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ON BISEMILATTICES, II

1. Introduction

In [5] J.Płonka introduced the concept of a quasilattice, later R.Padmanabhan (see [4]) called it a bisemilattice. Recall that an algebra $(B,+,\cdot)$ of type $(2,2)$ is said to be a bisemilattice (or a quasilattice) if it satisfies the following identities:

(1) $x+x = x,$ $\quad x\cdot x = x$

(2) $x+y = y+x,$ $\quad x\cdot y = y\cdot x$

(3) $(x+y)+z = x+(y+z),$ $\quad (x\cdot y)\cdot z = x\cdot(y\cdot z)$

(in the sequel we shall write xy instead of $x\cdot y$).

By $V(+)$ we denote the variety of all idempotent and commutative groupoids $(G,+)$. The class of all algebras of type $(2,2)$ satisfying (1) and (2) is denoted by $V(+,\cdot)$ and by $B(+,\cdot)$ we denote the variety of all bisemilattices.

Let $f = f(x_1,\ldots,x_n)$ be a function on a set A. We say that f admits a permutation $\sigma \in S_n$ (where S_n denotes the symmetric group on n letters) of its variables if $f = f^{\sigma}$, i.e., $f(x_1,\ldots,x_n) = f(x_{\sigma 1},\ldots,x_{\sigma n})$ for all $x_1,\ldots,x_n \in A$ where $f^{\sigma}(x_1,\ldots,x_n) = f(x_{\sigma 1},\ldots,x_{\sigma n})$. The group of all admissible permutations of f is denoted by $G(f)$ (see [3]). A function $f = f(x_1,\ldots,x_n)$ is called symmetric if $G(f) = S_n$, and, f is idempotent if $f(x,\ldots,x) = x$ for all $x \in A$. Recall that an algebra $\mathfrak{A} = (A,F)$ is symmetric (idempotent) if every $f \in F$ (depends on all its variables) is symmetric (idempotent), respectively.

Let $f(x_1,\dots,x_n)$ be a polynomial in the variety $B(+,\cdot)$ and let $\sigma \in S_n$. Then σ is said to be trivial for f (with respect to $B(+,\cdot)$) if the identity $f = f^{\sigma}$ follows from the identities of the variety $B(+,\cdot)$ (of course, in this definition $B(+,\cdot)$ can be replaced by any variety of algebras).

An algebra (A,F) is said to be proper if all f's from F are different and every non-nullary $f \in F$ depends on all its variables. For example, every at least two-element lattice is proper in the variety $B(+,\cdot)$.

By $p_n = p_n(\mathfrak{A})$ we denote the number of all essentially n-ary polynomials over $\mathfrak{A}$.

For the definitions and notations used here see [2].

In this note we prove the following

T h e o r e m . If $(B,+,\cdot)$ is a proper bisemilattice, then $p_n(B,+,\cdot) \geqslant 2+n!$ for all $n \geqslant 3$.

2. Lemmas and the proof of the theorem

In this section we present all lemmas being needed to prove the above theorem.

Let $(B,+,\cdot)$ be a bisemilattice. Consider two sequences $\{f_n\}$ and $\{g_n\}$ of polynomials over $(B,+,\cdot)$. The sequences are defined as follows

$f_2(x_1,x_2) = x_1x_2, f_3(x_1,x_2,x_3) = x_1x_2+x_3$, and in general

$$(*)\quad f_n(x_1,\dots,x_{n-1},x_n) = \begin{cases} f_{n-1}(x_1,\dots,x_{n-1})x_n & \text{if } n \text{ is even} \\ f_{n-1}(x_1,\dots,x_{n-1})+x_n & \text{if } n \text{ is odd} \end{cases}$$

and

$g_2(x_1,x_2) = x_1+x_2, g_3(x_1,x_2,x_3) = (x_1+x_2)x_3$, and in general,

we put

$$(**)\quad g_n(x_1,\dots,x_{n-1},x_n) = \begin{cases} g_{n-1}(x_1,\dots,x_{n-1})+x_n & \text{if } n \text{ is even} \\ g_{n-1}(x_1,\dots,x_{n-1})x_n & \text{if } n \text{ is odd.} \end{cases}$$

Further, we also consider the following polynomials: $s_n(+) = x_1 + \dots + x_n$ and $s_n(\cdot) = x_1 \dots x_n$ for $n \geqslant 1$.

