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ONE-SIDED APPROXIMATION BY ENTIRE FUNCTIONS

1. Preliminaries

Let $L^p(a,b)$, $1 \leqslant p < \infty$, be the space of all measurable complex-valued functions Lebesgue-integrable with p-th power on the interval (a,b). Denote by $L^\infty(a,b)$ the space of all complex-valued functions essentially bounded in (a,b). As usually, the norm of $f \in L^p(a,b)$ is given by the formula

$$\|f\|_{L^p(a,b)} = \|f(\cdot)\|_{L^p(a,b)} = \begin{cases} \left\{\int_a^b |f(x)|^p \, dx\right\}^{\frac{1}{p}} & \text{if} \quad p < \infty, \\ \operatorname*{ess\,sup}_{x \in (a,b)} |f(x)| & \text{if} \quad p = \infty. \end{cases}$$

We shall write L^p instead of $L^p(-\infty,\infty)$.

For functions f belonging to all spaces $L^p(a,b)$ with finite a, b $(a < b)$, the limit

$$\|f\|_p = \|f(\cdot)\|_p = \lim_{\substack{a \to -\infty \\ b \to \infty}} \|f\|_{L^p(a,b)}$$

is finite or infinite. If $f \in L^p$, then

$$\|f\|_p = \|f\|_{L^p}.$$

Suppose that $f \in L^p(a,b)$ for each finite (a,b). Introduce the differences

$$\Delta_h^k f(t) = \sum_{\nu=0}^{k} (-1)^{k-\nu}\binom{k}{\nu} f(t+\nu h) \quad (k = 1,2,\ldots),$$

with real increments h, and the moduli of smoothness

$$\omega_k(\delta;f)_p = \sup_{h\in\langle 0,\delta\rangle} \left\|\Delta_h^k f(\cdot)\right\|_p \quad (0 \leqslant \delta < \infty).$$

These moduli are finite or infinite, $\omega_k(0;f)_p = 0$, and

$$\omega_k(\delta;f)_p \leqslant \omega_k(\lambda;f)_p \quad \text{if} \quad 0 \leqslant \delta \leqslant \lambda < \infty.$$

Let E_σ be the class of all entire functions

$$F(z) = \sum_{k=0}^{\infty} a_k z^k \quad (z = x + iy)$$

of exponential type, of the order σ at most. Consider a complex-valued function f belonging to $L^p(a,b)$ for each finite interval (a,b). Denote by $H_{\sigma,p}(f)$ the set of all functions $G \in E_\sigma$ such that $f - G \in L^p$.

The quantity

$$A_\sigma(f)_p = \begin{cases} \inf\limits_{G\in H_{\sigma,p}(f)} \|f-G\|_p & \text{if } H_{\sigma,p}(f) \text{ is not empty,} \\ \infty & \text{otherwise} \end{cases}$$

is called the best approximation of f by entire functions of class $H_{\sigma,p}(f)$. It is well-known that

$$(1.1)\quad A_\sigma(f)_p \leqslant C(k)\,\omega_k\left(\frac{1}{\sigma}; f\right)_p \quad (0<\sigma<\infty,\ 1\leqslant p\leqslant\infty,\ k=1,2,\ldots),$$

where the positive number $C(k)$ depends only on k (see [1], Sect. 99-105, also [7], p.272-274).

Suppose that f is a real-valued function essentially bounded in each finite interval. Denote by $K^+_{\sigma,p}(f)$ [resp. $K^-_{\sigma,p}(f)$] the set of all entire functions $P\in E_\sigma$ [$Q\in E_\sigma$] real--valued in $(-\infty,\infty)$, such that

$$(1.2)\qquad P(x)\geqslant f(x)\quad [Q(x)\leqslant f(x)]$$

almost everywhere in $(-\infty,\infty)$ and $P-f\in L^p$ [$f-Q\in L^p$]. In the case of f bounded in each finite interval, we also introduce the sets $H^+_{\sigma,p}(f)$, $H^-_{\sigma,p}(f)$ of these $P\in K^+_{\sigma,p}(f)$, $Q\in K^-_{\sigma,p}(f)$, respectively, for which the inequalities (1.2) hold everywhere in $(-\infty,\infty)$.

The useful characteristics of the best one-sided approximation of real-valued functions f belonging to $L^p(a,b)$, bounded or essentially bounded in (a,b), for all finite a,b are given by the formulae

$$\tilde{A}_\sigma(f)_p = \begin{cases} \inf\limits_{P\in H^+_{\sigma,p}(f),\,Q\in H^-_{\sigma,p}(f)} \|P-Q\|_p & \text{if } H^\pm_{\sigma,p}(f) \text{ are not empty,}\\ \infty & \text{otherwise,}\end{cases}$$

$$\hat{A}_\sigma(f)_p = \begin{cases} \inf\limits_{P\in K^+_{\sigma,p}(f),\,Q\in K^-_{\sigma,p}(f)} \|P-Q\|_p & \text{if } K^\pm_{\sigma,p}(f) \text{ are not empty,}\\ \infty & \text{otherwise.}\end{cases}$$

In these cases,

$$\hat{A}_\sigma(f)_p \leqslant \tilde{A}_\sigma(f)_p,\quad A_\sigma(f)_p\leqslant \hat{A}_\sigma(f)_p \quad (0\leqslant\sigma<\infty,\ 1\leqslant p\leqslant\infty),$$

respectively, and

$$\hat{A}_\delta(f)_\infty \leqslant 4A_\delta(f)_\infty \qquad (0 \leqslant \delta < \infty).$$

For continuous functions f, $\tilde{A}_\delta(f)_p = \hat{A}_\delta(f)_p$.

Using some analogues of the moduli of smoothness defined in Sections 2 and 3, we shall give the estimates for $\tilde{A}_\delta(f)_p$ and $\hat{A}_\delta(f)_p$ $(1 \leqslant p < \infty)$ similar to that of (1.1). Moreover, the converse approximation theorems will be presented. They correspond to the recent results concerning one-sided trigonometric approximation announced in [2], [3], [4] and [5].

The symbols C_j, $C_1(k)$, with integer indices, will mean positive constants absolute or depending on the parameter k, only.

