{a)}\ g_{ij}(x,\dot{x})\, B^i_\alpha N^j_\mu = 0; \\ \text{b)}\ g_{ij}(x,\dot{x})\, N^i_\mu N^j_\nu = S_u \qquad (\mu,\nu = m+1,\dots,n) \end{cases}$$

are called normal vectors, g_{ij} being the components of the metric tensor of F_n.

The $\overset{\circ}{D}$-derivative of the projection factors B^i ([2], p.167)

$$(1.4)\qquad \overset{\circ}{D}B^i_\beta = \overset{\circ}{H}{}^i_{\nu\beta} = B^i_{\beta\nu} + \Gamma^i_h B^h - \bar{\Gamma}^{*\alpha}_{\beta\nu} B^i_\alpha,$$

$i,j,h,k = 1,\dots,n;\ \alpha,\beta,\nu,\delta,\varepsilon = 1,\dots,m,$ where

$$(1.5)\quad \begin{cases} \text{a)}\ \Gamma^i_{h\nu} \overset{\text{def}}{=\!=} \bar{\Gamma}^{*i}_{hk} B^k_\nu + A^i_{hk} H^k_\nu, \\ \text{b)}\ \bar{\Gamma}^{*\alpha}_{\beta\nu} = \Gamma^{*\alpha}_{\beta\nu} + B^\alpha_i A^i_{hk} B^h_\beta H^k_\nu, \\ \text{c)}\ \Gamma^{*\alpha}_{\beta\nu} = B^\alpha_i (B^i_{\beta\nu} + \Gamma^{*i}_{hk} B^h_\beta B^k_\nu), \\ \text{d)}\ B^\alpha_i = g^{\alpha\varepsilon} g_{ij} B^j_\varepsilon,\quad B^i_{\beta\nu} = \partial^2 x^i/\partial u^\beta \partial u^\nu,\quad FH^i_\nu = \overset{\circ}{H}{}^i_{\nu\beta}\, \dot{u}^\beta \end{cases}$$

and $g^{\alpha\varepsilon}$ being the contravariant components of the metric tensor of F_m and the symbols A^i_{hk} and Γ^{*i}_{hk} have their usual meanings.

Since $\overset{\circ}{H}{}^i_{\nu\beta}$ is normal to F_m, we have

$$(1.6)\qquad \overset{\circ}{H}{}^i_{\nu\beta} = N^i_\mu \overset{\circ}{H}{}^\mu_{\nu\beta} \qquad (\mu = m+1,\dots,n),$$

where

$$(1.7)\qquad \overset{\circ}{H}{}^\mu_{\nu\beta} = N^\mu_i H^i_{\nu\beta}.$$

2. Mean curvature vectors and minimal subspaces

Definition 2.1. The vector field

$$\overset{\circ}{K}{}^{i}_{(x,\dot{x})} \cdot \left\{F_m, F_n\right\} \overset{\text{def}}{=} \frac{1}{m}\, g^{\nu\beta} H^{i}_{\nu\beta} \tag{2.1}$$

of F_n is called the mean curvature vector of F_m. This vector field is normal to F_m. If $\overset{\circ}{K}{}^{i}_{(x,\dot{x})} \cdot \left\{F_m, F_n\right\}$ vanishes, then F_m is called a minimal subspace of F_n.

