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APPROXIMATION OF SOLUTIONS OF THE SYSTEM OF EQUATIONS x'=Ax+b IN THE SPACE $L^2(0,\infty)$

1. Introduction

We are going to consider the system of linear differential equations with constant coefficients

(1.1) $$x' = Ax + b$$

under the initial conditions

(1.2) $$x(0) = x_0,$$

where

$A = [a_{ij}] \quad i,j = 1,\dots,k$

$b = [b_1(t),\dots,b_k(t)]^T$

$x = [x_1(t),\dots,x_k(t)]^T$

$x_0 = [x_{10},\dots,x_{k0}]^T.$

The solution of the problem (1.1), (1.2) may be written ([1], [2]) in the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-u)}b(u)du.$$

This requires, however, a rather troublesome calculation of the matrix e^{At}. The essential difficulty consists here in the cumulation of the rounding errors.

In this paper we present a method of approximate solution in the sense of the metric of the space $L^2(0,\infty)$ by means of orthonormal exponential polynomials ([3], [4]).

2. Approximation in the space $L^2(-i\infty, i\infty)$

In paper [4] we have considered the space $L^2(-i\infty,i\infty)$ with the scalar product

$$(2.1) \qquad (F,G) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(s)\overline{G(-\bar{s})}ds$$

and the norm

$$(2.2) \qquad \|F\|_s = \sqrt{(F,F)}$$

as well as one of the possible orthonormal bases, i.e. the set of functions

$$(2.3) \qquad \begin{cases} U_1(s) = A_1 \cdot \dfrac{1}{s-s_1} \\ U_n(s) = A_n \cdot \dfrac{\prod_{l=1}^{n-1}(s+\bar{s}_l)}{\prod_{l=1}^{n}(s-s_l)}, \quad n = 2,3,\dots \end{cases}$$

where

$$A_n = \sqrt{-2 \operatorname{Re} s_n},$$

under the assumption that the poles s_n satisfy the conditions

$$(2.4)\qquad \begin{cases} s_n \neq s_m \quad \text{if} \quad n \neq m \\ \operatorname{Re} s_n < 0 \\ \displaystyle\sum_{n=1}^{\infty} \frac{-\operatorname{Re} s_n}{1 + \left| s_n + \frac{1}{2} \right|^2} = \infty . \end{cases}$$

Every function $F(s)$ analytic in the half-plane $\operatorname{Re} s \geqslant 0$, which is an element of the space $L^2(-i\infty, i\infty)$, may be represented in the form

$$(2.5)\qquad F(s) = f_1 U_1(s) + \dots + f_N U_N(s) + E_N(s)$$

with the Fourier coefficients

$$f_n = (F, U_n)$$

and the approximation error $E_N(s)$ satisfying the condition

$$\|E_N\|_s \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty .$$

In paper [5] the following formulas of algebraic character have been established for the Fourier coefficients

$$(2.6)\qquad \begin{cases} f_1 = A_1 F(-\bar{s}_1) \\ f_n = A_n \displaystyle\sum_{m=1}^{n} F(-\bar{s}_m) \cdot \frac{\prod_{l=1}^{n-1} (\bar{s}_m + s_l)}{\prod_{\substack{l=1 \\ l \neq m}}^{n} (\bar{s}_m - \bar{s}_l)} , \qquad n=2,3,\dots \end{cases}$$

3. Approximation of the solution of a system of differential equations

The problem (1.1), (1.2) can be solved by means of the Laplace transformation [6]. Introducing the transforms

$$X(s) = \mathcal{L}[x(t)], \quad B(s) = \mathcal{L}[b(t)]$$

we get

$$X(s) = [s\,I - A]^{-1}[x_0 + B(s)]. \tag{3.1}$$

Finding the exact solution by means of formula (3.1) may be an onerous task because of the difficulties connected with the calculation of the inversed functional matrix. This obstacle can be circumvented if we content ourselves with an approximate solution.

To each of the components $X_j(s)$ we shall apply the results established in section 2. For this purpose we shall calculate the vectors

$$X(-\bar{s}_m) = [-\bar{s}_m\,I - A]^{-1}[x_0 + B(s)], \quad m=1,\dots,N. \tag{3.2}$$

To do it we have, it is true, to find the matrix inverse to $-s_m I - A$, but this is indeed a much easier task than to inverse the functional matrix $s\,I - A$.