2. Modified moduli of smoothness

Consider here a complex-valued function f defined and bounded in each finite interval of $(-\infty, \infty)$. Introduce the intervals

$$I_\delta^k(x) = \left\langle x - k\frac{\delta}{2},\ x + k\frac{\delta}{2} \right\rangle \qquad (0 \leqslant \delta < \infty, \quad k = 1,2,\ldots)$$

and the quantity

$$\omega_k(\delta;x,f) = \sup_{t,\,t+kh \in I_\delta^k(x)} \left|\Delta_h^k f(t)\right| \qquad (-\infty < x < \infty).$$

It can easily be observed that, for every finite $\delta \geqslant 0$ and all real x,

$$(2.1) \qquad \omega_k(\delta;x,f) \leqslant 2\,\omega_{k-1}\left(\frac{k}{k-1}\,\delta;x,f\right) \quad \text{if} \quad k = 2,3,\ldots,$$

$$\omega_1(\delta;x,f) \leqslant 2 \sup_{s \in I_\delta^1(x)} |f(s)|.$$

In particular, the functions

$$(2.2) \qquad \varphi_k(x) = \omega_k(\delta; x, f) \qquad (\delta = \text{const} \geqslant 0,\ k = 1, 2, \ldots)$$

are bounded in finite intervals.

Evidently, for every positive integer k and all real x,

$$\varphi_k(x) = 0 \quad \text{when} \quad \delta = 0.$$

Considering positive δ, we have the following

L e m m a 2.1. (i) Let f be continuous almost everywhere in $(-\infty, \infty)$. Then φ_1 is continuous almost everywhere, too. (ii) If f is a real-valued function in $(-\infty, \infty)$, then φ_1 is measurable in this interval.

P r o o f of (i). Let ε be a positive number. Denote by X the set of all real x such that $x \pm \delta/2$ are the points of continuity of f. Choose an arbitrary $x_0 \in X$ and $\eta \in (-\delta/2, \delta/2)$; write

$$\varrho_\eta = \varphi_1(x_0) - \varphi_1(x_0+\eta) =$$

$$= \sup_{s,t \in I_\delta^1(x_0)} |f(s) - f(t)| - \sup_{u,v \in I_\delta^1(x_0+\eta)} |f(u) - f(v)|.$$

1° If $\varrho_\eta \geqslant 0$, then

$$\varrho_\eta < \varepsilon + \left|f(s_\varepsilon) - f(t_\varepsilon)\right| - \left|f(u) - f(v)\right| \leqslant$$

$$\leqslant \varepsilon + \left|f(s_\varepsilon) - f(t_\varepsilon) - f(u) + f(v)\right|$$

for some $s_\varepsilon, t_\varepsilon \in I_\delta^1(x_0)$ and each $u, v \in I_\delta^1(x_0+\eta)$. Hence

$$\varrho_\eta < \varepsilon + |f(s_\varepsilon) - f(u)| + |f(v) - f(t_\varepsilon)| < 3\varepsilon,$$

provided $|\eta|$ is small enough (u, v are taken correspondingly).

2^0 In the case $\varrho_\eta \leqslant 0$, under the restriction $s,t \in I_\delta^1(x_0)$,

$$-\varrho_\eta \leqslant \sup_{u,v\in I_\delta^1(x_0+\eta)} |f(u)-f(v)| - |f(s)-f(t)|.$$

Therefore

$$-\varrho_\eta < \varepsilon + |f(u_{\varepsilon,\eta}) - f(v_{\varepsilon,\eta}) - f(s)+f(t)|$$

for some $u_{\varepsilon,\eta}$, $v_{\varepsilon,\eta} \in I_\delta^1(x_0+\eta)$ and all $s,t \in I_\delta^1(x_0)$. Consequently,

$$-\varrho_\eta < \varepsilon + |f(u_{\varepsilon,\eta}) - f(s)| + |f(t) - f(v_{\varepsilon,\eta})| < 3\varepsilon,$$

whenever $|\eta|$ is small and s, t are near to $u_{\varepsilon,\eta}$, $v_{\varepsilon,\eta}$, respectively.

In view of 1^0 and 2^0, $\varrho_\eta \to 0$ as $\eta \to 0$, i.e.

$$\lim_{\eta\to 0} \varphi_1(x_0+\eta) = \varphi_1(x_0),$$

and the proof is completed.

P r o o f of (ii). For every real x,

$$\varphi_1(x) = \sup_{s,t\in I_\delta^1(x)} |f(s) - f(t)| =$$

$$= \sup_{s\in I_\delta^1(x)} f(s) - \inf_{t\in I_\delta^1(x)} f(t) \equiv f_1(x) - f_2(x).$$

To prove that φ_1 is measurable in $(-\infty,\infty)$, we introduce the collections

$$X_c = \left\{x : \sup_{s\in I_\delta^1(x)} f(s) \leqslant c\right\} = \left\{x : f(s) \leqslant c \ \forall s \in I_\delta^1(x)\right\},$$

$$Y_c = \left\{s : f(s) \leqslant c, \ -\infty < s < \infty\right\}$$

with real c.

1^0 If Y_c contains no open interval of length $\geqslant \delta$, the set X_c is empty.

2^0 If Y_c contains some separated intervals of length $\geqslant \delta$, X_c can be represented as a finite or enumerable sum of the open or closed intervals degenerated eventually to individual points.

In both these cases, the sets X_c are Lebesgue-measurable. Theorefore, the functions f_1 and $-f_2$ are measurable in $(-\infty,\infty)$. Hence the result follows.

Also, a simple calculation leads to

L e m m a 2.2. (i) If f is continuous in $(-\infty,\infty)$, then φ_k $(k = 1,2,\ldots)$ are continuous everywhere. (ii) Suppose that every open interval (a,b) of length not greater than $k\delta$ contains no more than one discontinuity point of f. Then the set of discontinuity points of φ_k is enumerable at most.

Assuming that φ_k is measurable in $(-\infty,\infty)$, for every fixed $\delta \geqslant 0$, we can define the k-th modified modulus of smoothness of f (in L^p-metric) by the formula

$$\tau_k(\delta;f)_p = \|\varphi_k\|_p = \|\omega_k(\delta;\cdot,f)\|_p \quad (0 \leqslant \delta < \infty).$$

For example, under the assumption on f given in Lemma 2.1 [Lemma 2.2], the modified moduli $\tau_1(\delta;f)_p$ $[\tau_k(\delta;f)_p]$ exist for each $p \geqslant 1$ and $\delta \geqslant 0$.

It is clear that if $0 \leqslant \delta \leqslant \lambda < \infty$, then $\tau_k(\delta;f)_p \leqslant \leqslant \tau_k(\lambda;f)_p$ and $\tau_k(0;f)_p = 0$ always. There are functions f, bounded or unbounded in $(-\infty,\infty)$, such that $\tau_k(\delta;f)_p < \infty$ for some $p \geqslant 1$ and all non-negative δ. But, sometimes,

$$\tau_k(\delta;f)_p = \infty \quad \text{for all positive } \delta.$$

This case has no practical interest.