L e m m a 1. Let $f(x_1,\dots,x_n)$ be essentially n-ary on a set A and let $(A,+)$ be proper in $V(+)$. Then the following polynomial $g = f(x_1,\dots,x_n) + f(y_1,\dots,y_n)$ is essentially 2n-ary on A.

P r o o f . Assume that g is not essentially 2n-ary. Therefore there exists a variable, say x_k such that the polynomial g does not depend on x_k. Using the fact that $g(x_1,\dots,x_n, y_1,\dots,y_n) = g(y_1,\dots,y_n, x_1,\dots,x_n)$ we infer that g does not depend on both variables x_k and y_k. Thus the following polynomial

$$g(x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n,x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n) =$$

$$= f(x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_n)$$

also does not depend on x_k, which is impossible.

L e m m a 2. Under the assumption of Lemma 1, the polynomial $f(x_1,\dots,x_n)+x_{n+1}$ is essentially n+1-ary.

P r o o f . It follows from Lemma 1 if we put $f(y_1,\dots,y_n)$ for the variable x_{n+1}

L e m m a 3. If $(B,+,\cdot)$ is proper in $B(+,\cdot)$, then the polynomials $f_n, g_n, s_n(+)$ and $s_n(\cdot)$ are essentially n-ary for all $n \geqslant 2$.

P r o o f . It follows from Lemma 2 and an easy induction on n.

L e m m a 4. If $(B,+,\cdot)$ is a proper algebra from $B(+,\cdot)$, then both polynomials $(x+y)z$ and $xy+z$ do not admit any nontrivial permutation.

P r o o f . Observe that if $(x+y)z$ admits a nontrivial permutation, then the polynomial is symmetric. Hence we have

$x+y = (x+y)(x+y) = ((x+y)+y)x = (x+y)x = (x+x)y = xy$, a contradiction. Analogously one can prove the statement for the polynomial $xy+z$.

L e m m a 5. If $(B,+,\cdot)$ is a proper bisemilattice and n is the smallest number such that f_n admits a nontrivial permutation $\sigma \in S_n$, then $\sigma n \neq n$ (the same is true for the sequence $\{g_n\}$).

P r o o f . By the assumption we have

$$(*_*^*) \qquad f_n(x_1,\dots,x_n) = f_n(x_{\sigma 1},\dots,x_{\sigma n})$$

for some nontrivial $\sigma \in S_n$ and all $x_1,\dots,x_n \in B$. Applying Lemma 4 we infer that $n \geqslant 4$. Assume, contrary that $\sigma n = n$. Then putting $f_{n-1}(x_1,\dots,x_{n-1})$ for the variable x_n in the latter identity and using the idempotency and the commutativity of $+$ and $\cdot$, and, of course using $(*_*^*)$ we obtain

$$f_{n-1}(x_1,\dots,x_{n-1}) = f_n(x_1,\dots,x_{n-1},f_{n-1}(x_1,\dots,x_{n-1})) =$$

$$= f_n(x_{\sigma 1},\dots,x_{\sigma(n-1)},f_{n-1}(x_1,\dots,x_{n-1})) =$$

$$= f_n(x_1,\dots,x_{n-1},f_{n-1}(x_{\sigma 1},\dots,x_{\sigma(n-1)})) =$$

$$= f_n(x_{\sigma 1},\dots,x_{\sigma(n-1)},f_{n-1}(x_{\sigma 1},\dots,x_{\sigma(n-1)})) =$$

$$= f_{n-1}(x_{\sigma 1},\dots,x_{\sigma(n-1)}).$$

Hence f_{n-1} (where $n-1 \geqslant 3$) admits a nontrivial permutation of its variables which gives a contradiction with the minimality of n. The proof of the lemma is completed.

The next lemma can be found in [1]. We quote here this lemma for convenience of the reader.

