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ON SOME CLASS OF SYSTEMS OF NONLINEAR PROCESSES WITH A MEMORY

Introduction

In [1] we have introduced the notion of a CGFk-process as a continuous function $x: \langle 0;+\infty) \to \mathbb{R}$ satisfying for all $t \geqslant k$ the condition

$$x(t) = G\left(t, \int_0^k F(s,t,x(t-k+s))ds\right), \qquad k > 0$$

where $G: \langle k;+\infty)\times\mathbb{R} \to \mathbb{R}$ and $F: \langle 0;k\rangle \times \langle k;+\infty)\times\mathbb{R} \to \mathbb{R}$ are given and continuous functions, satisfying some conditions of Lipschitz type. CGFk-processes seem to be useful in describing some real, generally nonlinear processes investigated in continuous time and which are characterized by a "memory" whose length is k time units. If $G(t,v) = v$ and $F(s,t,u) = \alpha(s)\cdot u$, where $\alpha:\mathbb{R} \to \mathbb{R}$ is a non-zero polynomial, then CGFk-process is a (α,k)-computation introduced in [2]. Such processes occur in many technical, economical and biological problems, e.g. in control theory, renewal theory, in the description of cells reproduction [3], [4] etc. Basic properties of (α,k)-computations and CGFk-processes were investigated by Żakowski in [5], [6] and [7].

In this paper we introduce the notion of a CGFk-system of processes (continuous GFk-system of processes) as some systems of n continuous functions satisfyings n integral conditions (4), generally nonlinear. The functions of a CGFk-systems seem to be useful in the description of some continuous processes, generally nonlinear and interrelated. These processes have a "memory" with the length $k_1,\dots,k_n$. We also consider some qualitative properties of CGFk-systems.

1. Basic notations and definitions

Let $\mathbf{R}$ denote the set of all real numbers and n - an arbitrary but fixed natural integer $(n \geqslant 1)$. By $k = [k_1,k_2,\dots,k_n]$ we denote the system of real numbers such that

$$0 < k_1 \leqslant k_2 \leqslant \dots \leqslant k_n < +\infty. \tag{1}$$

Let $\tau_i = k_n - k_i$, $i = 1,2,\dots,n$. We define the sets Δ_j, $j = 1,2,\dots,n$ and Ω as follows

$$\Delta_j = \left\{(s,t,u) \in \mathbf{R}^3 : 0 \leqslant s \leqslant k_j \wedge t \geqslant k_n \wedge u \in \mathbf{R}\right\}$$

and

$$\Omega = \left\{(t,v_1,\dots,v_n) \in \mathbf{R}^{n+1} : t \geqslant k_n \wedge v_i \in \mathbf{R},\quad i=1,2,\dots,n\right\}.$$

Let $F_{ij} : \Delta_j \longrightarrow \mathbf{R}$ and $G_i : \Omega \longrightarrow \mathbf{R}$, $i = 1,2,\dots,n$ be continuous functions. We assume that for every $\delta > k_n$ there exist positive numbers

$$L_{ij}^{(F,\delta)} \quad \text{and} \quad L_i^{(G,\delta)}, \qquad i,j = 1,2,\dots,n,$$

such that for every $s \in \langle 0;k_i\rangle$, $t \geqslant k_n$ and $\tilde{u},\tilde{\tilde{u}} \in \mathbf{R}$ the inequalities

$$(2)\quad |F_{ij}(s,t,\tilde{u})-F_{ij}(s,t,\tilde{\tilde{u}})| \leqslant L_{ij}^{(F,\delta)}\cdot|\tilde{u}-\tilde{\tilde{u}}|, \quad i,j=1,2,\dots,n,$$

hold, and for every $t \geqslant k_n$ and $\tilde{v}_1,\tilde{\tilde{v}}_1,\dots,\tilde{v}_n,\tilde{\tilde{v}}_n \in \mathbf{R}$ the inequalities

$$(3)\quad |G_i(t,\tilde{v}_1,\dots,\tilde{v}_n)-G_i(t,\tilde{\tilde{v}}_1,\dots,\tilde{\tilde{v}}_n)| \leqslant L_i^{(G,\delta)}\cdot\sum_{j=1}^{n}|\tilde{v}_j-\tilde{\tilde{v}}_j|$$

$i=1,2,\dots,n$, hold.

D e f i n i t i o n . The system $[x_1,x_2,\dots,x_n]$ of continuous functions $x_i : \langle\tau_i;+\infty)\to\mathbf{R}$, $i=1,2,\dots,n$, fulfilling for all $t \geqslant k_n$ the conditions

$$(4)\quad x_i(t) = G_i\left(t, \int_0^{k_1} F_{i1}(s,t,x_1(t-k_1+s))ds,\dots \right.$$
$$\left. \dots, \int_0^{k_n} F_{in}(s,t,x_n(t-k_n+s))ds\right)$$

is said to be CGFk-system of processes (shorthy: CGFk-system).