It may be shown (see [3], p.793-794), that, for measurable f and finite or infinite $p \geqslant 1$,

$$(2.3) \qquad \tau_k(\delta;f)_p \geqslant \omega_k(\delta;f)_p \quad (\delta \geqslant 0, \quad k = 1,2,\ldots).$$

P r o p o s i t i o n 2.3. Under the assumptions $p \geqslant 1$ and $\delta,\lambda \geqslant 0$,

(i) $\quad \tau_k(\delta;f+g)_p \leqslant \tau_k(\delta;f)_p + \tau_k(\delta;g)_p \quad (k \geqslant 1)$,

(ii) $\quad \tau_1(\delta+\lambda;f)_p \leqslant \tau_1(\delta;f)_p + \tau_1(\lambda;f)_p$,

provided that these moduli exist. In particular,

$$\tau_1(n\delta;f)_p \leqslant n\, \tau_1(\delta;f)_p \quad \text{for} \quad n = 2,3,\ldots$$

The inequality (i) follows at once from the identity

$$\Delta_h^k (f+g)(t) = \Delta_h^k f(t) + \Delta_h^k g(t).$$

The proof of (ii) is similar to that of Proposition 3.3 (ii).

P r o p o s i t i o n 2.4. (i) Let f be as in Lemma 2.2 (ii). Then, for every $p \geqslant 1$

$$\tau_k(\delta;f)_p \leqslant 2\tau_{k-1}\left(\frac{k}{k-1}\,\delta;\, f\right)_p \qquad (k \geqslant 2).$$

(ii) Suppose that f possesses the derivative f' continuous in $(-\infty, \infty)$. Then

$$\tau_k(\delta;f)_p \leqslant \delta\tau_{k-1}\left(\frac{k}{k-1}\,\delta;f'\right)_p$$

whenever $p \geqslant 1$, $\delta \geqslant 0$ and $k = 2,3,\ldots$

The assertion of (i) is an immediate consequence of the inequality (2.1).

To prove (ii), we observe that

$$I^k_\delta(x) = I^{k-1}_{k\delta/(k-1)}(x) \quad (-\infty < x < \infty)$$

and that, by Lemma 2.2(i), the function

$$\Phi(x,u) = \sup_{s,s+(k-1)h\in I^k_\delta(x+u)} \left|\Delta^{k-1}_h f'(s)\right|$$

is continuous on the plane. Further,

$$\omega_k(\delta;x,f) = \sup_{t,t+kh\in I^k_\delta(x)} \left|\int_0^h \Delta^{k-1}_h f'(u+t)du\right| \leqslant$$

$$\leqslant \int_0^\delta \sup_{t,t+(k-1)h\in I^k_\delta(x)} \left|\Delta^{k-1}_h f'(u+t)\right| du = \int_0^\delta \Phi(x,u)\,du.$$

Thus

$$\left\|\omega_k(\delta;\cdot,f)\right\|_p \leqslant \int_0^\delta \left\|\Phi(\cdot,u)\right\|_p du,$$

and the result follows.

P r o p o s i t i o n 2.5. Suppose that, for a certain function f continuous almost everywhere and for some finite numbers $p \geq 1$, $\delta_0 > 0$ and positive integer k,

$$(2.4)\qquad \int_{-\infty}^{\infty} \left|\omega_k(\delta;x,f)\right|^p dx < \infty \quad \text{when} \quad 0<\delta\leq\delta_0.$$

Then

$$(2.5)\qquad \lim_{\delta\to 0+} \tau_k(\delta;f)_p = 0.$$

P r o o f . The sequence of non-negative measurable functions

$$(2.6)\qquad \psi_n(x) = \omega_k\left(\frac{1}{n}\,;\,x,f\right) \quad \left(n \geq \frac{1}{\delta_0}\right)$$

tends monotonely to zero for almost every x. Moreover, for every real x,

$$\psi_n(x) \leq \psi_{n_0}(x) \quad \text{when} \quad n \geq n_0 \geq 1/\delta_0.$$

In view of (2.4),

$$\int_{-\infty}^{\infty} \left|\psi_{n_0}(x)\right|^p dx < \infty.$$

Hence, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left|\psi_n(x)\right|^p dx = 0,$$

i.e.

$$\lim_{n\to\infty} \tau_k\left(\frac{1}{n}\,;\,f\right)_p = 0.$$

Now, the assertion (2.5) is quite evident.

R e m a r k . In the case $k = 1$ or $k = 2$, the conditions (2.4) and (2.5) imply that f is continuous almost everywhere in $(-\infty, \infty)$.

Indeed, let f be a function discontinuous in a set E of positive measure, for which the relations (2.4), (2.5) are fulfilled. If ψ_n is defined by (2.6), with $k = 1$ or 2, then

$$\tau_k\left(\frac{1}{n}; f\right)_p \geqslant \left\{\int_E |\psi_n(x)|^p \, dx\right\}^{\frac{1}{p}} \geqslant 0.$$

Consequently,

$$\lim_{n\to\infty} \int_E |\psi_n(x)|^p \, dx = 0. \tag{2.7}$$

On the other hand, in view of monotonity of the sequence $\{\psi_n(x)\}$, a finite limit

$$\lim_{n\to\infty} \psi_n(x) = \psi(x)$$

exists and $\psi(x) > 0$ for all $x \in E$. Therefore, by Fatou's lemma,

$$\lim_{n\to\infty} \int_E |\psi_n(x)|^p \, dx \geqslant \int_E |\psi(x)|^p \, dx > 0,$$

which contradicts to (2.7).

P r o p o s i t i o n 2.6. If a function f is absolutely continuous in each finite interval and its derivative f' belongs to the space L^p, with a finite $p \geqslant 1$, then

$$\tau_1(\delta; f)_p \leqslant \delta \| f' \|_p \quad (\delta \geqslant 0).$$

The proof runs as in [3], p.795-796.

Let us define the p-th variation of f in finite intervals $\langle a,b\rangle$ as

$$V_p(f;a,b) = \sup_{\Pi}\left\{\sum_{j=0}^{n-1}\left|f(x_{j+1}) - f(x_j)\right|^p\right\}^{\frac{1}{p}} \qquad (p > 0),$$

where Π denotes partitions $\{a = x_0 < x_1 < \dots < x_n = b\}$. Write

$$V_p(f) = \sup_{a,b} V_p(f;a,b).$$

A simple calculation, which will not be presented here, yields

P r o p o s i t i o n 2.7. Suppose that f is of bounded p-th variation $V_p(f)$, $1 \leqslant p < \infty$. Then

$$\tau_1(\delta;f)_p \leqslant (5\delta)^{\frac{1}{p}} V_p(f) \qquad (0 < \delta < \infty).$$

Finally, we shall give the estimate needed in Section 4.