L e m m a 6. (Lemma 2 in [1]). If $(B,+,\cdot)$ is a bisemilattice and $(x+y)y$ is commutative, then $(x+y)y = x+y$ (the dual version is true for the polynomial $xy+y$).

L e m m a 7. If $(B,+,\cdot)$ is a proper bisemilattice, then the polynomials f_{2m+1} and g_{2m+1} do not admit the transposition $(1,2m+1)$.

P r o o f . We give only the proof for the polynomial f_{2m+1} (the proof for the polynomial g_{2m+1} runs similarly). Assume that

$$(**) \quad f_{2m+1}(x_1,x_2,\dots,x_{2m},x_{2m+1}) = f_{2m+1}(x_{2m+1},x_2,\dots,x_{2m},x_1)$$

holds in $(B,+,\cdot)$.

Then we have

$$x+xy = f_{2m+1}(x,x,\dots,x,xy) =$$

$$= f_{2m+1}(xy,x,\dots,x,x) = (\dots((xy)x+x)x+\dots+x)x+x =$$

$$= (\dots(xy+x)x+\dots+x)x+x = f_{2m+1}(x,y,x,x,\dots,x,x) =$$

$$= f_{2m+1}(y,x,x,x,\dots,x,x) = f_{2m+1}(x,x,x,\dots,x,y) =$$

$$= f_{2m}(x,x,\dots,x,x)+y = x+y.$$

Using Lemma 6 we infer that $x+y = xy$ which is impossible in a proper bisemilattice.

L e m m a 8. If $(B,+,\cdot)$ is a proper bisemilattice satisfying $(x+y)y = x+y$, then the polynomial $f_4(x_1,x_2,x_3,x_4) = (x_1x_2+x_3)x_4$ does not admit the transposition $(1,4)$ (the dual version is also true).

P r o o f . Indeed, if $(x_1x_2+x_3)x_4 = (x_4x_2+x_3)x_1$, then we have $x+y = (x+y)y = ((xx)+y)y = ((yx)+y)x = (xy+y)x = ((yy)+(xy))x = ((xy)+(xy))y = (xy)y = xy$, a contradiction.

L e m m a 9. If $(B,+,\cdot)$ is a proper bisemilattice for which $(x+y)y = x+y$, then f_{2m} does not admit the transposition $(1,2m)$ of its variables for $m \geqslant 2$.

P r o o f . Assume, contrary that f_{2m} admits $(1,2m)$ where $m \geqslant 2$ and let m be the smallest number such that

$$f_{2m}(x_1,x_2,\ldots,x_{2m-1},x_{2m}) = f_{2m}(x_{2m},x_2,\ldots,x_{2m-1},x_1).$$

Using Lemma 8 we infer that $m \geqslant 3$. From $(x+y)y = x+y$ and the latter identity we get

$$f_{2m-2}(x_1,x_2,\ldots,x_{2m-3},x_{2m-2}) =$$

$$= f_{2m}(x_1,x_2,x_3,x_3,x_3,x_4,\ldots,x_{2m-3},x_{2m-2}) =$$

$$= f_{2m}(x_{2m-2},x_2,x_3,x_3,x_3,x_4,\ldots,x_{2m-3},x_1) =$$

$$= f_{2m-2}(x_{2m-2},x_2,x_3,x_4,\ldots,x_{2m-3},x_1).$$

Hence

$$f_{2m-2}(x_1,x_2,\ldots,x_{2m-3},x_{2m-2}) =$$

$$= f_{2m-2}(x_{2m-2},x_2,\ldots,x_{2m-3},x_1)$$

which gives a contradiction with the minimality of $m (2m-2 \geqslant 4)$.

We should mention here that the previous lemma is also true for the polynomials g_n, of course, replacing $(x+y)y = x+y$ by $xy+y = xy$.

L e m m a 10. If $(B,+,\cdot)$ is a proper bisemilattice and f_{2m} admits the transposition $(1,2m)$ for some $m \geqslant 2$, then $(x+y)y = x+y$ holds in $(B,+,\cdot)$. (Analogously, for g_{2m}).

