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STABILITY OF A LINEAR RETARDED EQUATION

We consider the scalar retarded equation

$$\dot{x}(t) = -ax(t) - b\int_{-r}^{0} x(t+\tau)dg(\tau), \tag{1}$$

where $a,b \in R$ and $\mathrm{Var}_{[-r,0]}\, g = 1$.

We are interested in finding some sufficient conditions the zero solution of (1) to be asymptotically stable. The concept of asymptotical stability of the solutions of (1) is defined as in Hale [1] or Krasovskii [2]. It is known [2], [5] that this problem reduces to the Routh-Hurwitz problem for the characteristic function of (1) given by

$$\Delta(z) = z + a + b\int_{-r}^{0} \exp(z\tau)dg(\tau). \tag{2}$$

The Routh-Hurwitz problem for entire functions is posed and studied by Meiman and Cebotarev in [3], where it is said that we cannot expect an algebraic effective criterion to state for each a, b and g whether the zero solution of (1) is asymptotically stable or not. In some cases Pontryagin [4] gives and equivalent form of the Routh-Hurwitz problem, however even in this cases we hardly get an effective criterion solving the problem.

We put $c_1 = \int_{-r}^{0} |\tau|\,|dg(\tau)|$, $c_2 = \int_{-r}^{0} \tau^2 |dg(\tau)|$,
$w = \int_{-r}^{0} dg(\tau) \geqslant 0$, the latter inequality can be obtained by changing eventually the sign of b and g.

L e m m a . If functions $f(z)$ and $g(z)$ are analytic along the smooth curve C, and if $g(z) < \alpha|f(z)|$ for $z \in C$ and some $\alpha \in (0,1\rangle$, then

$$(3) \qquad |\Delta_C \operatorname{Arg}(f+g) - \Delta_C \operatorname{Arg} f| \leqslant 2 \arccos \frac{1-\alpha}{1+\alpha}.$$

Here the symbol $\Delta_C \operatorname{Arg}$ denotes the growth of argument of the respective function along C.

P r o o f . We see that neither f nor $f+g$ vanishes along C. Let $\varphi = g/f$. Using logarythmic derivative we have

$$\Delta_C \operatorname{Arg}(f+g) - \Delta_C \operatorname{Arg} f = \operatorname{Im} \int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz - \operatorname{Im} \int_C \frac{f'(z)}{f(z)}\,dz =$$

$$= \operatorname{Im} \int_C \frac{\varphi'(z)}{1+\varphi(z)}\,dz = \Delta_C \operatorname{Arg}(1+\varphi).$$

Since $|\varphi| < \alpha \leqslant 1$, we have $\operatorname{Re}(1+\varphi) \geqslant 1-\alpha \geqslant 0$, $|1+\varphi| \leqslant 1+\alpha$ and $\cos \arg(1+\varphi) = \frac{\operatorname{Re}(1+\varphi)}{|1+\varphi|} \geqslant \frac{1-\alpha}{1+\alpha} \geqslant 0$. Hence $|\Delta_C \operatorname{Arg}(1+\varphi)| \leqslant$ $\leqslant 2 \arccos \frac{1-\alpha}{1+\alpha}$.

T h e o r e m 1. Suppose N is the number of zeros of $\Delta(z)$ with positive real part and $\Delta(z)$ has no zeros on the imaginary axis. Then the growth of argument of $\Delta(z)$ along the imaginary axis $i\omega$, $-\infty < \omega < +\infty$, is $\pi - 2\pi N$, and along the semi-axis $i\omega$, $0 \leqslant \omega < \infty$, it is $\frac{\pi}{2} - \pi N$.

P r o o f . We consider the closed and positively oriented curve $\Gamma(R) = \Gamma_1(R) \cup \{-\Gamma_2(R)\}$, where $\Gamma_1(R): z = R\exp(i\omega)$, $-\frac{\pi}{2} \leqslant \omega \leqslant \frac{\pi}{2}$, and $\Gamma_2(R): z = i\omega$, $-R \leqslant \omega \leqslant R$. If

N_R is the number of zeros of $\Delta(z)$ inside $\Gamma(R)$ and if $\Delta(z)$ has no zeros on $\Gamma(R)$, then according to the principle of argument

