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THE ESTIMATIONS OF COEFFICIENTS FOR SOME SUBCLASSES OF SPIRALLIKE FUNCTIONS

Let H denote the class of all functions holomorphic in the unit disc $U = \{z: |z| < 1\}$ and let Ω denote the class of such functions $\omega \in H$, that $\omega(0) = 0$, $|\omega(z)| < 1$ in U. Let $f, F \in H$. We say that f is subordinate to F in U and write $f \prec F$, if there exists a function $\omega \in \Omega$ such that $f(z) = F(\omega(z))$.

In paper [5] the following class of functions was introduced

$$P_k(A,B) = \left\{ p(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \ldots \in H : p(z) \prec \frac{1+Az}{1-Bz} \right\},$$

where A,B are arbitrary complex numbers such that $|A| \leqslant 1$, $|B| \leqslant 1$ and k is an arbitrary natural number.

Now we introduce the following class

$$S_k(A,B) = \left\{ f(z) = z + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \ldots \ H : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1-Bz} \right\}.$$

It is easy to observe that if $f \in S_k(A,B)$ then $zf'(z)/f(z)$ takes values in some domain included in a half-plane with

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zero on the boundary and therefore f is univalent spirallike function. We also have that if f is normalized, then

(1) $$f \in S_k(A,B) \iff p(z) = zf'(z)/f(z) \in P_k(A,B).$$

The main purpose of this paper is to give some estimations for the coefficients of the functions f which belong to the class $S_k(A,B)$. The estimations of $|f(z)|$ and $|f'(z)|$ in this class are also given.

Theorem 1. If $f \in S_k(A,B)$ and $|z| = r < 1$, then

(2) $$\frac{r(1-|B|r^k)^{\alpha-\beta}}{(1+|B|r^k)^{\alpha+\beta}} \leqslant |f(z)| \leqslant \frac{r(1+|B|r^k)^{\alpha-\beta}}{(1-|B|r^k)^{\alpha+\beta}} \quad \text{if} \quad B \neq 0,$$

(3) $$r \exp\left\{-\frac{|A|r^k}{k}\right\} \leqslant |f(z)| \leqslant r \exp\left\{\frac{|A|r^k}{k}\right\} \text{ if } B=0$$

and

(4) $$\frac{(|1+A\bar{B}r^{2k}|-|A+B|r^k)(1-|B|r^k)^{\alpha-\beta-1}}{(1+|B|r^k)^{\alpha+\beta+1}} \leqslant |f'(z)| \leqslant$$

$$\leqslant \frac{(|1+A\bar{B}r^{2k}|+|A+B|r^k)(1+|B|r^k)^{\alpha-\beta-1}}{(1-|B|r^k)^{\alpha+\beta+1}} \quad \text{if} \quad B \neq 0,$$

(5) $$(1-|A|r^k)^{-\frac{|A|r^k}{k}} \leqslant |f'(z)| \leqslant (1+|A|r^k)^{\frac{|A|r^k}{k}} \quad \text{if} \quad B = 0,$$

where for $B \neq 0$ we put

(6) $$\alpha = \frac{1}{2} k^{-1} |1+A/B|, \quad \beta = \frac{1}{2} k^{-1} \operatorname{Re}\{1+A/B\}.$$

The results are sharp and we obtain equalities in (2)-(5) for such functions f which satisfy the following equation

$$\frac{zf'(z)}{f(z)} = \frac{1 + A\varepsilon z^k}{1 - B\varepsilon z^k}, \qquad |\varepsilon| = 1.$$

P r o o f . In [5] it was proved that if $f \in P_k(A,B)$ and $|z| = r < 1$, then

$$(7) \quad \frac{1-|A+B|r^k + \frac{1}{2}(A\bar{B}+\bar{A}B)r^{2k}}{1-|B|^2r^{2k}} \leqslant \operatorname{Re} p(z) \leqslant \frac{1+|A+B|r^k + \frac{1}{2}(A\bar{B}+\bar{A}B)r^{2k}}{1-|B|^2r^{2k}},$$

$$(8) \quad \frac{|1+A\bar{B}r^{2k}| - |A+B|r^k}{1-|B|^2r^{2k}} \leqslant |p(z)| \leqslant \frac{|1+A\bar{B}r^{2k}| + |A+B|r^k}{1-|B|^2r^{2k}}.$$