P r o o f . Assume that in the proper bisemilattice $(B,+,\cdot)$ the identity $f_{2m}(x_1,x_2,\ldots,x_{2m-1},x_{2m}) = f_{2m}(x_{2m},x_2,\ldots,x_{2m-1},x_1)$ holds. Then we have

$$(x+y)y = f_{2m}(x,x,\ldots,x,y,y) =$$

$$= f_{2m}(y,x,x,\ldots,x,y,x) = ((\ldots((xy)+x)x + \ldots + x)x+y)x =$$

$$= ((\ldots((xy)x+x)x + \ldots + x)x+y)x = f_{2m}(xy,x,x,\ldots,x,y,x) =$$

$$= f_{2m}(x,x,\ldots,x,y,xy) = f_{2m-1}(x,x,\ldots,x,y)(xy) = (x+y)(xy).$$

Hence $(x+y)y$ is symmetric. Using Lemma 6 we get $(x+y)y = x+y$ which finishes the proof.

L e m m a 11. If $(B,+,\cdot)$ is a proper bisemilattice, then f_{2m} and g_{2m} do not admit the transposition $(1,2m)$ for $m \geqslant 2$.

P r o o f . If so, for the polynomial f_{2m}, then applying Lemma 10 we get that $(x+y)y = x+y$ holds in $(B,+,\cdot)$. Now using Lemma 9 we obtain the required assertion.

L e m m a 12. If $(B,+,\cdot)$ is a proper bisemilattice, then f_n and g_n do not admit the transposition $(1,n)$ for $n \geqslant 3$.

P r o o f . It follows from Lemmas 7 and 11.

L e m m a 13. If $(B,+,\cdot)$ is proper in $B(+,\cdot)$ and n is the smallest number such that $(^{*}_{*}{}^{*})$ holds in $(B,+,\cdot)$, then $\{1,2\} \neq \{\sigma 1, \sigma 2\}$ (the same is true for g_n).

P r o o f . Assume that $\{1,2\} = \{\sigma 1,\sigma 2\}$ in the identity $(^{*}_{*}{}^{*})$. Using Lemma 4 we infer that $n \geqslant 4$. Now putting $x_1 = x_2$ in $(^{*}_{*}{}^{*})$ we get a contradiction with the minimality of n.

L e m m a 14. If $(B,+,\cdot)$ is a proper bisemilattice, then f_k and g_k do not admit any nontrivial permutations of their variables for $3 \leqslant k \leqslant 4$.

P r o o f . For $k = 3$, the proof follows from Lemma 4. Let now

(+) $$(x_1x_2+x_3)x_4 = (x_{\sigma1}x_{\sigma2}+x_{\sigma3})x_{\sigma4}$$

for some nontrivial $\sigma \in S_4$ (Analogously, we prove the assertion for g_4).

If $\{1,2\} = \{\sigma1,\sigma2\}$, then putting $x_1 = x_2$ in (+) we get a contradiction with Lemma 4. If $\sigma4 = 4$, then using the same method as in Lemma 5 we infer that $x_1x_2+x_3$ admits a nontrivial permutation, a contradiction with Lemma 4. Because of the commutativity of the polynomial xy we infer from (+) that the polynomial $(x_1x_2+x_3)x_4$ admits a nontrivial transposition $(\sigma1,\sigma2)$ (we know that $\{1,2\} \neq \{\sigma1,\sigma2\}$). We have the following possibilities for the set $\{\sigma1,\sigma2\}$, namely, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ and $\{3,4\}$. Using the commutativity of $\cdot$ and Lemma 12 we infer that the possibilities $\{1,4\}$ and $\{2,4\}$ do not occur. If $\{1,3\} = \{\sigma1,\sigma2\}$ or $\{2,3\} = \{\sigma1,\sigma2\}$, then the polynomial $f_4(x_1,x_2,x_3,x_4) = (x_1x_2+x_3)x_4$ admits a nontrivial permutation ϱ with $\varrho4 = 4$. Now using Lemmas 4 and 5 we get a contradiction. If $\{3,4\} = \{\sigma1,\sigma2\}$, then $(x_1x_2+x_3)x_4$ admits the transposition $(3,4)$. Putting in the latter polynomial $x_1 = x_2$ we get a contradiction with Lemma 4.

