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Shyam K. Bajpai

# ON THE COEFFICIENTS OF FUNCTIONS WHOSE REAL PART IS BOUNDED

## 1. Introduction

In this note we obtain the estimations of the coefficients of the generalised class of functions of Caratheodory class of functions and the family generated by Caratheodory functions by using a wellknown method of Clunie (J. London Math. Soc. 34 (1959), p.215-216).

Let $P(z)$ be a function regular in $E = \{z \mid |z| < 1\}$ and be such that $P(0) = 1$, i.e.,

$$(1.1) \qquad P(z) = 1 + \sum_{k=n}^{\infty} p_k z^k, \quad n = 1,2,\ldots$$

We say that $P \in \mathcal{P}_{m,M}[\mu,\vartheta,\alpha,t]$ if and only if the following condition is satisfied

$$(1.2) \qquad \left| \frac{P(z)e^{i\vartheta} - i\sin\vartheta - \mu\cos\vartheta}{(1-\mu)\cos\vartheta} - m - \alpha - it \right| < M$$

for all $z \in E$, $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$, $\mu < 1$, $\frac{1}{2} < m$, $0 < M$, $-\infty < \alpha < 1$, $-\infty < t < \infty$ and $D = (A+B)(1-\mu)\cos\vartheta > 0$ where $MA = [M^2-m^2+m(1-2\alpha)+\alpha(1-\alpha)+it-t^2]$ and $MB = [-1+m+\alpha-it]$.

The following lemma gives the representation theorem for $P \in \mathcal{P}_{m,M}(\mu,\vartheta,\alpha,t)$.

L e m m a. $P \in P_{m,M}(\mu,\vartheta,\alpha,t)$ if and only if there exists some $\omega(z)$ regular in $E$, $\omega(0) = 0$, $|\omega(z)| < 1$ and that

$$P(z) = \frac{1 + (De^{-i\vartheta} - B)\omega(z)}{1 - B\omega(z)} \tag{1.3}$$

P r o o f . Let us write

$$f(z) = \frac{1}{M}\left[\frac{P(z)e^{i\vartheta} - i\sin\vartheta - \mu\cos\vartheta}{(1-\mu)\cos\vartheta} - m - \alpha - it\right].$$

Then, $|f(z)| < 1$ for $z \in E$ and $f(0) = \frac{1}{M}[1-m-\alpha-it] = -\bar{B}$. Then, by Schwarz's lemma, there exists a $\omega(z)$ regzlar in E, $\omega(0) = 0$ with $|\omega(z)| < 1$ such that

$$\omega(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)} .$$

Hence, by solving the above equation, we have

$$f(z) = \frac{f(0) + \omega(z)}{1 + \overline{f(0)}\omega(z)} .$$

This in turn is equivalent to (1.3). Converse also follows.

As a consequence of our lemma we obtain the classes introduced in ([1], [4], [5]).

Also, if we let $P(z) = -\frac{zF'(z)}{F(z)}$, $F(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k$ and $F$ regular in $E_R = \{z \mid 0 < |z| < 1\}$, in (1.2), then this generalizes the classes introduced in ([1], [2], [3]). The class of such functions we denote by $F_{m,M}(\mu,\vartheta,\alpha,t)$. Hence $F \in F_{m,M}(\mu,\vartheta,\alpha,t)$ if and only if

$$(1.4)\qquad -\frac{zF'(z)}{F(z)} = \frac{1 + (De^{-i\nu} - B)\omega(z)}{1 - B\omega(z)}$$

## 2. Two theorems

We prove the following:

Theorem 1. If $P \in \mathcal{P}_{m,M}(\mu,\nu,\alpha,t)$ and

$$P(z) = 1 + \sum_{k=n}^{\infty} a_k z^k, \qquad n = 0,1,2,\ldots$$

then, we have

$$(2.1)\qquad |a_p| \leqslant D \quad \text{for} \quad p \geqslant n, \quad n = 0,1,\ldots$$

and

$$(2.2)\qquad \sum_{k=1}^{\infty} |a_k|^2 \leqslant MD(1-\mu)\cos\nu .$$

The inequality (2.1) is sharp for all $p$.