In the case of $n=1$, the CGFk-system is a $CG_1F_{11}k_1$-process introduced in [1]. Any CGFk-system describes some system of real and interrelated processes in continuous time. These processes are generally nonlinear with a "memory" of the finite length $k_1,k_2,\dots,k_n$.

We observe that for every system $[f_1,f_2,\dots,f_n]$ of continuous functions $f_i : \langle\tau_i;+\infty)\to\mathbf{R}$, $i = 1,2,\dots,n$, and for every system $k = [k_1,k_2,\dots,k_n]$ of real numbers such that $k_i = k_n-\tau_i$, $i = 1,2,\dots,n$ satisfying inequalities (1), the system $[f_1,f_2,\dots,f_n]$ is a CGFk-system if, for example, $F_{ij} \equiv 0$ for $i,j = 1,2,\dots,n$ and $G_i(t,v_1,\dots,v_n) = f_i(t)$ for all $t \geqslant k_n$ and $v_i \in \mathbf{R}$, $i = 1,2,\dots,n$.

The dependence of the functions G_i, $i = 1,2,\dots,n$, on the first variable and the dependence of the functions F_{ij},

$i,j = 1,2,\dots,n$, on the second variable describes the fact that CGFk-system is generally time-varying (in particular, controlled). If these functions are fixed and constant in the variable t (i.e. if G_i are constant with respect to the first variable and F_{ij} are constant with respect to the second variable, $i,j = 1,2,\dots,n$), then the values of $x_i(t)$, $i = 1,2,\dots,n$, $t \geqslant k_n$, depend only on the values of the functions $x_1,x_2,\dots,x_n$ on the intervals $\langle t-k_1;t),\langle t-k_2;t),\dots$ $\dots,\langle t-k_n;t)$ respectively. This is expressed by saying that the CGFk-system "remembers" k_n time units baqkwards. We observe, that in this case the values of $x_i(t)$, $i=1,2,\dots,n$, $t \geqslant k_n$, depend on the values of this function only on the interval $\langle t-k_i;t)$. This is expressed by saying that the process x_i "remembers" k_i time units backwards, i.e., that this process have a "memory" with the length k_i.

If $x : \langle a;+\infty) \longrightarrow \mathbb{R}$, then the restriction of x to the set $U \subset \langle a;+\infty)$ is denoted by $x|U$. In particular, if $[x_1,x_2,\dots,x_n]$ is a CGFk-system, then

$$\left[x_1|\langle \tau_1;k_n\rangle,\ x_2|\langle \tau_2;k_n\rangle,\dots,x_n|\langle 0;k_n\rangle\right]$$

is called the initial state of this CGFk-system.

The set of all real and continuous functions on the intervall $\langle a;b\rangle$ we denote by $\mathbf{C}_{\langle a;b\rangle}$.

2. Basic properties of the CGFk-systems

From the conditions (4) it follows, that if $[f_1,f_2,\dots,f_n]$ is the initial state of any CGFk-system then

$$(5)\qquad f_i(k_n) = $$

$$= G_i\left(k_n, \int_0^{k_1} F_{i1}(s,k_n,f_1(\tau_1+s))ds,\dots,\int_0^{k_n} F_{in}(s,k_n,f_n(s))ds\right)$$

$$i=1,2,\dots,n.$$

T h e o r e m 1. If the functions $f_i \in \mathbf{C}_{\langle \tau_i;k_n \rangle}$, $i = 1,2,\dots,n$, satisfy the conditions (5) then there exists exactly one CGFk-system $[x_1,x_2,\dots,x_n]$ such that $x_i \mid \langle \tau_i;k_n \rangle = f_i$ for $i = 1,2,\dots,n$. This CGFk-system is a limit of the sequence

$$(6)\qquad \left[x_1^{(m)},x_2^{(m)},\dots,x_n^{(m)}\right], \quad m = 0,1,2,\dots$$

of succesive approximations, defined as follows

$$(7)\qquad x_i^{(0)}(t) = \begin{cases} f_i(t) & \text{for } t \in \langle \tau_i;k_n \rangle \\ f_i(k_n) & \text{for } t > k_n, \end{cases}$$

$i = 1,2,\dots,n$, and

$$(8)\qquad x_i^{(m)}(t) = \begin{cases} f_i(t) & \text{for } t \in \langle \tau_i;k_n \rangle \\ G_i\Big(t, \int_0^{k_1} F_{i1}\big(s,t,x_1^{(m-1)}(t-k_1+s)\big)ds,\dots \\ \dots, \int_0^{k_n} F_{in}\big(s,t,x_n^{(m-1)}(t-k_n+s)\big)ds\Big) & \text{for } t>k_n, \end{cases}$$

$i = 1,2,\dots,n$ and $m = 1,2,\dots$. For every $i = 1,2,\dots,n$, the sequence $(x_i^{(m)})$, $m = 1,2,\dots$, is almost uniformly convergent on the interval $\langle \tau_i;+\infty)$.

