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SET STABILITY FOR A FUNCTIONAL EQUATION OF ITERATIVE TYPE

1. G.A.Shanholt has proved in [1] stability theorems for a difference equation. It is the aim of this paper to prove similar results for the equation of iterative type

$$(1) \qquad \varphi(f(x)) = g(x,\varphi(x)).$$

Throughout the paper we will assume the hypothesis

(H_1) $\varphi : I \to H$, $f : I \to I$, $g : I \times X \to H$, $I = (0,b)$, H is an open connected subset of Banach space B and f, g are continuous functions. Moreover f is strictly increasing in $(0,b)$ and $0 < f(x) < b$ for $x \in I$.

In this paper we adopt the following notation
$N(A,\varepsilon) = \{x: d(x,A) < \varepsilon\}$;
$K = \{\Phi : \Phi : R_+ \cup \{0\} \to R_+ \cup \{0\}$, Φ is strictly increasing, continuous function and $\Phi(0) = 0\}$; for a function $\varphi: I \to H$, a set $G \subset H$ and $\varepsilon > 0$ the relation $d(\varphi,G) < \varepsilon$ denotes that for every $x \in I$ we have $d(\varphi(x),G) < \varepsilon$;

$I_0 = [f(x_0), x_0]$ for $x_0 \in I$;

$\varphi_0 : I_0 \to H$ will denote a continuous function such that $\varphi_0(f(x_0)) = g(x_0,y_0)$, where y_0 is an arbitrary point of the set H and $\varphi_0(x_0) = y_0$;

$\varphi(x,x_0,y_0,\varphi_0)$ will denote a continuous solution of equation (1) defined on $(0,x_0]$ and such that $\varphi/_{I_0} = \varphi_0$.

R e m a r k 1. Observe that under the hypothesis (H_1) for given $x_0 \in I$, $y_0 \in H$ a solution $\varphi(x,x_0,y_0,\varphi_0)$ exists, because H as an open and connected subset of Banach space is arc-connected, then there exists an arc from the point (x_0,y_0) to the point $(y_0,g(x_0,y_0))$ which may be extended "by equation (1)" to the interval $(0,x_0]$.

In the sequel we will assume the hypothesis:

(H_2) There exist a closed and connected set G and $\alpha > 0$ such that $N(G,\alpha) \subset H$ for which there exists a $k > 0$ such that for $\varepsilon \in (0,\alpha)$ the inequality $d(y_0,G) < \varepsilon$ implies $d(g(x,y_0),G) < k\varepsilon$.

R e m a r k 2. If hypothesis (H_2) is satisfied and G is a connected set, then for any x_0, taking y_0 such that $d(y_0,G) < r$, $r := \min(\frac{1}{k}\varepsilon, \varepsilon)$, we have $d(g(x_0,y_0),G) < \varepsilon$. It implies, as $N(G,\varepsilon)$ is arc-connected set, that we may take φ_0 in $N(G,\varepsilon)$ and for such a φ_0 there exists $\varphi(x,x_0,y_0,\varphi_0)$ (see Remark 1).

2. We will adopt following definitions, which are in the spirit of definitions given by G.A.Shanholt in [1], with necessary and natural modifications.

D e f i n i t i o n 1. Let $G \subset H$ be a closed subset of H. We say

(i) G is stable if for every $x_0 \in I$ and $\varepsilon > 0$ there exists a $\delta = \delta(x_0,\varepsilon) > 0$ such that $d(\varphi_0,G) < \delta$ implies that $\varphi(x,x_0,y_0,\varphi_0)$ exists and $d(\varphi(x,x_0,y_0,\varphi_0),G) < \varepsilon$;

(ii) G is uniformly stable if it is stable and δ in (i) is independent of x_0;

(iii) G is asymptotically stable if it is stable and if for every $x_0 \in I$ there exists $\eta = \eta(x_0) > 0$ such that $d(\varphi_0,G) < \eta$ implies $d(\varphi(x,x_0,y_0,\varphi_0),G) \to 0$ as $x \to 0$;

(iv) G is uniformly asymptotically stable if it is uniformly stable, and η in (iii) is independent of x_0 and limit is uniform in $x_0,y_0,\varphi_0(t)$ $(t \in I_0)$, for $(x_0,y_0,\varphi_0(t)) \in I \times N(G,\eta) \times N(G,\eta)$.