Proof. By lemma, we have

$$(2.3)\qquad P(z)-1 = \left\{De^{-i\nu} - B + BP(z)\right\}\omega(z) \quad \text{and} \quad \omega(z) = \sum_{k=n}^{\infty} b_k z^k.$$

Also, by equating the coefficient of $z^n$, we have

$$(2.4)\qquad |a_n| \leqslant De^{-i\nu}.$$

The equation (2.3) can be written in the following form for $p > n$

$$(2.5)\qquad \sum_{k=n} a_k z^k + \sum_{k=p+1}^{\infty} d_k z^k = \left\{De^{-i\nu} - B\sum_{k=n}^{p-1} a_k z^k\right\}\omega(z).$$

The equation (2.5) gives us

$$|D|^2 + |B|^2 \sum_{k=n}^{p-1} |a_k|^2 r^{2k} \geqslant \sum_{k=n}^{p} |a_k|^2 r^{2k} + \sum_{k=p+1}^{\infty} |d_k|^2 r^{2k}.$$

Since $(1-|B|^2) > 0$, it gives $|a_p| \leqslant D$. The result is sharp whenever $\omega(z) = z^p$.

Again from (2.3) we have

$$\sum_{k=n}^{\infty} |a_k|^2 r^{2k} \leqslant D^2 + |B|^2 \sum_{k=n}^{\infty} |a_k|^2 r^{2k}.$$

Hence, we get

$$\sum_{k=n}^{\infty} |a_k|^2 \leqslant \frac{D^2}{1-|B|^2} = \left\{\frac{MD^2}{A+B}\right\}.$$

This completes the proof of the theorem.

Theorem 2. Let

$$F(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k, \quad n = 0,1,2,\ldots,$$

$F \in \mathcal{F}_{m,M}(\mu,\nu,\alpha,t)$ and also $1+2(1-\mu)\cos\nu \ \mathrm{Re}\{MBe^{i\nu}\} - M(A+B)(1-\mu)^2\cos^2\nu \geqslant 0$ and $1+(1-\mu)\cos\nu \ \mathrm{Re}\{MBe^{i\nu}\} > 0$. Then we have

$$(2.6) \qquad |a_0| \leqslant D \ \text{ if } \ n = 0,$$

$$(2.7) \qquad |a_m| \leqslant \frac{D}{m+1} \ \text{ if } \ m \geqslant n, \ \ n = 0,1,\ldots$$

$$(2.8)\qquad \sum_{k=n}^{2n} (k+1)^2 |a_k|^2 \leqslant D^2 \quad \text{if} \quad a_0 = 0,$$

$$(2.9)\qquad \sum_{k=mn}^{(m+1)n} (k+1)^2 |a_k|^2 \leqslant D^2 \quad \text{if} \quad a_0 = 0,\ m = 2,3,\ldots$$

and finally if $a_0 = 0$, then

$$(2.10)\qquad \sum_{n=k}^{\infty} \left\{ (k+1)^2 - |De^{-i\vartheta} - B(k+1)|^2 \right\} |a_k|^2 \leqslant D^2.$$

P r o o f . From (1.4), we have

$$(2.11)\qquad -a_0 - \sum_{k=n}^{\infty} a_k (k+1) z^k = \left[ \frac{De^{-i\vartheta}}{z} + a_0 (De^{-i\vartheta} - B) + \right.$$

$$\left. + \sum_{k=n}^{\infty} \left\{ De^{-i\vartheta} - B(k+1) \right\} a_k z^k \right] \omega(z),$$

where $\omega(z) = z \sum_{k=n}^{\infty} b_k z^k$ and $n \geqslant 1$.

Comparing the coefficients of $z^k$ for $k = 0, n, \ldots, 2n$, we have

$$(2.12)\qquad -a_0 = De^{-i\vartheta} \omega'(0) = De^{-i\vartheta} b_0$$

$$(2.13)\qquad -(n+p+1) a_{n+p} = De^{-i\vartheta} b_p, \quad p = 0,1,2,\ldots n \quad \text{and} \quad a_0 = 0.$$