P r o o f . Let δ denotes an arbitrary positive number greater then k_n. We define the metric space

$$(9)\qquad \mathbf{C}_f^{(k,\delta)} = \Big\{[x_1,x_2,\dots,x_n] : x_i \in \mathbf{C}_{\langle \tau_i;\delta \rangle} \wedge x_i \mid \langle \tau_i;k_n \rangle = f_i, \quad i=1,2,\dots,n\Big\}$$

with the metric

$$(10)\qquad \varrho(\tilde{x},\tilde{\tilde{x}}) = \max_{1\leq i\leq n}\ \sup_{\langle\tau_i;\delta\rangle}\left(e^{\lambda t}\cdot\left|\tilde{x}_i(t)-\tilde{\tilde{x}}_i(t)\right|\right)$$

where λ is a negative number such that

$$(11)\qquad \max_{1\leq i\leq n}\left(L_i^{(G,\delta)}\cdot\sum_{j=1}^{n} k_j\cdot L_{ij}^{(F,\delta)}\right)\frac{e^{\lambda k_1}-1}{\lambda k_1} < 1.$$

The space (9) is complete. On this space we define an operator A as follows

$$A\left([x_1,x_2,\ldots,x_n]\right) = [A_1(x),A_2(x),\ldots,A_n(x)],$$

where $x = [x_1,x_2,\ldots,x_n]$ and

$$(12)\qquad A_i[x](t) = \begin{cases} f_i(t) & \text{for}\quad t\in\langle\tau_i;k_n\rangle \\ G_i\left(t,\int_0^{k_1} F_{i1}(s,t,x_1(t-k_1+s))ds,\ldots\right. & \\ \left.\ldots,\int_0^{k_n} F_{in}(s,t,x_n(t-k_n+s))ds\right) & \text{for}\ t\in(k_n;\delta\rangle, \end{cases}$$

$i = 1,\ldots,n$. On the basis of (8) we observe that the operator A transformes the space (9) into itself. In view of (12), (2) and (3) we have for every $t\in(k_n;\delta\rangle$ and for every $\tilde{x},\tilde{\tilde{x}}\in C_f^{(k,\delta)}$:

$$e^{\lambda t}\cdot\left|A_i[\tilde{x}](t) - A_i[\tilde{\tilde{x}}](t)\right| \leqslant$$

$$\leqslant e^{\lambda t}\cdot L_i^{(G,\delta)}\cdot\sum_{j=1}^{n}\int_0^{k_j} L_{ij}^{(F,\delta)}\left|\tilde{x}_j(t-k_j+s) - \tilde{\tilde{x}}_j(t-k_j+s)\right| ds =$$

$$= L_i^{(G,\delta)}\cdot\sum_{j=1}^{n} L_{ij}^{(F,\delta)}\int_0^{k_j} e^{\lambda(k_j-s)} e^{\lambda(t-k_j+s)}\left|\tilde{x}_j(t-k_j+s) - \tilde{\tilde{x}}_j(t-k_j+s)\right| ds.$$

$i = 1,2,\dots,n$. From the above and basing on (10) and (12) we have for every $t \in \langle \tau_i ; \delta \rangle$

$$e^{\lambda t}\cdot\left|A_i[\tilde{x}](t) - A_i[\tilde{\tilde{x}}](t)\right| \leqslant$$

$$\leqslant L_i^{(G,\delta)}\cdot\varrho(\tilde{x},\tilde{\tilde{x}})\cdot\sum_{j=1}^{n} L_{ij}^{(F,\delta)}\,\frac{e^{\lambda k_j} - 1}{\lambda k_j}\cdot k_j \leqslant$$

$$\leqslant \frac{e^{\lambda k_1} - 1}{\lambda k_1}\cdot L_i^{(G,\delta)}\cdot\varrho(\tilde{x},\tilde{\tilde{x}})\cdot\sum_{j=1}^{n} k_j\cdot L_{ij}^{(F,\delta)},$$