(4) $$\Delta_{\Gamma(R)} \operatorname{Arg} \Delta(z) = 2\pi N_R.$$

For $z \in \Gamma_1(R)$ we have

$$\Delta(z) = \Delta(R\, e^{i\omega}) = R\, e^{i\omega} + a + b \int_{-r}^{0} \exp(R\tau\, e^{i\omega}) dg(\tau),$$

and

$$\left| a+b \int_{-r}^{0} \exp(R\tau\, e^{i\omega}) dg(\tau)\right| \leqslant |a| + |b| \int_{-r}^{0} \exp(R\tau \cos\omega) \, |dg(\tau)| \leqslant$$
$$\leqslant \frac{|a|+|b|}{R} |R\, e^{i\omega}|.$$

Using the Lemma we see that if $R > |a|+|b|$, then

$$\left|\Delta_{\Gamma_1(R)} \operatorname{Arg} \Delta(z) - \pi\right| = \left|\Delta_{\Gamma_1(R)} \operatorname{Arg} \Delta(z) - \Delta_{\Gamma_1(R)} \operatorname{Arg} z\right| \leqslant$$
$$\leqslant 2 \arccos \frac{1-(|a|+|b|)/R}{1+(|a|+|b|)/R},$$

whence

(5) $$\Delta_{\Gamma_1(R)} \operatorname{Arg} \Delta(z) = \pi + \varepsilon(R), \quad \varepsilon(R) \to 0 \quad \text{as} \quad R \to \infty.$$

From (4) and (5) we obtain

(6) $$\Delta_{\Gamma_2(R)} \operatorname{Arg} \Delta(z) = \pi + \varepsilon(R) - 2\pi N_R.$$

The Lemma yields that for sufficiently large R

$$\left|\Delta_{\Gamma_2(R)} \operatorname{Arg} \Delta(z)\right| \leqslant 2 \arccos \frac{1-(|a|+|b|)/R}{1+(|a|+|b|)/R}.$$

Hence the right-hand side of (6) has a limit as $R \to \infty$. Hence N is a finite number and passing to a limit in (6) we obtain our thesis. In the case of the semi-axis involved we obtain that $\Delta(\bar{z}) = \overline{\Delta(z)}$, whence we get our conclusion.

C o r o l l a r y . a. The function $\Delta(z)$ has an odd number N of zeros with positive real part if

$$a + b w < 0. \tag{7}$$

Since the conjugate of a zero of $\Delta(z)$ is a zero of $\Delta(z)$, then at least one zero of $\Delta(z)$ is real and positive.

b. The function $\Delta(z)$ has only zeros with negative real parts if either

$$a > |b|, \tag{8}$$

or

$$a + bw > 0 \quad \text{and} \quad 1 - |b| c_1 > 0. \tag{9}$$

P r o o f . a) Let $\Delta(z)$ has no zeros along the imaginary axis. Since $\arg \Delta(0) = \arg(a+bw) = \pi$ and $\lim_{\omega \to \infty} \arg i\omega = \frac{\pi}{2}$, we see that $2k\pi - \frac{\pi}{2}$, k integer, is the growth of argument of $\Delta(z)$ along $i\omega$, $0 \leqslant \omega < +\infty$. We see that $2k\pi - \frac{\pi}{2} = \frac{\pi}{2} - \pi N$, whence $N = 2k+1$.

In the case of zeros on the imaginary axis, we make the same reasoning about the function $\Delta_1(z) = \Delta(z+\varepsilon)$ for sufficiently small positive ε.

b) If (8) is true, then

$$F(\omega) \overset{df}{=} \operatorname{Re} \Delta(i\omega) = a + b \int_{-r}^{0} \cos(\omega\tau)\, dg(\tau) \geqslant a - |b| > 0 \quad \text{for} \quad \omega \in R.$$

Hence $\Delta(z)$ has no imaginary zeros. Since

$$\Delta(0) = a+bw \geqslant a-|b| > 0, \quad \lim_{\omega\to\infty} \arg\Delta(i\omega) = \frac{\pi}{2} \quad \text{and} \quad F(i\omega) > 0,$$

we obtain that $\frac{\pi}{2}$ is the growth of argument of $\Delta(z)$ along $i\omega$, $0 \leqslant \omega < +\infty$, whence $N = 0$.

Let (9) be true and let

$$G(\omega) \overset{df}{=} \operatorname{Im}\Delta(i\omega) = \omega + b\int_{-r}^{0} \sin(\omega\tau)dg(\tau) \quad \text{for } \omega \in R.$$

Then

$$G'(\omega) = 1 + b\int_{-r}^{0} \tau \cos\omega\tau\, dg(\tau) \geqslant 1-|b|c_1 > 0.$$

Hence $\Delta(z)$ has no imaginary zeros and $G(\omega)$ is an increasing function. Since $\arg\Delta(0) = \arg(a+bw) = 0$ and $\lim_{\omega\to\infty} \arg\Delta(i\omega) = \frac{\pi}{2}$, we obtain $N = 0$.