Also, if $a_k = 0$ for $k < n$, $n = 0,1,2,\ldots$ then for $p > n$, we have

$$(2.14)\qquad -\sum_{k=n}^{n+p}(k+1)a_k z^k + \sum_{k=n+p+1}^{\infty} d_k z^k =$$

$$= \left[\frac{De^{-i\nu}}{z} + \sum_{k=n}^{p-1}\left[De^{-i\nu} - B(k+1)\right] a_k z^k\right]\omega(z),$$

where $\sum_{k=n+p+1}^{\infty} d_k z^k$ is absolutely and uniformly convergent on compact subsets of $\{z \mid |z| < 1\}$. Letting $z = re^{i\theta}$ and using the assumption that $\omega(z)$ is bounded by 1, we obtain

$$(2.15)\qquad |a_0| \leqslant D \text{ by (2.12) and } n = 0$$

$$(2.16)\qquad |a_{n+p+1}| \leqslant D \quad \text{if} \quad p=0,1,\ldots,n$$

and $a_0 = 0$ by (2.13).

$$(2.17)\qquad \sum_{k=n}^{2n}(k+1)^2\,|a_k|^2 \leqslant D^2 \sum_{k=1}^{n-1}|b_k|^2 \leqslant D^2$$

if $a_0 = 0$ by (2.13)

$$(2.18)\qquad \sum_{k=p}^{p+n}(k+1)^2|a_k|^2 \leqslant D^2 + \sum_{k=n}^{p-1}\left\{|De^{-i\nu} - B(k+1)|^2 - (k+1)^2\right\}|a_k|^2 \leqslant D^2$$

for $p \geqslant n+1$ by (2.14)

$$(2.19) \qquad \sum_{k=n}^{m} (k+1)^2 |a_k|^2 \leqslant D^2 + \sum_{k=n}^{m-1} \{|De^{-i\vartheta} -$$

$$- B(k+1)|^2\} |a_k|^2 \quad \text{by (2.14).}$$

Also, (2.18) and (2.19) give

$$(2.20) \qquad \sum_{k=mn}^{(m+1)n} (k+1)^2 |a_k|^2 \leqslant D^2, \quad m=2,3,\ldots$$

and

$$(2.21) \qquad |a_m| \leqslant \left[\left(\frac{1}{m+1}\right)^2 \left\{D^2 + \sum_{k=n}^{m-1} \left[|De^{-i\vartheta} - B(k+1)|^2 -\right.\right.\right.$$

$$\left.\left.\left. - (k+1)^2\right] |a_k|^2\right\}\right]^{1/2} \leqslant \frac{D}{m+1}$$

for $m \geqslant n$.

The inequalities (2.15) through (2.21) prove Theorem 2.

As an application of Theorem 2, we deduce the result of Libera and Livingston ([5], p.201). Let $0 \leqslant \alpha < 1$, $\vartheta = 0 = \mu$, $\varrho = m = M$, then, we find the expressions

$$1 + (1-\mu)\cos\vartheta \ \operatorname{Re}\{MBe^{i\vartheta}\} = \varrho + \alpha > 0$$

and

$$1 + 2(1-\mu)\cos\vartheta \ \operatorname{Re}\{MBe^{i\vartheta}\} - M(A+B)(1-\mu)^2 \cos^2\vartheta =$$

$$= 2\varrho\alpha + \alpha^2 + t^2 > 0$$

are positive, hence no more assumption. Also in this case (2.7) coincides Theorem 3 of ([5], p.201). Hence, our theorem is a generalization. Similarly, our Theorem 2, contains the results proved in ([1], [2], [3]). Also, Theorem 1, contains the results proved in ([1], [4], [5]).

## REFERENCES

[1] A. Szynal, J. Szynal, J. Zygmunt: On the coefficients of functions whose real part is bounded, Ann. Univ. Marie Curie-Skłodowska, Lublin, 26 (1972) 63-69.

[2] H.S. Gopalkrishana, V.S. Shetiya: Coefficient estimates for spiral like functions, Ann. Polon. Math. 35 (1977) 1-9.

[3] Z.J. Jakubowski: On the coefficients of starlike functions of some classes, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. 19 (1971) 811-815.

[4] Z.J. Jakubowski: On the coefficients of Caratheodory functions. Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys., 19 (1971) 805-809.

[5] B.J. Libera, A.E. Livingston: Bounded functions with positive real part, Czechoslovak Math. J. 22 (1972) 195-209.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, PURDUE UNIVERSITY AT FORT WAYNE, 2101 COLISEUM BOULEVARD EAST, FORT WAYNE, INDIANA, 46805, USA
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