L e m m a 2.8. For functions φ_k defined by (2.2) and $p \geqslant 1$,

$$\tau_1(\lambda;\varphi_k)_p \leqslant \tau_k\left(\delta + \frac{\lambda}{k}\,;\,f\right)_p \qquad (0 \leqslant \delta,\lambda < \infty),$$

provided that the right-hand side exists.

P r o o f . Since $\varphi_k(x) \geqslant 0$ always, we have

$$\omega_1(\lambda;x,\varphi_k) \leqslant \sup_{u\in I_\lambda^1(x)} \varphi_k(u) =$$

$$\sup_{u\in I_\lambda^1(x)}\ \sup_{t,t+kh\in I_\delta^k(u)} \left|\Delta_h^k f(t)\right| \leqslant \sup_{t,t+kh\in I^k_{\delta+\lambda/k}(x)} \left|\Delta_h^k f(t)\right|.$$

Therefore, under the assumption $x \in (-\infty,\infty)$ and $\lambda \geq 0$,

$$\omega_1(\lambda;x,\varphi_k) \leq \omega_k(\delta + \frac{\lambda}{k};x,f).$$

Thus, by 2.1(ii), our result follows.

3. Further basic quantities

Now let f be an arbitrary complex-valued function of a real variable, essentially bounded in each finite interval. Retain the symbols $I_\delta^k(x)$, $\omega_k(\delta;x,f)$ and $\tau_k(\delta;f)_p$ used in Section 2.

Write, for every real x, $\omega_*(0;x,f) = 0$ and

$$\omega_*(\delta;x,f) = \inf_E \sup_{t,t+h\in I_\delta^1(x)\setminus E} \left|\Delta_h^1 f(t)\right| \quad (0<\delta<\infty),$$

where the infimum is taken over all sets E, $E \subset I_\delta^1(x)$, of the Lebesgue measure zero (mes $E = 0$). In the case of f's bounded in finite intervals,

$$(3.1) \qquad \omega_*(\delta;x,f) \leq \omega_1(\delta;x,f) < \infty \quad (\delta \geq 0,\ -\infty<x<\infty);$$

the sign $\leq$ can be replaced by $=$ if f is continuous in $(-\infty,\infty)$.

L e m m a 3.1. For every real x,

$$\omega_*(\delta;x,f) \leq 2\,\|f\|_{L^\infty(I_\delta^1(x))} \qquad (\delta > 0).$$

P r o o f . Clearly,

$$\omega_*(\delta;x,f) \leqslant \inf_E\left\{\sup_{t,t+h\in I_\delta^1(x)\setminus E}\left(|f(t+h)| + |f(t)|\right)\right\} \leqslant$$

$$\leqslant \inf_E\left\{\sup_{t+h\in I_\delta^1(x)\setminus E}|f(t+h)| + \sup_{t\in I_\delta^1(x)\setminus E}|f(t)|\right\}.$$

Hence our result follows.

In particular, the function φ_* defined by the formula

$$(3.2) \qquad \varphi_*(x) = \omega_*(\delta;x,f) \qquad (\delta = \text{const} \geqslant 0)$$

is bounded in each finite interval.

L e m m a 3.2. If a real-valued f is continuous almost everywhere in $(-\infty,\infty)$, then φ_* is measurable in this interval.

P r o o f . In the non-trivial case $\delta > 0$, for all real x,

$$\varphi_*(x) = \operatorname*{ess\,sup}_{s\in I_\delta^1(x)} f(s) - \operatorname*{ess\,inf}_{t\in I_\delta^1(x)} f(t) \equiv g_1(x) - g_2(x),$$

where

$$\operatorname*{ess\,inf}_{t\in I_\delta^1(x)} f(t) = \sup_{D_x} \inf_{t\in I_\delta^1(x)\setminus D_x} f(t) \quad \begin{pmatrix} D_x \subset I_\delta^1(x), \\ \text{mes } D_x = 0 \end{pmatrix}.$$

To prove that g_1 is measurable in $(-\infty,\infty)$, we introduce the set

$$Z_c = \left\{x : g_1(x) \geqslant c, \ -\infty < x < \infty\right\} =$$

$$= \left\{x : \sup_{s\in I_\delta^1(x)\setminus E_x} f(s) \geqslant c \ \forall E_x \subset I_\delta^1(x), \ \text{mes } E_x = 0\right\}$$

with a real parameter c.

Take an accumulation point x' of Z_c, such that $x' \pm \delta/2$ are the points of continuity of f. In this case, there is a sequence $\{x_n\}_1^\infty$ tending to x' for which

$$\sup_{s\in I_\delta^1(x_n)\setminus E_{x_n}} f(s) \geqslant c \quad \text{when} \quad E_{x_n} \subset I_\delta^1(x_n),\ \text{mes}\ E_{x_n} = 0.$$

Suppose that $x_n \geqslant x'$. Then

$$\sup_{s\in\langle x_n-\frac{\delta}{2},\ x'+\frac{\delta}{2}\rangle\setminus E_{x_n}^1} f(s) \geqslant c \tag{3.3}$$

or

$$\sup_{s\in\langle x'+\frac{\delta}{2},\ x_n+\frac{\delta}{2}\rangle\setminus E_{x_n}^2} f(s) \geqslant c, \tag{3.4}$$

where $E_{x_n}^1$, $E_{x_n}^2$ signify the sets of the suitable intervals,

$$E_{x_n}^1 \cup E_{x_n}^2 = E_{x_n}.$$

For sufficiently large n, the left-hand side of (3.4) is near to $f(x'+\delta/2)$.

If the inequality (3.3) holds, then

$$\sup_{s\in I_\delta^1(x')\setminus E_{x'}^1} f(s) \geqslant c \begin{pmatrix} E_{x'}^1 \subset I_\delta^1(x'),\ \text{mes}\ E_{x'}^1 = 0, \\ E_{x'}^1 = E_{x_n}^1 \ \text{in}\ \langle x_n-\frac{\delta}{2},\ x'+\frac{\delta}{2}\rangle \end{pmatrix}.$$

If, for infinitely many n, the estimate (3.4) remains valid, then $f\left(x'+\frac{\delta}{2}\right) > c - \varepsilon$ ($\varepsilon > 0$). Consequently,

$$\sup_{s\in I_\delta^1(x')\setminus E_{x'}^2} f(s) \geqslant c \quad (E_{x'}^2 \subset I_\delta^1(x'),\ \text{mes}\ E_{x'}^2 = 0).$$

In the case $x_n \leqslant x'$ two similar inequalities hold. Hence $x' \in Z_c$.