Now, let us consider a differentiable subspace F_l, of dimension l, of F_m represented parametrically by the equations

$$u^{\alpha} = u^{\alpha}(z^{\theta}); \quad (l < m \leqslant n) \quad (\alpha, \beta, \nu, \delta, \ \varepsilon = 1, \ldots, m); \quad \theta, \phi, \ \psi = 1, \ldots, l). \tag{2.2}$$

Assuming that the vector $\dot{u}^{\alpha}$ is tangent to F_m, the corresponding tangent vector to F_l will be $\dot{z}^{\theta}$ and these two are related by

$$\dot{u}^{\alpha} = B^{\alpha}_{\theta}\, \dot{z}^{\theta}, \tag{2.3}$$

where

$$B^{\alpha}_{\theta} = \partial u^{\alpha}/\partial z^{\theta}.$$

There exist (m-l) vectors $N^{\alpha}_{p(z,\dot{z})}$ $(p = l + 1, \ldots, m)$ normal to F_l satisfying the conditions

$$\begin{cases} \text{a)} \quad g_{\alpha\beta}(u, \dot{u}) B^{\alpha}_{\theta} N^{\beta}_{p} = 0 \\ \\ \text{b)} \quad g_{\alpha\beta}(u, \dot{u})\, N^{\alpha}_{p} N^{\beta}_{q} = \delta^{q}_{p} \ (p, q = l+1, \ldots, m). \end{cases} \tag{2.4}$$

The $\overset{\circ}{D}$-derivative of the projection factors B^{α}_{θ} will be given by

(2.5) $$\overset{\circ}{D}B^{\alpha}_{\theta} = \overset{\circ}{H}{}^{\alpha}_{\phi\theta} = B^{\alpha}_{\theta\phi} + \Gamma^{\alpha}_{\delta\phi}B^{\delta}_{\theta} - \bar{\Gamma}^{*\psi}_{\theta\phi}B^{\alpha}_{\psi},$$

where the quantities $\Gamma^{\alpha}_{\delta\phi}$, $\bar{\Gamma}^{*\psi}_{\theta\phi}$, $\Gamma^{*\psi}_{\theta\phi}$, B^{ψ}_{α}, $B^{\alpha}_{\theta\phi}$, H^{ℓ}_{ϕ} and $g^{\psi\phi}$ in F_l may be defined in the same manner as the corresponding quantities are defined in F_m.

Since $\overset{\circ}{H}{}^{\alpha}_{\phi}$ is normal to F_l, we may write

(2.6) $$\overset{\circ}{H}{}^{\alpha}_{\phi\theta} = N^{\alpha}_{p}\overset{\circ}{H}{}^{p}_{\phi\theta}, \quad p = l+1,\dots,m,$$

where

(2.7) $$\overset{\circ}{H}{}^{p}_{\phi\theta} = N_{p\alpha}\overset{\circ}{H}{}^{\alpha}_{\phi\theta}.$$

Definition 2.2. The vector field

(2.8) $$\overset{\circ}{K}{}^{\alpha}_{(u,\dot{u})}\cdot\left\{F_l,F_m\right\} \overset{\text{def}}{=\!=} \frac{1}{l}\, g^{\phi\theta}\overset{\circ}{H}{}^{\alpha}_{\phi\theta}$$

($g^{\phi\theta}$ being the contravariant components of the metric tensor of F_l) of F_m is normal to F_l. This is called the mean curvature vector of F_l in F_m. If $\overset{\circ}{K}{}^{\alpha}_{(u,\dot{u})}\cdot\left\{F_l,F_m\right\} = 0$, then F_l is called minimal subspace of F_m.

A subspace F_l of the subspace F_m can be regarded as a subspace of the enveloping space F_n and it can be expressed by the equations

(2.9) $$x^i = x^i(u^{\alpha}(z^{\theta})); \quad i=1,\dots,n;\ \theta=1,\dots,m;\ (l < m \leqslant n)$$

and consequently

(2.10) $$B^{i}_{\theta} = B^{i}_{\alpha}B^{\alpha}_{\theta}.$$

The components $g_{\theta\phi}(z,\dot{z})$, $g_{\alpha\beta}(u,\dot{u})$, $g_{ij}(x,\dot{x})$ of the metric tensors of F_l, F_m and F_n respectively are related by

(2.11) $$g_{\phi\theta}(z,\dot{z}) = g_{\alpha\beta}(u,\dot{u})B^{\alpha}_{\theta}B^{\beta}_{\phi} = g_{ij}(x,\dot{x})B^{i}_{\theta}B^{j}_{\phi}.$$