It is known that for $z = \varrho e^{i\theta}$ we have the equality:

$$(9) \quad \varrho \frac{\partial}{\partial \varrho} \log |f(z)/z| + 1 = \operatorname{Re}\{zf'(z)/f(z)\}.$$

Using (1), (7) and (9) we have

$$(10) \quad \frac{\partial}{\partial \varrho} \log |f(z)/z| \geqslant \frac{-|A+B|\varrho^{k-1} + [\operatorname{Re}(A\bar{B}) + |B|^2]\varrho^{2k-1}}{1-|B|^2\varrho^{2k}}.$$

Now if $B \neq 0$ then

$$\log|f(re^{i\theta})/(re^{i\theta})| = \int_0^r \frac{\partial}{\partial \varrho} \log|f(\varrho e^{i\theta})/(\varrho e^{i\theta})| \, d\varrho \geqslant$$

$$\geqslant \int_0^r \frac{-|A+B| + [\operatorname{Re}(A\bar{B} + |B|^2]\varrho^k}{1-|B|^2\varrho^{2k}} \varrho^{k-1} \, d\varrho = \log \frac{(1-|B|r^k)^{\alpha-\beta}}{(1+|B|r^k)^{\alpha+\beta}},$$

which gives the left side of inequality (2). If $B = 0$, then

$$\log|f(re^{i\theta})/(re^{i\theta})| \geqslant \int_0^r -|A|\varrho^{k-1}d\varrho = -|A|r^k/k$$

which gives the left side of inequality (3).

In the same way, using (1), (9) and the right side of (7), we obtain the right sides of (2) and (3). Now noting that $|f'(z)| = |f(z)/z|\ |zf'(z)/f(z)|$ and using (1), (8), (2) and (3) we obtain immediately (4) and (5).

Now, to estimate the coefficients we will need the following lemmas:

L e m m a 1. Let $A, B, |B| \leqslant 1$ be some given complex numbers and put

$$w_l := B + (A+B)/l, \quad l = 1,2,3,\dots .$$

1°. If $|B| < 1$, then there exists a such natural number l_0 that $|w_{l_0}| \leqslant 1$.

2°. If for some natural number l_0 we have $|w_{l_0}| \leqslant 1$, then for every $l \geqslant l_0$, we also have $|w_l| \leqslant 1$.

3°. If for some natural number l_0 we have $|w_{l_0}| > 1$, then for every $l \geqslant l_0$ we have

$$(11) \qquad \frac{|w_l|^2 - 1}{|w_{l_0}|^2 - 1} \leqslant 1.$$

P r o o f . 1°. Since $\lim_{l\to\infty} \frac{A+B}{l} = 0$, then for every ε, in particular for $\varepsilon = 1-|B| > 0$, there exists l_0 such that for every $l \geqslant l_0$ we have $\left|\frac{A+B}{l}\right| < \varepsilon = 1-|B|$. Thus $|w_l| = \left|B + \frac{A+B}{l}\right| \leqslant |B| + \left|\frac{A+B}{l}\right| < |B|+1-|B| = 1$.

2°. For every $l > l_0$ the points w_l may be written in the form

(12) $w_l = (1-\lambda)\,B + \lambda w_{l_0}$, where $\lambda = l/l_0 \in (0,\ 1]$.

Because B and w_{l_0} belong to the closed unit disc, then w_l also belong to it.

3^o. Since $|B| \leqslant 1 < |w_{l_0}|$, then by (12) we have

$$|w_l| \leqslant (1-\lambda)|B| + \lambda|w_{l_0}| \leqslant (1-\lambda)|w_{l_0}| + \lambda|w_{l_0}| = |w_{l_0}|$$

which is equivalent to (11).

L e m m a 2. For any complex numbers A, B and natural numbers k, q the following equality

$$(13)\quad |A+B|^2 + \sum_{v=1}^{q-1}\left\{\left[|vkB+A+B|^2-(vk)^2\right]\left[\frac{1}{v!}\prod_{\mu=0}^{v-1}\left|\mu B + \frac{A+B}{k}\right|\right]^2\right\} =$$

$$= \left[\frac{k}{(q-1)!}\prod_{\mu=0}^{q-1}\left|\mu B + \frac{A+B}{k}\right|\right]^2$$

holds, where by the definition we put $\sum_{v=m}^{n}\gamma_v := 0$ for $n < m$.