But, the set of all remaining accumulation points x'' of Z_c has the Lebesgue measure zero. Hence Z_c is measurable, i.e. the function g_1 is measurable in $(-\infty,\infty)$. Analogously, g_2 is measurable. Thus, the proof is completed.

Considering these f's for which the functions φ_* are measurable in $(-\infty,\infty)$, we introduce the modified modulus of smoothness (of the second kind)

$$\tau_*(\delta;f)_p = \|\varphi_*\|_p = \|\omega_*(\delta;\cdot,f)\|_p \quad (p \geqslant 1,\ 0 \leqslant \delta < \infty).$$

It may be finite or infinite.

If $0 \leqslant \delta \leqslant \lambda < \infty$, then $\tau_*(\delta;f)_p \leqslant \tau_*(\lambda;f)_p$ and $\tau_*(0;f)_p = 0$ always. For every f bounded in finite intervals, the estimate (3.1) implies

$$\tau_*(\delta;f)_p \leqslant \tau_1(\delta;f)_p \quad (p \geqslant 1,\ \delta \geqslant 0).$$

The last two moduli coincide when f is continuous in $(-\infty,\infty)$.

P r o p o s i t i o n 3.3. Suppose that $p \geqslant 1$ and $\delta, \lambda \geqslant 0$. Then

(i) $$\tau_*(\delta;f+g)_p \leqslant \tau_*(\delta;f)_p + \tau_*(\delta;g)_p,$$

provided these moduli exist, and

(ii) $$\tau_*(\delta+\lambda;f)_p \leqslant \tau_*(\delta;f)_p + \tau_*(\lambda;f)_p,$$

whenever f is continuous almost everywhere.

P r o o f . The assertion (i) follows at once from the inequality

$$\omega_*(\delta;x,f+g) \leqslant \omega_*(\delta;x,f) + \omega_*(\delta;x,g) \quad (-\infty < x < \infty).$$

To prove (iii), let us choose a real x such that $x - (\delta-\lambda)/2$ is a point of continuity of f. Denote by A, B two arbitrary sets of the Lebesgue measure zero, lying in the intervals

$$\left\langle x - \frac{1}{2}(\delta+\lambda),\ x - \frac{1}{2}(\delta-\lambda)\right\rangle,\ \left\langle x - \frac{1}{2}(\delta-\lambda),\ x+\frac{1}{2}(\delta+\lambda)\right\rangle,$$

respectively. Then, putting $D = A \cup B$, we have

$$\omega_*(\delta+\lambda;x,f) = \inf_{D} \sup_{t,t+h\in I^1_{\delta+\lambda}(x)\setminus D} \left|\Delta^1_h f(t)\right| \leqslant$$

$$\leqslant \inf_{A,B}\left\{ \sup_{u,v\in I^1_{\lambda}(x-\frac{\delta}{2})\setminus A} \left|f(u)-f(v)\right| + \sup_{u,v\in I^1_{\delta}(x+\frac{\lambda}{2})\setminus B} \left|f(u)-f(v)\right|\right\}.$$

Consequently,

$$(3.5)\quad \omega_*(\delta+\lambda;x,f) \leqslant \omega_*(\lambda;x - \frac{\delta}{2}, f) + \omega_*(\delta;x + \frac{\lambda}{2}, f)$$

for almost every $x \in (-\infty,\infty)$.

Applying (3.5) and Lemma 3.2, we conclude that $\tau_*(\eta;f)_p$ is a subadditive function of $\eta \geqslant 0$.

P r o p o s i t i o n 3.4. Let f be such that, for some finite numbers $p \geqslant 1$, $\delta_0 > 0$,

$$(3.6)\qquad \int_{-\infty}^{\infty} \left|\omega_*(\delta;x,f)\right|^p dx < \infty \quad \text{when} \quad 0 < \delta \leqslant \delta_0$$

and

$$\lim_{\delta\to 0+} \omega_*(\delta;x,f) = 0 \quad \text{for almost every } x.$$

Then

$$\lim_{\delta\to 0+} \tau_*(\delta;f)_p = 0.$$

P r o o f . We put

$$\psi_n(x) = \omega_*(\tfrac{1}{n};\, x,f) \qquad (n \geqslant \tfrac{1}{\delta_0})$$

and proceed as in the case of Proposition 2.5.

R e m a r k . If f is continuous almost everywhere, the condition (3.6) implies

$$\lim_{\eta\to 0} \tau_*(\delta+\eta;f)_p = \tau_*(\delta;f)_p$$

for all positive δ. This is an immediate consequence of Propositions 3.3(ii) and 3.4.

Taking a positive integer k, positve δ and arbitrary real x, we also introduce the auxiliary quantity

$$\bar{\omega}_k(\delta;x,f) = \operatorname*{ess\,sup}_{h\in I^1_\delta(0)} \left|\Delta^k_h f(x)\right| =$$

$$= \inf_S \sup_{h\in\langle-\delta/2,\delta/2\rangle\setminus S} \left|\Delta^k_h f(x)\right| \quad \left(\begin{matrix} S\subset\langle-\delta/2,\ \delta/2\rangle\ , \\ \operatorname{mes} S = 0\end{matrix}\right).$$

It can easily be observed that the functions

$$\bar{\varphi}_k(x) = \bar{\omega}_k(\delta;x,f) \qquad (\delta = \mathrm{const} > 0,\ k = 1,2,\ldots)$$

are essentially bounded in each finite interval and, for f's bounded in finite intervals,

$$(3.7) \qquad \bar{\varphi}_k(x) \leqslant \omega_k(\delta;x,f) \qquad (-\infty < x < \infty).$$

If f is continuous in $(-\infty,\infty)$, the functions $\bar{\varphi}_k$ are continuous in this interval, too.

4. Estimates of the Jackson type

Throughout this Section the considered functions f are real-valued, defined and measurable in the interval $(-\infty,\infty)$.

First, we shall formulate the fundamental result proved in Section 3 of [6].

T h e o r e m 4.1. Let f be a function bounded in each finite interval. Then, for every finite $p \geqslant 1$,

$$\tilde{A}_\delta(f)_p \leqslant C_1\tau_1\left(\frac{1}{\delta}; f\right)_p \qquad (0<\delta<\infty).$$

Analogously, the following estimate can be obtained (in definitions of S_δ, J_δ given in [6] the upper and lower bounds of f should be replaced by ess sup and ess inf, respectively).