By means of the vectors (3.2) we find the vectors of the Fourier coefficients

$$\begin{cases} x_1 = A_1 X(-\bar{s}_1) \\[2ex] x_n = A_n \sum\limits_{m=1}^{n} X(-\bar{s}_m) \cdot \dfrac{\prod\limits_{l=1}^{n-1} (\bar{s}_m + s_l)}{\prod\limits_{\substack{l=1 \\ l\neq m}}^{n} (\bar{s}_m - \bar{s}_l)} \end{cases} . \tag{3.3}$$

T h e o r e m 3.1. If
1. the eigenvalues of the matrix A lie inside the half-plane $\operatorname{Re} s < 0$
2. the components of the vector-function $b(t)$ are continuous and are elements of the space $L^2(0,\infty)$
3. the numbers s_n satisfy conditions (2.4),
then the Laplace-transform of the solution of the problem (1.1), (1.2) can be written in the form

$$X(s) = x_1 U_1(s) + \dots + x_N U_N(s) + E_N(s), \tag{3.4}$$

where the coefficients x_n are expressed by formulas (3.3) and the approximation error $E_N(s)$ satisfies the condition

$$\|E_{jN}\|_s \longrightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \quad j = 1,\dots,k. \tag{3.5}$$

P r o o f . By assumptions 1 and 2 there exists a solution $x(t) \in L^2(0,\infty)$ of the problem (1.1), (1.2) and its transform $X(s) \in L^2(-i\infty, i\infty)$. From assumptions 1 and 3 it follows that it is possible to determine the vectors $X(-s_m)$ by means of formula (3.2) as well as the Fourier coefficients x_n (3.3). Furthermore, assumption 3 enables us to construct the orthonormal base $U_1(s)$, $U_2(s)$,... in the space $L^2(-i\infty, i\infty)$. This, of course, implies that equality (3.4) and condition (3.5) hold and the proof is thus completed.

To the base $U_1(s)$, $U_2(s)$,... in the space $L^2(-i\infty, i\infty)$ corresponds a base $u_1(t)$, $u_2(t)$,... in the space $L^2(0,\infty)$, where

$$u_n(t) = \mathcal{L}^{-1}[U_n(s)]$$

are orthonormal exponential polynomials ([3], [4]).

T h e o r e m 3.2. Under the assumptions of Theorem 3.1 the solution of the problem (1.1), (1.2) may be written in the form

$$x(t) = x_1 u_1(t) + \dots + x_N u_N(t) + e_N(t), \tag{3.6}$$

where the coefficients x_n are given by formulas (3.3) and the approximation error $e_N(t)$ satisfies the condition

$$\|e_{jN}\|_t \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty, \quad j = 1,\dots,k. \tag{3.7}$$

P r o o f . Equality (3.6) follows from equality (3.4) by means of the inverse Laplace-transform, whereas condition (3.7) follows from Parseval's formula

$$\|f\|_t = \|F\|_s,$$

where $f(t) = \mathcal{L}^{-1}[F(s)]$, q.e.d.

4. Final remarks

Formula (3.6) enables us to find the approximate solution of the problem (1.1), (1.2) in form of the partial sum of the orthonormal series

$$x(t) \approx x_1 u_1(t) + \dots + x_N u_N(t).$$

According to Parseval's formula the approximation error $e_N(t)$ satisfies conditions

$$\|e_{jN}\|_t^2 = \|x_j\|_t^2 - |x_{j1}|^2 - \dots - |x_{jN}|^2.$$

The norm of the solution $\|x_j\|_t$ is, however, unknown, so that the approximation error cannot be determined by means of this latter condition; nevertheless, the order of magnitude of this error may be estimated from the variation of decimal digits in the sum of squares of the Fourier coefficients

$$|x_{j1}|^2 + \dots + |x_{jN}|^2.$$

This error depends on the number N and, of course, on the choice of the numbers s_n used in constructing the base

$$U_1(s),\ U_2(s),\ \dots$$

In order to simplify the calculations it would be convenient to put, say,

$$s_n = -n.$$

On the other hand, one may expect that a suitable choice of the numbers s_n would lead to the desired order of the approximation error with a lesser number N. The numbers s_n may, for example, be taken as a decreasing arithmetic sequence whose first term would be the real part of the eigenvalue of the matrix A lying next to the imaginary axis.

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