P r o o f . It is easy to check that for $q = 1$ the equality (13) holds for any A, B and k. Suppose now that (13) holds for every $q \leqslant p$. Then for $q = p+1$ the left side of (13) takes the form (using (13) with $q = p$)

$$|A+B|^2 + \sum_{v=1}^{p-1}\left\{\left[|vkB+A+B|^2 - (vk)^2\right]\left[\frac{1}{v!}\prod_{\mu=0}^{v-1}\left|\mu B + \frac{A+B}{k}\right|\right]^2\right\} +$$

$$+ \left[|pkB+A+B|^2 - (pk)^2\right]\left[\frac{1}{p!}\prod_{\mu=0}^{p-1}\left|\mu B + \frac{A+B}{k}\right|\right]^2 =$$

$$= \left[\frac{k}{(p-1)!}\prod_{\mu=0}^{p-1}\left|\mu B+\frac{A+B}{k}\right|\right]^2 \left\{1+\left[|pkB+A+B|^2-(pk)^2\right](pk)^{-2}\right\} =$$

$$= \left\{\frac{k}{p!}\prod_{\mu=0}^{p}\left|\mu B+\frac{A+B}{k}\right|\right\}^2.$$

Thus (13) holds for $q = p+1$ and the proof of Lemma 2 is completed by induction argument.

Now we prove two lemmas concerning the sums of coefficients for functions in the class $S_k(A,B)$.

Lemma 3. If $f(z) = z+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\dots$ $\dots \in S_k(A,B)$, then

$$\sum_{l=k+1}^{2k} (l-1)^2 |a_l|^2 \leqslant |A+B|^2. \tag{14}$$

In particular for every natural number n such that $k+1 \leqslant n \leqslant 2k$ we have

$$|a_n| \leqslant |A+B|/(n-1). \tag{15}$$

The estimations (14) and (15) are sharp and the extremal functions have the form

$$f_n(z) = z(1-B\varepsilon z^{n-1})^{\frac{A+B}{(1-n)B}}, \quad |\varepsilon| = 1. \tag{16}$$

Lemma 4. Let $f(z) = z+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\dots$ $\dots \in S_k(A,B)$ and let for every natural v a natural number v_1 be defined as follows: $v_1 = \min(v,v_0)$, where v_0 is the smallest natural number such that the inequality $|B+(A+B)/(kv_0)| \leqslant 1$ holds (if there is no such finite number v_0, then we put $v_0 = +\infty$ i.e. $v_1 = v$). Then for every natural number v we have

(17) $$\sum_{l=vk+1}^{(v+1)k} (l-1)^2 \, |a_l|^2 \leqslant \Big[\frac{k}{(v_1-1)!} \prod_{\mu=0}^{v_1-1} \Big|\mu B + \frac{A+B}{k}\Big|\Big]^2 =$$

$$= \begin{cases} \Big[\frac{k}{(v-1)!} \prod_{\mu=0}^{v-1} \Big|\mu B + \frac{A+B}{k}\Big|\Big]^2 & \text{for} \quad v \leqslant v_0 \\ \Big[\frac{k}{(v_0-1)!} \prod_{\mu=0}^{v_0-1} \Big|\mu B + \frac{A+B}{k}\Big|\Big]^2 & \text{for} \quad v > v_0 \end{cases}$$

and

(18) $$\sum_{l=vk+1}^{(v+1)k} \Big[|A+lB|^2 - (l-1)^2\Big] |a_l|^2 \leqslant$$

$$\leqslant \begin{cases} \leqslant (vk)^2 \Big[\Big|B + \frac{A+B}{vk}\Big|^2 - 1\Big] \Big\{\frac{1}{v!} \prod_{\mu=0}^{v-1} \Big|\mu B + \frac{A+B}{k}\Big|\Big\}^2 & \text{for } v<v_0; \\ 0 & \text{for } v \geqslant v_0. \end{cases}$$

P r o o f s of Lemmas 3 and 4. By the definition of $S_k(A,B)$ every function $f \in S_k(A,B)$ satisfies the following equation