T h e o r e m 4.2. Suppose that f is essentially bounded in each finite interval, and that φ_* defined by (3.2) is measurable in $(-\infty,\infty)$ for all finite $\delta > 0$. Then, for every finite $p \geqslant 1$,

$$\hat{A}\,(f)_p \leqslant C_2\,\tau_*\left(\frac{1}{\delta}; f\right)_p \qquad (0<\delta<\infty).$$

For example, in the case of f continuous almost everywhere φ_* is measurable, by Lemma 3.2. Therefore $\tau_*(1/\delta;f)_p$ exists. If this quantity is finite for some positive δ, Proposition 3.4 implies

$$\lim_{\delta\to\infty}\tau_*\left(\frac{1}{\delta}; f\right)_p = 0$$

(Lemma 2.1 and Proposition 2.5 explain the behaviour of $\tau_1(1/\delta;f)_p$ as $\delta\to\infty$).

Given a positive integer k and positive δ, let us introduce the Steklov functions

$$f_{\delta,k}(x)=\frac{1}{\delta^k}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}\sum_{\nu=1}^{k}(-1)^{\nu-1}\binom{k}{\nu}f\left(x+\frac{\nu}{k}(t_1+\ldots+t_k)\right)dt_1\ldots dt_k$$

generated by the function f essentially bounded in finite intervals.

L e m m a 4.3. The following estimates hold:

(i) $$\left|f(x)-f_{\delta,k}(x)\right|\leqslant\bar{\omega}_k(\delta;x,f)\qquad(-\infty<x<\infty),$$

(ii) $$\left\|f_{\delta,k}^{(k)}\right\|_p\leqslant(2k/\delta)^k\,\omega_k(\delta;f)_p\qquad(p\geqslant 1).$$

P r o o f of (i). Clearly,

$$f(x)-f_{\delta,k}(x)=\frac{(-1)^k}{\delta^k}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}f\left(x+\frac{\nu}{k}(t_1+\ldots+t_k)\right)dt_1\ldots dt_k.$$

Hence

$$\left|f(x)-f_{\delta,k}(x)\right|\leqslant\frac{1}{\delta^k}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}\left|\Delta^k_{\frac{1}{k}(t_1+\ldots+t_k)}f(x)\right|dt_1\ldots dt_k\leqslant$$

$$\leqslant\frac{1}{\delta^k}\sup_{\eta\in\langle-\delta/2,\delta/2\rangle\setminus S}\left|\Delta^k_\eta f(x)\right|\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}dt_1\ldots dt_k\,(-\infty<x<\infty)$$

for all sets S, $S\subset\langle-\delta/2,\delta/2\rangle$, of the Lebesgue measure zero.

Passing to infimum over S, we get the desired result.

The inequality (ii) follows at once from the identity,

$$f_{\delta,k}^{(k)}(x)=(-1)^{k+1}\sum_{\nu=1}^{k}(-1)^{k-\nu}\binom{k}{\nu}\frac{1}{(\nu\delta/k)^k}\Delta^k_{\nu\delta/k}f\left(x-k\,\frac{\nu\delta}{2k}\right),$$

which is valid for almost every x.

L e m m a 4.4. Suppose that the real-valued functions f, g, ψ are measurable and bounded in each finite interval. Moreover, let

$$|f(x) - g(x)| \leqslant \psi(x) \quad \text{when} \quad x \in (-\infty, \infty).$$

Then, for every $p \geqslant 1$ and all finite $\sigma > 0$,

$$\tilde{A}_\sigma(f)_p \leqslant \tilde{A}_\sigma(g)_p + 2\tilde{A}_\sigma(\psi)_p + 2\|\psi\|_p.$$

P r o o f . It will be assumed that the right-hand side of the last inequality is finite.

Consider the functions P_j, Q_j $(j = 1,2)$ of class E_σ, such that

$$P_1(x) \geqslant g(x) \geqslant Q_1(x), \; P_2(x) \geqslant \psi(x) \geqslant Q_2(x) \quad (-\infty < x < \infty)$$

and

$$P_1 - Q_1 \in L^p, \quad P_2 - Q_2 \in L^p.$$

Clearly,

$$-P_2(x) \leqslant f(x) - g(x) \leqslant P_2(x) \quad \text{always}.$$

Therefore

$$Q_1(x) - P_2(x) \leqslant f(x) \leqslant P_1(x) + P_2(x) \; \text{always}.$$

Denote by $P_3(x)$, $Q_3(x)$ the right- and the left-hand sides of this estimate. Then

$$\tilde{A}_\sigma(f)_p \leqslant \|P_3 - Q_3\|_p \leqslant \|P_1 - Q_1\|_p + \|2P_2\|_p \leqslant$$
$$\leqslant \|P_1 - Q_1\|_p + 2(\|P_2 - \psi\|_p + \|\psi\|_p) \leqslant$$
$$\leqslant \|P_1 - Q_1\|_p + 2\|P_2 - Q_2\|_p + 2\|\psi\|_p.$$

Now, the assertion follows immediately.

Theorem 4.5. Let f be a function bounded in finite intervals. Suppose that φ_k defined by (2.2), with some positive integer k, is measurable in $(-\infty,\infty)$ for each finite $\delta > 0$. Then, for every finite $p \geqslant 1$,

$$\tilde{A}_\sigma(f)_p \leqslant C_3(k)\, \tau_k\left(\tfrac{1}{\sigma}; f\right)_p \quad (0 < \sigma < \infty).$$

Proof. Write

$$g(x) = f_{1/2\sigma,k}(x), \quad \psi(x) = \omega_k\left(\tfrac{1}{2\sigma}; x, f\right) \quad (-\infty < x < \infty).$$

In view of Lemma 4.3(i) and (3.7),

$$|f(x) - g(x)| \leqslant \psi(x) \quad \text{for all real } x.$$

Therefore, Lemma 4.4 and Theorem 4.1 lead to

$$\tilde{A}_\sigma(f)_p \leqslant \tilde{A}_\sigma(g)_p + 4\, C_1 \tau_1\left(\tfrac{1}{2\sigma}; \psi\right)_p + 2\|\psi\|_p$$

(see also Proposition 2.3(ii)).

Theorem 3.3 of [6] and Lemma 4.3(ii) yield

$$\tilde{A}_\sigma(g)_p \leqslant \frac{C_4(k)}{\sigma^k}\, \|g^{(k)}\|_p \leqslant C_4(k)(4k)^k\, \omega_k\left(\tfrac{1}{2\sigma}; f\right)_p.$$

By Lemma 2.8,

$$\tau_1\left(\tfrac{1}{2\sigma}; \psi\right)_p \leqslant \tau_k\left(\tfrac{1}{\sigma}; f\right)_p,$$

and

$$\|\psi\|_p = \left\|\omega_k\left(\frac{1}{2\sigma};\cdot,f\right)\right\|_p \leqslant \tau_k\left(\frac{1}{\sigma};f\right)_p.$$

Hence

$$\widetilde{A}_\sigma(f)_p \leqslant (4k)^k\, C_4(k)\, \omega_k(\tfrac{1}{\sigma};f)_p + (4C_1+2)\,\tau_k(\tfrac{1}{\sigma};f)_p.$$

Applying the estimate (2.3), we get the desired result.