The orthogonal unit normal vectors of F_l in F_n are $N^i_p = N^\alpha_p B^i_\alpha$ and N^i_μ. N^i_p is tangent to F_m and N^i_μ is normal to F_m. The $\overset{\circ}{D}$-derivative of B^i_θ is

$$(2.12)\qquad \overset{\circ}{D}_\phi B^i_\theta = \overset{\circ}{H}{}^i_{\phi\theta} = B^i_{\theta\phi} + \Gamma^i_{h\phi}B^h_\theta - \bar{\Gamma}^{*\psi}_{\theta\phi}B^i_\psi,$$

where

$$(2.13)\qquad \Gamma^i_{h\phi} \overset{\text{def}}{=\!=} \Gamma^i_{hk}B^k_\phi + A^i_{hk}H^k_\phi,$$

$$(2.14)\qquad FH^k_\phi = \overset{\circ}{H}{}^k_{\phi\theta}\dot{z}^\theta, \quad B^i_{\theta\phi} = \partial^2 x^i/\partial z^\theta\,\partial z^\phi$$

and $\bar{\Gamma}^{*\psi}_{\theta\phi}$ have the same meaning as in (2.5).

Since $\overset{\circ}{H}{}^i_{\phi\theta}$ is normal to F_l, we have

$$(2.15)\qquad \overset{\circ}{H}{}^i_{\phi\theta} = N^i_\mu \overset{\circ}{H}{}^\mu_{\phi\theta},$$

$$(2.16)\qquad \overset{\circ}{H}{}^\mu_{\phi\theta} = N_{\mu i}\overset{\circ}{H}{}^i_{\phi\theta}.$$

3. Relation between mean curvatures vectors

The $\overset{\circ}{D}$-derivative of the equation (2.10) gives us

$$(3.1)\qquad \overset{\circ}{H}{}^i_{\phi\theta} = \overset{\circ}{H}{}^\nu_{\phi\theta}B^i_\nu + \overset{\circ}{H}{}^i_{\nu\beta}B^\nu_\theta B^\beta_\phi,$$

where we have used the equations (1.4), (2.5) and (2.12). On multiplying (3.1) by $\frac{1}{l}\cdot g^{\phi\theta}$, we get

$$(3.2)\qquad \frac{1}{l}\, g^{\phi\theta}\,\overset{\circ}{H}{}^i_{\phi\theta} = \left(\frac{1}{l}\, g^{\phi\theta}\overset{\circ}{H}{}^\nu_{\phi\theta}\right)B^i_\nu + \left(\frac{1}{l}\, g^{\phi\theta}B^\nu_\theta B^\beta_\phi\right)\overset{\circ}{H}{}^i_{\nu\beta}.$$

D e f i n i t i o n 3.1. The vector field $\overset{\circ}{K}{}^i_{(x,\dot{x})}\cdot\{F_l,F_m,F_n\}$ defined by

(3.3) $$\overset{\circ}{K}{}^{i}_{(x,\dot{x})}\cdot\left\{F_l,F_m,F_n\right\} \overset{\text{def}}{=\!=} \left(\frac{1}{l}\, g^{\phi\theta} B^{\gamma}_{\theta} B^{\beta}_{\phi}\right) \overset{\circ}{H}{}^{i}_{\gamma\beta}$$

is said to be the relative mean curvature vector of F_l with respect to F_m and F_n. It is obvious that the above vector field is normal to F_m.

Using the equations (2.8), (3.2) and (3.3), we obtain

(3.4) $$\overset{\circ}{K}{}^{i}_{(x,\dot{x})}\cdot\left\{F_l,F_n\right\} = \overset{\circ}{K}_{(u,\dot{u})}\cdot\left\{F_l,F_m\right\}B^{i}_{\nu} + \overset{\circ}{K}{}^{i}_{(x,\dot{x})}\cdot\left\{F_l,F_m,F_n\right\}$$

which yields the following theorem.