T h e o r e m 2. Let $a+bw > 0$ and let ω_1 be the smallest positive root of the equation $F(\omega) = 0$. If

$$a^2 + \omega_1^2 > b^2, \tag{10}$$

then all the zeros of $\Delta(z)$ have negative real parts.

The thesis is also true, if F has no real roots.

P r o o f . The function $\Delta(z)$ has no imaginary zeros. Indeed, for $0 \leqslant \omega < \omega_1$ we have $F(\omega) \neq 0$. If $\Delta(i\omega) = 0$ for some $\omega \geqslant \omega_1$, then $a+i\omega = -b\int_{-r}^{0} \exp(i\omega\tau)dg(\tau)$, whence $b^2 \geqslant \left|b\int_{-r}^{0} \exp(i\omega\tau)dg(\tau)\right|^2 = |a+i\omega|^2 = a^2+\omega^2 \geqslant a^2+\omega_1^2$, a contradiction with (10).

Suppose that $\Delta(z)$ has zeros with positive real parts. According to Theorem 1 we see that the growth of the argument of $\Delta(z)$ along the semi-axis $\{i\omega,\ 0\leqslant\omega<+\infty\}$, is at most $-\frac{\pi}{2}$. Since $\arg\Delta(0)=\arg(a+bw)=0$ and $\lim_{\omega\to\infty}\arg\Delta(i\omega)=$ $=\frac{\pi}{2}$, we see that $\Delta(i\omega_2)=\varkappa(a+i\omega_2)$ for some $\omega_2\geqslant\omega_1$ and $\varkappa\leqslant 0$. Then

$$|a+i\omega_2-\Delta(i\omega_2)| = (1-\varkappa)(a+i\omega_2)\geqslant\sqrt{a^2+\omega_2^2}\geqslant\sqrt{a^2+\omega_1^2}>|b|$$

and on the other hand

$$|a+i\omega_2-\Delta(i\omega_2)| = |b|\int_{-r}^{0}\exp(i\omega\tau)dg(\tau)\leqslant|b|,$$

a contradiction. The proof is completed.

We find some estimates of ω_1.

1^o. $$F'(\omega) = -b\int_{-r}^{0}\tau\ \sin\omega\tau\, dg(\tau)\geqslant -bc_1,$$

whence

$$F(\omega) = F(0)+\int_0^{\omega}F'(\omega)d\omega\geqslant a+bw-c_1\omega,$$

so $\omega_1\geqslant(a+bw)/c_1$.

2^o. $$F'(\omega) = -b\int_{-r}^{0}\tau\ \sin\omega\tau\, dg(\tau)\geqslant|b|\int_{-r}^{0}\tau\ |\sin\omega\tau||dg(\tau)|\geqslant -\omega c_2,$$

whence

$$F(\omega) = F(0)+\int_0^{\omega}F'(\omega)d\omega\geqslant a+bw-\tfrac{1}{2}\omega^2c_2,$$

so $\omega_1\geqslant\{2(a+bw)/c_2\}^{1/2}$.

We collect the obtained results using stability notions for eq. (1). Since (1) is linear the stability notions can be related to the whole equation.

T h e o r e m 3. Equation (1) is unstable if inequality (7) is satisfied and it is asymptotically stable in each of the cases:

a) $a > |b|$,

b) $a+bw > 0$ and $|b|c_1 < 1$,

c) $a+bw > 0$ and either the smallest positive root ω_1 of

$$a + b \int_{-r}^{0} \cos \omega\tau \, dg(\tau) = 0 \tag{11}$$

satisfies the inequality (10), or equation (11) has no real roots.

d) $a+bw > 0$ and

$$(a+bw)^2 > b^2(b^2-a^2)c_1^2, \tag{12}$$

e) $a+bw > 0$ and

$$2(a+bw) > |b|(b^2-a^2)c_2. \tag{13}$$

If g is an increasing function, then $w = 1$ and instead of (12) and (13) we can take respectively

$$a+b > b^2(b-a)c_1^2 \tag{12'}$$

and

$$2 > |b|(b-a)c_2. \tag{13'}$$

P r o o f . Follows from the previous considerations. Let us mention that cases d) and e) are corollaries of c). If g is increasing, then a+bw becomes a+b and (12') and (13') are obtained from (12) and (13) respectively devided by $a+b > 0$.

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