L e m m a 15. If $(B,+,\cdot)$ is a proper bisemilattice, then the polynomials f_n and g_n $(n \geqslant 3)$ do not admit any nontrivial transposition (i,j) of its variables.

P r o o f . We again give the proof only for f_n. Let n be the smallest number such that

(++) $$f_n(x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_{j-1},x_j,x_{j+1},\dots,x_{n-1},x_n) =$$
$$= f_n(x_1,x_2,\dots,x_{i-1},x_j,x_{i+1},\dots,x_{j-1},x_i,x_{j+1},\dots,x_{n-1},x_n)$$

for some nontrivial transposition (i,j) where $i < j$. Using Lemma 14 we infer that $n \geqslant 5$. We also may assume that $\{1,2\} \neq \{i,j\}$ since the transposition (i,j) is nontrivial. Now we prove that $j = n$. Indeed, if $j < n$, then $i,j \leqslant n-1$

and putting $f_{n-1}(x_1,\dots,x_{n-1})$ for x_n in the identity (++) and using the same method of proving as in Lemma 5 we get a contradiction that f_{n-1} $(n-1 \geqslant 4)$ admits a nontrivial transposition. Therefore $j = n$. If $i \geqslant 4$, then putting $x_1 = x_2 = x_3$ in the identity (++) we get a contradiction with the minimality of n. Let now $i = 3$. Then putting $x_1 = x_2$ in (++) we infer that $g_{n-1}(x_2,x_3,\dots,x_{n-1},x_n) = f_n(x_2,x_2,x_3,\dots,x_{n-1},x_n)$ admits the transposition $(2,n)$ which gives a contradiction with Lemma 12. Therefore we infer that $i \in \{1,2\}$ and $j = n$. Applying again Lemma 12 we get a contradiction. The proof of the lemma is completed.

L e m m a 16. If $(B,+,\cdot)$ is a proper bisemilattice, then f_n and g_n do not admit any nontrivial permutation of their variables for $n \geqslant 3$.

P r o o f . Let f_n satisfy the identity $(^{*}_{*}{}^{*})$, i.e.,

$$f_n(x_1,x_2,\dots,x_{n-1},x_n) = f_n(x_{\sigma 1},x_{\sigma 2},\dots,x_{\sigma(n-1)},x_{\sigma n})$$

for some nontrivial permutation $\sigma \in S_n$ $(n \geqslant 3)$. Of course, we may assume that n is the smallest number for which $(^{*}_{*}{}^{*})$ holds. Using Lemma 13, we infer that $\{1,2\} \neq \{\sigma 1,\sigma 2\}$. Because $\cdot$ is commutative we deduce from $(^{*}_{*}{}^{*})$ that f_n admits a nontrivial transposition $(\sigma 1,\sigma 2)$ which gives a contradiction with Lemma 15.

L e m m a 17. If $(B,+,\cdot)$ is a proper bisemilattice, then $G(f_n) \cong G(g_n) \cong S_2$ for all $n \geqslant 2$.

P r o o f . If $n = 2$, then it follows from the commutativity of $+$ and $\cdot$. Observe, using Lemma 16 that the only admissible permutations for the polynomials f_n and g_n are: the identity permutation and the transposition $(1,2)$. Of course, these permutations are the only trivial permutations for f_n and g_n in any proper bisemilattice. Therefore $G(f_n) \cong G(g_n) \cong S_2$.

L e m m a 18. If $(B,+,\cdot)$ is proper in $B(+,\cdot)$, then $f_n \neq g_n$ for all $n \geqslant 2$.

P r o o f . Let $f_n(x_1,x_2,\dots,x_{n-1},x_n) = g_n(x_1,x_2,\dots\dots,x_{n-1},x_n)$. Putting $x = x_1 = x_2 = \dots = x_{n-1}$ and $y = x_n$ in the latter identity we get $x+y = xy$, which is impossible in any proper bisemilattice.