5. Converse approximation theorem

Considering a real-valued function f measurable and bounded in each finite interval, we shall present two analogues of the Bernstein and Popov results given in [1] and [4].

Theorem 5.1. Let

$$0 < \widetilde{A}_\sigma(f)_p < \infty$$

for some finite $p \geqslant 1$ and all finite $\sigma \geqslant \sigma_0 > 0$, and let

$$\lim_{\sigma\to\infty} \widetilde{A}_\sigma(f)_p = 0.$$

Then

$$f(x) = F_{\sigma_0}(x) + \varphi(x) \qquad (-\infty < x < \infty), \tag{5.1}$$

where $F_{\sigma_0} \in E_{\sigma_0}$ and $\varphi \in L^p$. Moreover, for $k = 1,2,\ldots,$

$$\tau_k(\tfrac{1}{\sigma};\varphi)_p \leqslant \frac{C_5(k)}{[\sigma/\sigma_0]^k} \sum_{\mu=1}^{[\sigma/\sigma_0]} \mu^{k-1}\, \widetilde{A}_{\mu\sigma_0}(f)_p \qquad (\sigma_0 \leqslant \sigma < \infty), \tag{5.2}$$

whenever the left-hand side of (5.2) exists.

P r o o f . Denote by P_σ and Q_σ the entire functions belonging to the sets $H^{+}_{\sigma,p}(f)$ and $H^{-}_{\sigma,p}(f)$, respectively, such that

$$\| P_\sigma - Q_\sigma \|_p \leqslant 2\widetilde{A}_\sigma(f)_p \qquad (\sigma_0 \leqslant \sigma < \infty).$$

By the assumptions, both the series

$$P_{\sigma_0}(x) + \sum_{\nu=1}^{\infty} \left\{ P_{2^\nu\sigma_0}(x) - P_{2^{\nu-1}\sigma_0}(x) \right\} \equiv \sum_{\nu=0}^{\infty} U_\nu(x),$$

$$Q_{\sigma_0}(x) + \sum_{\nu=1}^{\infty} \left\{ Q_{2^\nu\sigma_0}(x) - Q_{2^{\nu-1}\sigma_0}(x) \right\} \equiv \sum_{\nu=0}^{\infty} V_\nu(x)$$

converge to $f(x)$ in L^p-metric. Therefore

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left| f(x) - \frac{1}{2}\left\{ P_{\sigma_0}(x) + Q_{\sigma_0}(x) \right\} - \frac{1}{2} \sum_{\nu=1}^{N} \left\{ U_\nu(x) + V_\nu(x) \right\} \right|^p dx = 0.$$

Putting

$$F_{\sigma_0}(z) = \frac{1}{2}\left\{ P_{\sigma_0}(z) + Q_{\sigma_0}(z) \right\}, \quad \varphi(x) = f(x) - F_{\sigma_0}(x),$$

we get the decomposition (5.1).

If $\Delta_h^k \varphi(t) \geqslant 0$ for some real x and $t, t+kh \in I_\delta^k(x)$, then

$$\Delta_h^k \varphi(t) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} \left\{ f(t+\nu h) - F_{\sigma_0}(t+\nu h) \right\} \leqslant$$

$$\leqslant \sum_{\substack{0\leqslant\nu\leqslant k \\ \nu=k(\mathrm{mod}\,2)}} \binom{k}{\nu} P^*_\sigma(t+\nu h) - \sum_{\substack{0\leqslant\nu\leqslant k \\ \nu=k-1(\mathrm{mod}\ 2)}} \binom{k}{\nu} Q^*_\sigma(t+\nu h),$$

where

$$P^*_\sigma(z) = P_\sigma(z) - F_{\sigma_0}(z),\ Q^*_\sigma(z) = Q_\sigma(z) - F_{\sigma_0}(z) \quad (\sigma \geqslant \sigma_0),$$

Consequently,

$$\Delta^k_h \varphi(t) \leqslant \Delta^k_h P^*_\sigma(t) + \sum_{\nu=0}^{k} \binom{k}{\nu} \left\{ P^*_\sigma(t+\nu h) - Q^*_\sigma(t+\nu h) \right\}$$

and whence

$$\left|\Delta^k_h \varphi(t)\right| \leqslant \left|\Delta^k_h P^*_\sigma(t)\right| + 2^k \left\{ \omega_1(k\delta;x,P^*_\sigma-Q^*_\sigma) + \left|P^*_\sigma(x)-Q^*_\sigma(x)\right| \right\}.$$

Analogously, in the case $\Delta^k_h \varphi(t) \leqslant 0$,

$$\left|\Delta^k_h \varphi(t)\right| \leqslant \left|\Delta^k_h Q^*_\sigma(t)\right| + 2^k \left\{ \omega_1(k\delta;x,P^*_\sigma-Q^*_\sigma) + \left|P^*_\sigma(x)-Q^*_\sigma(x)\right| \right\}.$$

Thus,

$$\omega_k(\delta;x,\varphi) \leqslant \omega_k(\delta;x,P^*_\sigma) + \omega_k(\delta;x,Q^*_\sigma) +$$

$$+ 2^k \left\{ \omega_1(k\delta;x,P_\sigma-Q_\sigma) + \left|P_\sigma(x)-Q_\sigma(x)\right| \right\} \qquad (-\infty < x < \infty).$$

Further, by Lemma 2.2(i) and Proposition 2.3(ii),

$$\tau_k(\delta;\varphi)_p \leqslant \tau_k(\delta;P^*_\sigma)_p + \tau_k(\delta;Q^*_\sigma)_p +$$

$$+ 2^k \left\{ k\, \tau_1(\delta;P_\sigma-Q_\sigma) + 2\tilde{A}_\sigma(f)_p \right\} \quad (\sigma \geqslant \sigma_0,\ 0 \leqslant \delta \leqslant 1/\sigma_0).$$

In view of Proposition 2.6 and the well-known Bernstein inequality ([7], p.232),

$$\tau_1(\delta;P_\sigma-Q_\sigma)_p \leqslant \delta\left\|(P_\sigma-Q_\sigma)'\right\|_p \leqslant \delta\sigma\cdot 2\,\tilde{A}_\sigma(f)_p.$$

Therefore

$$(5.3)\qquad \tau_k(\delta;\varphi)_p \leqslant \tau_k(\delta;P_\sigma^*)_p + \tau_k(\delta;Q_\sigma^*)_p +$$

$$+\ 2^{k+1}(k\delta\sigma+1)\ \tilde{A}_\sigma(f)_p \qquad (\sigma \geqslant \sigma_0,\quad 0 \leqslant \delta \leqslant 1/\sigma_0).$$