$i = 1,2,\dots,n$. This implies that for every $\tilde{x},\tilde{\tilde{x}} \in \mathbb{C}_f^{(k,\delta)}$

$$\varrho\left(A[\tilde{x}],A[\tilde{\tilde{x}}]\right) \leqslant \max_{1\leqslant i\leqslant n}\left(L_i^{(G,\delta)}\cdot\sum_{j=1}^{n} k_j\cdot L_{ij}^{(F,\delta)}\right)\frac{e^{\lambda k_1}-1}{\lambda k_1}\cdot\varrho(\tilde{x},\tilde{\tilde{x}}).$$

From this and from (11), applying the Banach fixed point theorem, it follows that there exists exactly one system of functions $[x_1^*,x_2^*,\dots,x_n^*] \in \mathbb{C}_f^{(k,\delta)}$ such that

$$A\left([x_1^*,x_2^*,\dots,x_n^*]\right) = [x_1^*,x_2^*,\dots,x_n^*].$$

Moreover, this system is a limit of the sequence (6), defined by (7) and (8). From (10) it follows that for every $i = 1,2,\dots,n$, the sequence $(x_i^{(m)})$, $m = 1,2,\dots,$ is uniformly convergent on the interval $\langle \tau_i;\delta\rangle$. Because δ denotes an arbitrary positive number greater then k_n, the proof is complete.

We observe, that the exponential factor in definition (10) gives that the numbers

$$L_i^{(G,\delta)} \quad \text{and} \quad L_{ij}^{(F,\delta)}, \qquad i,j = 1,2,\dots,n$$

in Lipschitz's conditions (2) and (3) may be arbitrarily large.

T h e o r e m 2. If $[x_1,x_2,\dots,x_n]$ is an CGFk-system, and moreover, if

1^o there exist continuous functions

$$M_{ij}^{(F)} : \left\{ (s,t) \in R^2 : s \in \langle \tau_j;k_j\rangle \wedge t \geqslant k_n \right\} \to \langle 0;+\infty)$$

$i,j = 1,2,\dots,n$, such that for every $(s,t,u) \in \Delta_{ij}$ the conditions

$$(13) \qquad |F_{ij}(s,t,u)| \leqslant M_{ij}^{(F)}(s,t)\cdot|u|$$

$i,j = 1,2,\dots,n$, hold

2^o there exist functions $M_i^{(G)}:\langle k_n;+\infty) \to \langle 0;+\infty)$, $i = 1,2,\dots,n$, such that for every $(t,v_1,\dots,v_n) \in \Omega$ the conditions

$$(14)\qquad |G_i(t,v_1,\dots,v_n)| \leqslant M_i^{(G)}(t)\cdot\sum_{j=1}^{n}|v_j|$$

$i = 1,2,\dots,n,$ hold

3^0 for every $t \geqslant k_n$ the conditions

$$(15)\qquad M_i^{(G)}(t)\cdot\sum_{j=1}^{n}\int_0^{k_j} M_{ij}^{(F)}(s,t)\,ds \leqslant 1$$

$i = 1,2,\dots,n,$ hold, then there exists an integer r, $1 \leqslant r \leqslant n$, and a number $c_r \in \langle \tau_r ; k_n \rangle$ such that for every $i = 1,2,\dots,n,$ and $t \geqslant \tau_i$

$$(16)\qquad |x_i(t)| \leqslant |x_r(c_r)|.$$

P r o o f . The CGFk-system $[x_1,x_2,\dots,x_n]$ is a limit of the sequence (6), defined by equalities (7) and (8), where $f_i = x_i | \langle \tau_i ; k_n \rangle$, $i = 1,2,\dots,n.$ Let

$$M = \max_{1 \leqslant i \leqslant n} \sup_{\langle \tau_i ; k_n \rangle} |x_i(t)|.$$

On the basis of (13) and (14) we have for every $t \geqslant k_n$, $i = 1,2,\dots,n,$ and $m = 1,2,\dots$

$$(17)\qquad |x_i^{(m)}(t)| \leqslant M_i^{(G)}(t)\cdot\sum_{j=1}^{n}\int_0^{k_j} M_{ij}^{(F)}(s,t)\,|x_j^{(m-1)}(t-k_j+s)|\,ds.$$

We observe that $|x_i^{(0)}(t)| \leqslant M$ for every $t \geqslant \tau_i$, $i=1,2,\dots,n.$ From this, on the basis of inequalities (15) and (17), applying mathematical induction, we get

$$(18) \qquad |x_i^{(m)}(t)| \leqslant M$$

for every $t \geqslant \tau_1$, $i = 1,2,\dots,n$, and $m = 0,1,2,\dots$. Passing in the inequalities (18) to the limit with $m \to +\infty$ we get

$$|x_i(t)| \leqslant M$$

for every $t \geqslant \tau_1$, $i = 1,2,\dots,n$. On the other hand, there exists an integer r, $1 \leqslant r \leqslant n$, such that $M = \sup_{\langle \tau_r;k_n \rangle} |x_r(t)|$. Obviously, there exists a number $c_r \in \langle \tau_r;k_n \rangle$ such that $|x_r(c_r)| = \sup_{\langle \tau_r;k_n \rangle} |x_r(t)|$, so we get inequality (16).