L e m m a 19. If $(B,+,\cdot)$ is proper in $B(+,\cdot)$, then $f_n \neq g_n^{\sigma}$ for all $\sigma \in S_n$ and all $n \geqslant 2$.

P r o o f . Let n be the smallest number such that

$$(\overset{++}{+})\quad f_n(x_1,x_2,\dots,x_{n-1},x_n) = g_n(x_{\sigma 1},x_{\sigma 2},\dots,x_{\sigma(n-1)},x_{\sigma n}).$$

If $\{1,2\} \neq \{\sigma 1,\sigma 2\}$, then by the commutativity of $+$ we infer that f_n admits a nontrivial transposition $(\sigma 1,\sigma 2)$ which is a contradiction with Lemma 16. If $\{1,2\} = \{\sigma 1,\sigma 2\}$ in $(\overset{++}{+})$, then putting $x_1 = x_2$ we get from $(\overset{++}{+})$ $f_n(x_2,x_2,x_3,\dots,\dots,x_{n-1},x_n) = g_{n-1}(x_2,x_3,\dots,x_{n-1},x_n) = g_n(x_2,x_2,x_{\sigma 3},\dots\dots,x_{\sigma(n-1)},x_{\sigma n}) = f_{n-1}(x_2,x_{\sigma 3},\dots,x_{\sigma(n-1)},x_{\sigma n})$. Hence we have $f_{n-1}(x_2,x_{\sigma 3},\dots,x_{\sigma(n-1)},x_{\sigma n}) = g_{n-1}(x_2,x_3,\dots,x_{n-1},x_n)$. If in the latter identity $\sigma 3 \neq 3$, then the polynomial $g_{n-1}(x_2,x_3,\dots,x_{n-1},x_n)$ admits a nontrivial transposition $(2, \sigma 3)$ and therefore applying Lemma 16 we get a contradiction. So, we have $\sigma 1 = 1$, $\sigma 2 = 2$ and $\sigma 3 = 3$ or $\sigma 1 = 2$, $\sigma 2 = 1$ and $\sigma 3 = 3$. In both cases putting $x_1 = x_2 = x_3$ in the identity $(\overset{++}{+})$ we get $f_{n-2}(x_3,x_4,\dots,x_{n-1},x_n) = f_n(x_3,x_3,x_3,x_4,\dots,x_{n-1},x_n) = g_n(x_3,x_3,x_3,x_{\sigma 4},\dots\dots,x_{\sigma(n-1)},x_{\sigma n}) = g_{n-2}(x_3,x_{\sigma 4},\dots,x_{\sigma(n-1)},x_{\sigma n})$. Continuing this process we infer that σ is or the identity permutation or the trivial transposition $(1,2)$. Now an application of Lemma 18 completes the proof of the lemma.

P r o o f of the Theorem. Let $(B,+,\cdot)$ be a proper bisemilattice. Using Lemma 3 we infer that all polynomials $s_n(+), s_n(\cdot)$, $f_n^{\sigma_1}$ and $f_n^{\sigma_2}$ are essentially n-ary for all $\sigma_1, \sigma_2 \in S_n$ and $n \geqslant 2$. It is clear that $G(s_n(+)) \cong G(s_n(\cdot)) \cong S_n$ for all n. It is also obvious that the identity $f_n^{\sigma_1} = g_n^{\sigma_2}$ is equivalent to the identity $f_n = g_n^{\sigma}$ for some

$\sigma \in S_n$. Now using the above facts and Lemma 19 we get permuting the variables in the polynomials f_n and g_n the required assertion since

$$p_n(B,+,\cdot) \geqslant \frac{n!}{\mathrm{card}G(f_n)} + \frac{n!}{\mathrm{card}G(g_n)} + \frac{n!}{\mathrm{card}G(s_n(+))} + \frac{n!}{\mathrm{card}G(s_n(\cdot))} =$$

$$= \frac{n!}{2} + \frac{n!}{2} + \frac{n!}{n!} + \frac{n!}{n!} = 2 + n!.$$

The proof of the Theorem is completed.

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