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AN INTRINSIC ISOSCELES FOUR POINT PROPERTY WHICH CHARACTERIZES HYPERBOLIC AND EUCLIDEAN SPACES

1. Introduction

In 1932, W.A.Wilson [9] showed that a complete, metrically and externally convex metric space is congruent to a generalized euclidean space if and only if each four points of the space are congruent to four points of euclidean space (see [1] for definitions and notation). Subsequent characterizations of euclidean spaces by Blumenthal, Day [3] and others have assumed weaker forms of the Wilson four point property which consider only quadruples containing a linear triple of points. (For a more complete summary of these and other subsequent characterizations see [2]).

Another class of four point properties, the isosceles euclidean four point properties, have been studied by Freese [4] and Valentine [5]. In these properties it is assumed only that quadruples containing an isosceles triple, e.g., a triple p, q, r with $pq = pr$, are embeddable in euclidean space.

In Valentine and Wayment [7] and Valentine and Andalafte [6] the concept of an intrinsic four point property is introduced. For example, a metric space M satisfies the intrinsic feeble four point property provided any congruence between triples of points p, q, r and p', q', r' of M is extendable to a congruence between p, q, r, m and p', q', r', m', where m and m' are the respective midpoints of q and r, q' and r'. In [6] it is shown that a finitely compact, convex, externally convex metric space which satisfies the two triple property is euclidean or hyperbolic if and only if it satisfies the intrinsic feeble four point property. It is natural to ask whether an intrinsic isosceles four point property suffices to characterize such spaces. In the present paper an affirmative answer is provided. Following the terminology of [6] we will say that a metric space satisfies the intrinsic isosceles weak four point property if for every pair of triples p, q, r, and p', q', r' of M, if $pq = pr$ and if $p, q, r \approx p', q', r'$ this congruence can be extended to a congruence $\{p\} \cup L(q, r) \approx \{p'\} \cup L(q', r')$. The following result is then obtained.

T h e o r e m . Let M be a finitely compact, metrically convex, externally convex metric space which satisfies the two triple property. The space M is euclidean or hyperbolic if and only if it satisfies the intrinsic isosceles weak four point property.

2. The characterization theorem

In the following, let M denote a finitely compact, metrically convex, externally convex metric space which satisfies the two triple property, and which satisfies the intrinsic isosceles weak four point property:

(*) If p, q, r, p', q', r' are points of M with $pq = pr$, and if $p, q, r \approx p', q', r'$ then this congruence can be extended to a congruence $\{p\} \cup L(q, r) \approx \{p'\} \cup L(q', r')$.

L e m m a 1. Let p, q, r be noncollinear points of M with $pq = pr$, and let m denote the midpoint of q and r. If s is a point of $L(q,r)$ and s^* is the reflection of s in m (the unique point of $L(q,r)$ such that sms^* holds and $sm = ms^*$) then $ps = ps^*$.

P r o o f . Since s^* is the reflection of s in m, it follows easily that $qs = rs^*$ and $qs^* = rs$. But $p, q, r \approx p, r, q$, so by Property (*) $p, q, r, s \approx p, r, q, s^*$ and the result follows.

L e m m a 2. The foot of a point p on a line L is unique, and if $q, r \in L$ with $pq = pr$, the foot of p on L is the midpoint of q and r.

P r o o f . Suppose $p \notin L$, and suppose f and f' are distinct feet of p on L. Since the distances px, $x \in L$ are not bounded above, the set of feet of p on L is a bounded subset of L and we may assume that f and f' are respectively the leftmost and rightmost feet of p on L. Thus if $ff't$ or $f'ft$, then t is not a foot of p on L.

Case 1: There is a foot $\bar{f}$ of p between f and f'. Then $pf = pf' = p\bar{f}$ and since $p, f, \bar{f} \approx p, \bar{f}, f$, we have by Property (*) $p, f, \bar{f}, f' \approx p, \bar{f}, f, t$ where t satisfies $\bar{f}ft$ and $ft = \bar{f}f'$. But then $pt = pf' = pf$ and t is a foot of p, contrary to the choice of f and f'.

Case 2. There is no foot of p between f and f'. In particular, $pm > pf$, where m is the midpoint of f and f', and by continuity of the metric, there is a point m^* of L such that the betweenness mfm^* holds and $pm^* = pm$. Now $p, m^*, m \approx p, m, m^*$ and by Property (*), $p, m^*, m, f' \approx p, m, m^*, t$ where t is the point of L which satisfies mm^*t and $m^*t = mf'$, and hence satisfies $f'ft$. But again $pt = pf' = pf$ and t is a foot of p on L, contrary to the choice of f and f'.