D e f i n i t i o n 2. Let $V : I \times N(G,\alpha) \to R_+ \cup \{0\}$. We say:

(i) V is positive definite with respect to the set G if there exists a $\Phi \in K$ such that $\Phi(d(y,G)) \leqslant V(x,y)$ for $(x,y) \in I \times N(G,\alpha)$;

(ii) V is decrescent with respect to the set G if there exists a $\psi \in K$ such that $\psi(d(y,G)) \geqslant V(x,y)$ for $(x,y) \in I \times N(G,\alpha)$;

(iii) V satisfies property (B) with respect to the set G if for each $\varepsilon > 0$ and $x_0 \in I$ there exists $\delta = \delta(x_0,\varepsilon) \in (0,\alpha)$ such that $d(y,G) < \delta \Longrightarrow V(x_0,y) < \varepsilon$;

(iv) V is Lapunov function for (1) on $I \times N(G,\alpha)$ if it satisfies property (B) with respect to G and $\Delta V(x,y) \leqslant 0$, where $\Delta V(x,y) := V(f(x),g(x,y)) - V(x,y)$ for $(x,y) \in I \times N(G,\alpha_0)$, $\alpha_0 = \min\{\alpha, \frac{\alpha}{k}\}$.

D e f i n i t i o n 3. A Lapunov function V for (1) on $N(G,\alpha)$ has a strongly negative difference along solutions of (1) if there exists a $\beta > 0$ such that $\Delta V(x,y) \leqslant -\beta|g(x,y) - y|$ for $(x,y) \in I \times N(G,\alpha_0)$, $\alpha_0 = \min\{\alpha, \frac{\alpha}{k}\}$.

T h e o r e m 1. If hypothesis (H_1) and (H_2) are satisfied and if there exists a Lapunov function V for (1) on $N(G,\alpha)$ and it has strongly negative difference along solutions of (1), then G is stable. Moreover, for each $x_0 \in I$ there exists a $\vartheta > 0$ such that for an $y_0 \in N(G,\vartheta)$ the solution $\varphi(x,x_0,y_0,\varphi_0)$, where $d(\varphi_0,G) < \vartheta$, is bounded.

P r o o f . Suppose that G is not stable. Then there exist a $\varepsilon_0 \in (0,\alpha_0)$ and $x_0 \in I$ such that for any $\gamma \in (0,\varepsilon_0)$ there exist $y_0 \in N(G,\gamma)$ and $\bar{x} < x_0$ such that $d(\varphi(\bar{x},x_0,y_0,\varphi_0),G) \geqslant \varepsilon_0$ with $d(\varphi_0,G) < \gamma$ (by virtue of Remarks 1 and 2 suitable φ_0 and φ exist). As V has property (B), then for $\frac{\beta\varepsilon_0}{2}$ there exists $\delta\left(\frac{\beta\varepsilon_0}{2}, x_0\right) \in \left(0, \frac{\varepsilon_0}{2}\right)$ such that $d(y,G) < \delta$ implies $V(x_0 y) < \frac{\beta\varepsilon_0}{2}$. Let us take $\gamma < \min\left(-\frac{\varepsilon_0}{2}, \delta\right)$. Then $d(\varphi_0,G) < \gamma$ and $d(\varphi(\bar{x}),G) \geqslant \varepsilon_0$.

Let $\bar{x} = f^{\bar{n}}(t_0)$ for a $t_0 \in I_0$, $\bar{n}$ a positive integer (it follows from hypothesis (H_1) that x_0 and t_0 exist) and let $d(\varphi(f^n(t_0)),G) < \varepsilon_0$ for $\bar{n} > n \geqslant 0$. It implies that $V(\bar{x},\varphi(\bar{x}))$ exists.