T h e o r e m 3.1. The mean curvature vector of a Finsler subspace F_l in F_n is the sum of the mean curvature vector of F_l in F_m and the relative mean curvature vector of F_l with respect to F_m and F_n.

4. Concurrent vector field

A vector field of Riemannian manifold concurrent along a submanifold has been considered by B.Y.Chen and K.Yano ([3], p.551). In analogy to this, a vector field V^i of F_n will be called concurrent along F_l if

(4.1) $$B^{i}_{\theta} + \overset{\circ}{D}_{\theta}V^{i} = 0.$$

Taking the $\overset{\circ}{D}$-derivative of (4.1), we find

(4.2) $$\overset{\circ}{D}_{\phi}B^{i}_{\theta} + \overset{\circ}{D}_{\phi}\overset{\circ}{D}_{\theta}V^{i} = 0$$

or

(4.3) $$\overset{\circ}{H}{}^{i}_{\phi\theta} + \overset{\circ}{D}_{\phi}\overset{\circ}{D}_{\theta}V^{i} = 0,$$

where we have used equation (2.12).

On multiplying (4.3) by $\frac{1}{l}\cdot g^{\phi\theta}$ and using the equation (2.17) we obtain

(4.4) $$\overset{\circ}{K}{}^{i}_{(x,\dot{x})}\cdot\left\{F_1,F_n\right\} + \frac{1}{l}\, g^{\phi\theta}\overset{\circ}{D}_{\phi}\overset{\circ}{D}_{\theta}V^{i} = 0.$$

The equations (3.4) and (4.4) yields

(4.5) $$\overset{\circ}{K}_{(u,\dot{u})}\cdot\left\{F_1,F_m\right\} B^{i}_{\nu} + \overset{\circ}{K}{}^{i}_{(x,\dot{x})}\cdot\left\{F_1,F_m,F_n\right\} + \\ + \frac{1}{l}\, g^{\phi\theta}\overset{\circ}{D}_{\phi}\overset{\circ}{D}_{\theta}V^{i} = 0.$$

Since $\overset{\circ}{K}{}^{\nu}_{(x,\dot{x})}\left\{F_1,F_m,F_n\right\}$ is normal to F_m, we have

(4.6) $$\overset{\circ}{K}{}^{\nu}_{(u,\dot{u})}\cdot\left\{F_1,F_m\right\} = 0$$

iff $\frac{1}{l}\cdot g^{\phi\theta}\overset{\circ}{D}_{\phi}\overset{\circ}{D}_{\theta}V^{i}$ is normal to the subspace F_m.

Hence we have

Theorem 4.1. Suppose that there exists a vector field V^i of F_n satisfying (4.1). Then in order for F_1 to be minimal in F_m it is necessary and sufficient that $g^{\varphi\theta}\overset{\circ}{D}_{\varphi}\overset{\circ}{D}_{\theta}V^{i}$ to be normal to F_m.

Particular case: Let $F_m = F_n$. In this case the equation (4.4) gives the following theorem.

Theorem 4.2. Suppose that there exists a vector field of F_n which satisfies the equation of the type (4.1). Then in order for F_1 to be minimal in F_n it is necessary and sufficient that $g^{\varphi\theta}\overset{\circ}{D}_{\varphi}\overset{\circ}{D}_{\theta}V^{\alpha} = 0$.

REFERENCES

[1] C.E. Weatherburn: Riemannian geometry and the tensor calculus. Cambridge 1966.

[2] H. Rund: The differential geometry of Finsler spaces. Berlin 1959.

[3] B.Y. Chen, K. Yano: On submanifolds of a Riemannian manifold, J.Math.Soc.Japan 23 (1971) 548-554.

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