Thus in each case we reach a contradiction, so the foot of p on L is unique. To complete the proof, suppose $q, r \in L$ with $pq = pr$. Suppose the foot f of p on L is not the midpoint m of q and r. Then by Lemma 1 the reflection f^* of f in m is a second foot of p on L, a contradiction. Thus $f = m$, and the proof is complete.

L e m m a 3. (Monotone Property). If point $p \notin L$ has foot f on L, the distance px is monotone increasing as x recedes from f on either half line of L determined by f on L.

P r o o f . If not, by continuity of the metric, there exist points $q, r \in L$ with fqr holding and $pq = pr$. But then by Lemma 2, the midpoint m of q and r is the unique foot of p on L, contrary to fpq. This completes the proof.

L e m m a 4. Let p, q, r be noncollinear points of M. If $qr \geq pq$ and $qr \geq pr$ then the foot f of p on $L(q, r)$ satisfies qfr.

P r o o f . Since the conclusion follows from Lemma 2 if $pq = pr$, without loss of generality, we may assume $pq > pr$. Then if qfr does not hold, we have by Lemma 3 the relation frq or $f = r$. Let q^* denote the reflection of q in f. Then from Lemma 2 it follows that $pq = pq^*$ and since $qf \geq qr$ we have $qq^* = 2qf \geq 2qr \geq 2pq = qp + pq^*$, contrary to the trinagle inequality, since p, q, q^* are not linear. Thus qfr holds, and the proof is complete.

L e m m a 5. Let p, q, r and p', q', r' be noncollinear triples of points of M. If $p, q, r \approx p', q', r'$ and $qr \geq pq$, $qr \geq pr$, then the congruence $p, q, r \approx p', q', r'$ can be extended to $\{p\} \cup L(q, r) \approx \{p'\} \cup L(q', r')$.

P r o o f . We may assume that $pq > pr$, since the result is immediate if $pq = pr$. If f, f' denote the feet of p, p' on $L(q, r)$, $L(q', r')$ respectively, we have by Lemma 4 the betweennesses qfr and $q'f'r'$. We will show that $qf = q'f'$. If not, we may assume without loss of generality that $qf < q'f'$. Then let $\bar{q}$ satisfy $\bar{q}qf$ and $\bar{q}f = q'f'$. Now $p'q' > q'f'$, for otherwise, if q'^* denotes the reflection of q' in f', we would have $p'q'^* + p'q' = 2p'q' \leq 2q'f' = q'q'^*$, a contradiction. Thus $p\bar{q} > pq = p'q' > q'f' = \bar{q}f$, and there exists, by continuity of the metric, a point $\bar{p}$ satisfying $p\bar{p}f$ and $\bar{p}\bar{q} = pq = p'q'$. Clearly f is the foot of $\bar{p}$ on $L(q, r)$, and if $\bar{q}^*$ denotes the reflection of $\bar{q}$ in f, we have $\bar{p}\bar{q}^* = \bar{p}\bar{q} = p'q' = p'q'^*$, and $\bar{q}\bar{q}^* = 2\bar{q}f = 2q'f' = q'q'^*$ and by Property (*) the congruence $\bar{p}, \bar{q}, \bar{q}^* \approx p', q', q'^*$ can be extended to $\{\bar{p}\} \cup L(\bar{q}, \bar{q}^*) \approx \{p'\} \cup L(q', q'^*)$. Clearly $\bar{p}r < pr$, for otherwise the foot f_r of r on $L(p, f)$ would satisfy $f\bar{p}f_r$ and a contradiction would result as before by observing that $rf > r\bar{p}$ and if r^* is the reflection of r in f, $\bar{p}r^* = \bar{p}r$ and $r\bar{p} + \bar{p}r^* < rr^*$.

Thus the unique point $\bar{r}$ which satisfies $\bar{q}f\bar{r}$ and $p\bar{r} = pr$ also satisfies $fr\bar{r}$. But under the congruence $\{\bar{p}\} \cup L(\bar{q}, \bar{q}^*) \approx \{p'\} \cup L(q', q'^*)$ we have $\bar{p}, \bar{q}, \bar{r} \approx p', r'$ so $\bar{q}\bar{r} = q'r' = qr$, contrary to $\bar{q}qf$ and $fr\bar{r}$. A similar contradiction is obtained if $qf > q'f'$. Thus $qf = q'f'$, $rf = r'f'$, and as above, if q^* denotes the reflection of q in f, we have $\{p\} \cup L(q, q^*) \approx \{p'\} \cup L(q', q'^*)$, and since points r and r' correspond under this congruence we have extended $p, q, r \approx p', q', r'$ to $\{p\} \cup L(q, r) \approx \{p'\} \cup \cup L(q', r')$ as desired.