Clearly,

$$P^*_{2^m\sigma_0}(x) = P_{2^m\sigma_0}(x) - F_{\sigma_0}(x) = \sum_{\nu=1}^{m} U_\nu(x) + \frac{1}{2}\left\{P_{\sigma_0}(x)-Q_{\sigma_0}(x)\right\},$$

By Propositions 2.3(i), 2.4(ii) and 2.6,

$$\tau_k(\delta;P^*_{2^m\sigma_0})_p \leqslant \sum_{\nu=1}^{m} \tau_k(\delta;U_\nu)_p + \tau_k\Big(\delta;\ \frac{1}{2}\left\{P_{\sigma_0}-Q_{\sigma_0}\right\}\Big)_p \leqslant$$

$$\leqslant \sum_{\nu=1}^{m} C_6(k)\ \delta^k\left\|U_\nu^{(k)}\right\|_p + \frac{1}{2}\,C_6(k)\ \delta^k\left\|P_{\sigma_0}^{(k)} - Q_{\sigma_0}^{(k)}\right\|_p \leqslant$$

$$\leqslant C_6(k)\ \delta^k\left\{\sum_{\nu=1}^{m}(2^\nu\sigma_0)^k\left\|U_\nu\right\|_p + \frac{1}{2}\,\sigma_0^k\left\|P_{\sigma_0} - Q_{\sigma_0}\right\|_p\right\} \leqslant$$

$$\leqslant C_6(k)(\delta\sigma_0)^k\left\{4\sum_{\nu=1}^{m} 2^{\nu k}\,\tilde{A}_{2^{\nu-1}\sigma_0}(f)_p + \tilde{A}_{\sigma_0}(f)_p\right\} \qquad (\delta \geqslant 0).$$

Observing that

$$2^{\nu k}\,\tilde{A}_{2^{\nu-1}\sigma_0}(f)_p \leqslant C_7(k)\sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}} \mu^{k-1}\,\tilde{A}_{\mu\sigma_0}(f)_p \quad (\nu \geqslant 2),$$

we obtain

$$\tau_k\left(\delta; P^*_{2^m\sigma_0}\right)_p \leqslant C_6(k)(\delta\sigma_0)^k\left\{(4\cdot 2^k+1)\tilde{A}_{\sigma_0}(f)_p + \right.$$

$$\left. + 4C_7(k)\sum_{\mu=2}^{2^{m-1}} \mu^{k-1}\tilde{A}_{\mu\sigma_0}(f)_p\right\}.$$

Hence

$$\tau_k\left(\delta; P^*_{2^m\sigma_0}\right)_p \leqslant C_8(k)(\delta\sigma_0)^k\sum_{\mu=1}^{2^{m-1}} \mu^{k-1}\tilde{A}_{\mu\sigma_0}(f)_p \quad (\delta \geqslant 0, m=1,2,\ldots).$$

Given any $\sigma \geqslant \sigma_0$, let us choose the positive integers n and m such that

$$n\,\sigma_0 \leqslant \sigma < (n+1)\,\sigma_0, \qquad 2^{m-1} \leqslant n < 2^m.$$

Then, putting $\delta = (n\,\sigma_0)^{-1}$, we have

$$\tau_k\left(\delta; P^*_{2^m\sigma_0}\right)_p \leqslant \frac{C_8(k)}{n^k}\sum_{\mu=1}^{2m-1} \mu^{k-1}\,\tilde{A}_{\mu\sigma_0}(f)_p \quad (m=1,2,\ldots),$$

and the same estimate for $Q^*_{2^m\sigma_0}$ holds. Hence the inequality (5.3), in which σ is replaced by $2^m\sigma_0$, leads to

$$\tau_k\left(\frac{1}{n\,\sigma_0}; \varphi\right)_p \leqslant 2\,\frac{C_8(k)}{n^k}\sum_{\mu=1}^{2^{m-1}} \mu^{k-1}\,\tilde{A}_{\mu\sigma_0}(f)_p +$$

$$+ 2^{k+1}\left(k\,\frac{2^m}{n} + 1\right)\tilde{A}_{2^m\sigma_0}(f)_p.$$

It is easily seen,

$$\tilde{A}_{2^m\sigma_0}(f)_p \leqslant \tilde{A}_{n\sigma_0}(f)_p \leqslant \frac{C_9(k)}{n^k}\sum_{\mu=1}^{n}\mu^{k-1}\tilde{A}_{\mu\sigma_0}(f)_p.$$

Thus

$$\tau_k\left(\frac{1}{\sigma};\varphi\right)_p \leqslant \tau_k\left(\frac{1}{n\sigma_0};\varphi\right)_p \leqslant 2\,\frac{C_8(k)}{n^k}\sum_{\mu=1}^{n}\mu^{k-1}\tilde{A}_{\mu\sigma_0}(f)_p +$$

$$+\,2^{k+1}(2k+1)\,\frac{C_9(k)}{n^k}\sum_{\mu=1}^{n}\mu^{k-1}\tilde{A}_{\mu\sigma_0}(f)_p \quad (\sigma_0\leqslant\sigma<\infty),$$

and the estimate (5.2) is proved.

A similar calculation yields

Theorem 5.2. Suppose that $f\in L^p$ for some finite $p\geqslant 1$ and that $\tilde{A}_{\sigma_0}(f)_p<\infty$, σ_0 being a certain positive number. Then if $k=1,2,\ldots,$

$$(5.4)\quad \tau_k\left(\frac{1}{\sigma};f\right)_p \leqslant \frac{C_{10}(k)}{[\sigma/\sigma_0]^k}\left\{\|f\|_p+\sum_{\mu=1}^{[\sigma/\sigma_0]}\mu^{k-1}\tilde{A}_{\mu\sigma_0}(f)_p\right\} \quad (\sigma_0\leqslant\sigma<\infty),$$

whenever the left-hand side of (5.4) exists.

Also, it may be shown that for every real-valued function $f\in L^p$ $(1\leqslant p<\infty)$, essentially bounded in finite intervals and continuous almost everywhere,

$$\tau_*\left(\frac{1}{\sigma};f\right)_p \leqslant \frac{C_{11}}{[\sigma/\sigma_0]}\left\{\|f\|_p+\sum_{\mu=1}^{[\sigma/\sigma_0]}\hat{A}_{\mu\sigma_0}(f)_p\right\} \quad (0<\sigma_0\leqslant\sigma<\infty)$$

(see Lemma 3.2 and Proposition 3.3).

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