Now let us compute

$$V(\bar{x},\varphi(\bar{x})) = V(f^{\bar{n}}(t_0),\varphi(f^{\bar{n}}(t_0))) = V(f(f^{\bar{n}-1}(t_0)),\varphi(f(f^{\bar{n}-1}(t_0)))) =$$

$$= V(f(f^{\bar{n}-1}(t_0)),g(f^{\bar{n}-1}(t_0)),\varphi(f^{\bar{n}-1}(t_0)))) =$$

$$= V(f^{\bar{n}-1}(t_0),\varphi(f^{\bar{n}-1}(t_0))) + \Delta V(f^{\bar{n}-1}(t_0),\varphi(f^{\bar{n}-1}(t_0))) = \dots =$$

$$= V(t_0,\varphi(t_0)) + \sum_{i=0}^{\bar{n}-1} \Delta V(f^i(t_0),\varphi(f^i(t_0))) \leqslant$$

$$\leqslant V(t_0,\varphi(t_0)) - \beta \sum_{i=0}^{\bar{n}-1} \left| g(f^i(t_0),\varphi(f^i(t_0))) - \varphi(f^i(t_0)) \right| \leqslant$$

$$\leqslant V(t_0,\varphi(t_0)) - \beta \left| \sum_{i=0}^{\bar{n}-1} g(f^i(t_0),\varphi(f^i(t_0))) - \varphi(f^i(t_0)) \right| \leqslant$$

$$\leqslant V(t_0,\varphi(t_0)) - \beta \left| \sum_{i=0}^{\bar{n}-1} \varphi(f^{i+1}(t_0)) - \varphi(f^i(t_0)) \right| =$$

$$= V(t_0,\varphi(t_0)) - \beta \left| \varphi(f^{\bar{n}}(t_0)) - \varphi(t_0) \right| <$$

$$< \frac{\beta\varepsilon_0}{2} + \beta(d(\varphi(t_0),G) - d(\varphi(\bar{x}),G)) \leqslant \frac{\beta\varepsilon_0}{2} + \beta(\gamma-\varepsilon_0) \leqslant$$

$$\leqslant \frac{\beta\varepsilon_0}{2} - \beta\varepsilon_0 + \beta\gamma = \beta\left(-\frac{\varepsilon_0}{2} + \gamma\right) < 0,$$

which contradicts our supposition.

Now we prove the second part of the theorem.

For an x_0 we chose $\vartheta(x_0) < \min(\alpha,\delta)$ (where δ satisfies the definition of stability of G). For $y \in N(G,\vartheta)$ the solution $\varphi(x,x_0,y,\varphi_0)$ exists and it is in $N(G,\alpha)$, for $x \leqslant x_0$. Suppose that for some $y_0 \in N(G,\vartheta)$ the solution $\varphi(x,x_0,y_0,\varphi_0)$ is unbounded i.e. for some $x_k \to 0$ we have $|\varphi(x_k)| \to \infty$. Let us take a positive integer j such that $V(t_0,\varphi(t_0)) - \beta|\varphi(x_j) - \varphi(t_0)| < 0$, where $t_0 \in I_0$, $x_j = f^{n_j}(t_0)$ (this is possible because V is a Lapunov function). Computing as above we have

$$V(x_j,\varphi(x_j)) \leqslant V(t_0,\varphi(t_0)) - \beta|\varphi(x_j) - \varphi(t_0)| < 0,$$

because of this contradiction the proof is ended.

T h e o r e m 2. Under assumptions of Theorem 1, if moreover V is decrescent with respect to the set G, then G is uniformly stable.

P r o o f . Suppose that G is not uniformly stable. Then there exists $\varepsilon_0 \in (0,\alpha_0)$ such that for any $\gamma \in (0,\varepsilon_0)$ there exist $x_0 \in I$, $y_0 \in N(G,\alpha)$ and $\bar{x} < x_0$ such that

$$d(\varphi(\bar{x},x_0,y_0,\varphi_0),G) \geqslant \varepsilon_0 \quad \text{with} \quad d(\varphi_0,G) < \gamma .$$

As V is descrescent with respect to the set G, then for $\frac{\beta\varepsilon_0}{2}$ there exists $\delta\left(\frac{\beta\varepsilon_0}{2}\right) \in \left(0,\frac{\varepsilon_0}{2}\right)$ such that $\psi(t) \leqslant \frac{\beta\varepsilon_0}{2}$ (here ψ is an element of K satisfying Definition 2, part (ii)). It is easy to verify that if V is decrescent with respect to the set G, then it also fulfils property (B). Then $d(y,G) < \delta$ implies $V(x_0,y) < \frac{\beta\varepsilon_0}{2}$. Further argument does not differ from the one used to establish Theorem 1. This completes the proof.

We do not know under what assumptions the set G is asymptotically stable or uniformly asymptotically stable.

REFERENCE

[1] G.A. S h a n h o l t : Set stability for difference equations, Internat. J. Control, 19 (1974) 309-314.

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