L e m m a 6. The space M satisfies the intrinsic feeble four point property.

P r o o f . Since the property is clearly satisfied of p, q, r are linear, suppose p, q, r and p', q', r' are two noncollinear triples of points of M with $p, q, r \approx \approx p', q', r'$ and let m and m' denote the respective midpoints of q and r, q' and r'. We must show $p, q, r, m \approx \approx p', q', r', m'$. This follows from Lemma 5 if $qr \geqslant pq$ and $qr \geqslant pr$, and is immediate by Property (*) if $pq = pr$. We may therefore assume labelling so that $pq > pr$ and $pq > qr$. Then by Lemma 5 the congruence $r, p, q \approx r', p', q'$ can be extended to $\{r\} \cup L(p, q) \approx \{r'\} \cup L(p', q')$. Since $pq > pr$, by continuity of the metric there is a point $\bar{p}$ satisfying $q\bar{p}p$ and $\bar{p}q = \bar{p}r$. If $\bar{p}'$ denotes the corresponding point under the above congruence, we have $q'\bar{p}'p'$ and $\bar{p}, q, r \approx \approx \bar{p}', q', r'$ and by Property (*) this congruence can be extended to $\{\bar{p}\} \cup L(q, r) \approx \{\bar{p}'\} \cup L(q', r')$. In particular $\bar{p}m = \bar{p}'m'$. But $\bar{p}q > \bar{p}m$ since m is the foot of $\bar{p}$ on $L(q, r)$, and $\bar{p}q > qm$, for otherwise $qr = qm + mr \geqslant q\bar{p} + \bar{p}r$, a contradiction. Thus the hypotheses of Lemma 5 are satisfied and the congruence $m, \bar{p}, q \approx m', \bar{p}', q'$ can be extended to $\{m\} \cup L(\bar{p}, q) \approx \{m'\} \cup L(\bar{p}', q')$ and since p and p' correspond under this congruence we obtain $pm = p'm'$, and the desired result follows.

The main theorem is now an immediate consequence of the result of Valentine and Andalafte ([6], Theorem 1').

Theorem 1. Let M be a finitely compact, metrically convex, externally convex metric space which satisfies the two triple property. The space M is euclidean or hyperbolic if and only if it satisfies the intrinsic isosceles weak four point property.

3. Concluding remarks

It is not known whether the four point property in the hypothesis may be replaced by a weaker four point property, such as an intrinsic isosceles feeble or intrinsic external isosceles feeble four point property analogous to those used in [4] or [5]. The intrinsic isosceles weak four point property is satisfied by the convexly metrized tripod, the metric space whose points are the points of three closed euclidean half-lines with a common end point p in which distance qr is defined as the euclidean distance between q and r if both points are on the same closed half-line, and $qr = qp + pr$ if q and r are on different half-lines. Thus the hypothesis of the two triple property cannot be removed.

It is also noted that the result of Valentine, Wayment and Andalafte [8] permits an extension of Theorem 1. We will say a metric space M satisfies the intrinsic isosceles weak four point property at a point provided there exists a point $q_0 \in M$ such that for all $p_0, r_0 \in M$ with $p_0q_0 = p_0r_0$, if $p,q,r \approx p_0, q_0, r_0$, then this congruence can be extended to $\{p\} \cup L(q, r) \approx \{p_0\} \cup L(q_0, r_0)$.

Theorem 2. Let M be a finitely compact, metrically convex, externally convex metric space which satisfies the two triple property. The space M is euclidean or hyperbolic if and only if it satisfies the intrinsic isosceles weak four point property at a point.

Proof. It suffices to show that the intrinsic isosceles weak four point property is satisfied in M. Suppose $p, q, r, p', q', r' \in M$ with $p, q, r \approx p', q', r'$. Since the result is clear if p, q, r are linear, we may assume p, q, r are nonlinear. Let q_0 be the point whose existence is guaranteed by the intrinsic isosceles weak four point property at a point. By ([8], Theorem 2.2) there exist $p_0, r_0 \in M$ such that $p, q, r \approx p_0, q_0, r_0$. But then $p', q', r' \approx$ $\approx p_0, q_0, r_0$ and these congruences can be extended to $\{p\} \cup L(q, r) \approx \{p_0\} \cup L(q_0, r_0)$ and $\{p'\} \cup L(q', r') \approx \{p_0\} \cup$ $\cup L(q_0, r_0)$ respectively. But then it is clear that $\{p\} \cup$ $\cup L(q, r) \approx \{p'\} \cup L(q', r')$, so the intrinsic isosceles weak four point property is satisfied, and the conclusion follows by Theorem 1.

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