Corollary 1. If the hypotheses of the Theorem 2 hold, then any CGFk-system $[x_1,x_2,\dots,x_n]$ is bounded, i.e. there exists the positive numbers $M_1,M_2,\dots,M_n$ such that

$$|x_i(t)| \leqslant M_i$$

for every $t \geqslant \tau_1$, $i = 1,2,\dots,n$.

Theorem 3. If $[x_1,x_2,\dots,x_n]$ is an CGFk-system and the hypotheses 1° and 2° of Theorem 2 hold, and $k_i = k_0$ for $i = 1,2,\dots,n$, and there exists a number $t_0 \geqslant 0$ for which the functions $x_i \,|\, \langle t_0;t_0+k_0 \rangle$, $i = 1,2,\dots,n$, are nonnegative (nonpositive) and nonzero and, moreover

$$(19) \qquad M_i^{(G)}(t_0+k_0) \cdot \sum_{j=1}^{n} \int_0^{k_0} M_{ij}^{(F)}(s,t_0+k_0)\,ds < 1$$

$i = 1,2,\dots,n$, then there exists an integer r, $1 \leqslant r \leqslant n$, such that the function $x_r \,|\, \langle t_0;t_0+k_0 \rangle$ is not nondecreasing (or not nonincreasing, respectively).

Proof. If the functions $x_i \,|\, \langle t_0;t_0+k_0 \rangle$, $i = 1,2,\dots,n$, are nonnegative and nonzero, then in view of (4), (13) and (14) we have

$$(20)\quad x_0(t_0+k_0) \leqslant M_i^{(G)}(t_0+k_0)\cdot\sum_{j=1}^{n}\int_0^{k_0} M_{ij}^{(F)}(s,t_0+k_0)x_j(t_0+s)ds$$

$i = 1,2,\ldots,n$. Hence we get

$$(21)\quad x_i(t_0+k_0) \leqslant$$

$$\leqslant \max_{1\leqslant j\leqslant n}\ \sup_{\langle t_0;t_0+k_0\rangle} x_j\ M_i^{(G)}(t_0+k_0)\cdot\sum_{j=1}^{n}\int_0^{k_0} M_{ij}^{(F)}(s,t_0+k_0)ds$$

$i = 1,2,\ldots,n$. If there exists an integer r, $1\leqslant r\leqslant n$, such that

$$M_r^{(G)}(t_0+k_0)\cdot\sum_{j=1}^{n}\int_0^{k_0} M_{rj}^{(F)}(s,t_0+k_0)ds = 0$$

then by (21) we have $x_r(t_0+k_0) = 0$. Since the function $x_r \big| \langle t_0;t_0+k_0\rangle$ is nonnegative and nonzero, this implies that this function is not nondecreasing. If

$$0 < M_i^{(G)}(t_0+k_0)\cdot\sum_{j=1}^{n}\int_0^{k_0} M_{ij}^{(F)}(s,t_0+k_0)ds < 1$$

$i = 1,2,\ldots,n$, we have, on the basis of (19) and (21), the inequalities

$$(22)\quad x_i(t_0+k_0) < \max_{1\leqslant j\leqslant n}\ \sup_{\langle t_0;t_0+k_0\rangle} x_j$$

for $i = 1,2,\ldots,n$. On the other hand, there exist integers r, $1\leqslant r\leqslant n$ such that

(23) $$\sup_{\langle t_0;t_0+k_0\rangle} x_r = \max_{1\leqslant j\leqslant n} \sup_{\langle t_0;t_0+k_0\rangle} x_j .$$

From (23) and (22), by substitution $i = r$, we get

$$x_r(t_0+k_0) < \sup_{\langle t_0;t_0+k_0\rangle} x_r$$

which completes the proof for the case of nonnegative and nonzero functions $x_i \,|\, \langle t_0;t_0+k_0\rangle$, $i = 1,2,\ldots,n$. In the case when this functions are nonpositive and nonzero the proof is analogous.

Theorems 1, 2 and 3 generalize some analogous theorems of the papers [1] and [7].

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