(19) $$\frac{zf'(z)}{f(z)} = \frac{1 + A\,\omega(z)}{1 - B\,\omega(z)}$$

where

(20) $$\omega(z) = b_k z^k + b_{k+1} z^{k+1} + \ldots \in \Omega .$$

The equality (19) may be written in the form

$$zf'(z) - f(z) = [Af(z) + Bzf'(z)]\, \omega(z)$$

or in the equivalent form

$$(21) \qquad \sum_{l=k+1}^{\infty} (l-1)a_l z^l = \left[(A+B)z + \sum_{l=k+1}^{\infty} (A+lB)a_l z^l\right] \sum_{q=k}^{\infty} b_q z^q.$$

Now if we put

$$\sum_{l=2k+1}^{\infty} d_l z^l := \sum_{l=2k+1}^{\infty} (l-1)a_l z^l - \sum_{l=k+1}^{\infty} (A+lB)a_l z^l \sum_{q=k}^{\infty} b_q z^q,$$

$$\sum_{l=(v+1)k+1}^{\infty} h_l z^l := \sum_{l=(v+1)k+1}^{\infty} (l-1)a_l z^l - \sum_{l=vk+1}^{\infty} (A+lB)a_l z^l \cdot \sum_{q=k}^{\infty} b_q z^q$$

then (21) takes the form

$$(22) \qquad \sum_{l=k+1}^{2k} (l-1)a_l z^l + \sum_{l=2k+1}^{\infty} d_l z^l = (A+B)z\, \omega(z)$$

or

$$(23) \qquad \sum_{l=k+1}^{(v+1)k} (l-1)a_l z^l + \sum_{l=(v+1)k+1}^{\infty} h_l z^l =$$

$$= \left[(A+B)z + \sum_{l=k+1}^{vk} (A+lB)a_l z^l\right] \omega(z), \qquad v = 2,3,\ldots$$

Now, using Clunie's method (see e.g. [1]-[6]), that is integrating squares of modulus on both sides of (22) and (23) along circle $\{z:|z|=r\}$, applying the inequalities $|\omega(z)|<1$ and $|z\omega(z)|<1$ and taking $r\to 1$, we obtain

$$(24)\qquad \sum_{l=k+1}^{2k}(l-1)^2|a_l|^2\leqslant|A+B|^2$$

and

$$(25)\qquad \sum_{l=k+1}^{(v+1)k}(l-1)^2|a_l|^2\leqslant|A+B|^2+\sum_{l=k+1}^{vk}|A+lB|^2|a_l|^2,$$

$$v=2,3,\ldots .$$

Inequality (24) is equivalent to (14) and noting that

$$(26)\qquad (n-1)^2|a_n|^2\leqslant\sum_{l=vk+1}^{(v+1)k}(l-1)^2|a_l|^2,\ vk+1\leqslant n\leqslant(v+1)k$$

we obtain (15). It is easy to see that for the function $f_n(z)$, given by (16), we have

$$f_n(z)=z+\frac{A+B}{n-1}\varepsilon z^n+\ldots$$

and

$$zf_n'(z)/f_n(z)=(1+A\varepsilon z^{n-1})/(1-B\varepsilon z^{n-1}).$$

Thus $f_n(z)\in S_k(A,B)$ for $n\geqslant k+1$ and gives equality in (14) and (15). This completes the proof of Lemma 3.

To prove Lemma 4 we rewrite (25) in the form

$$(27)\quad \sum_{l=vk+1}^{(v+1)k} (l-1)^2|a_l|^2 \leqslant |A+B|^2 + \sum_{\mu=1}^{v-1} \sum_{l=\mu k+1}^{(\mu+1)k} \left[|A+lB|^2-(l-1)^2\right]|a_l|^2.$$

The inequalities (17) and (18) are proved by induction with respect to v.

For $v = 1$ the inequality (17) is equivalent to (24) and therefore it holds.

The inequality (18) in this case has the form

$$(28)\quad \sum_{l=k+1}^{2k} \left[|A+lB|^2 - (l-1)^2\right]|a_l|^2 \leqslant$$

$$\leqslant \begin{cases} \left[\left|B + \frac{A+B}{k}\right|^2 - 1\right]|A+B|^2 & \text{for } \left|B + \frac{A+B}{k}\right| > 1 \\ 0 & \text{for } \left|B + \frac{A+B}{k}\right| \leqslant 1. \end{cases}$$

In the first case, by (24) and Lemma 1 (3^0 with $l_0 = k$) we have

$$\sum_{l=k+1}^{2k} \left[|A+lB|^2-(l-1)^2\right]|a_l|^2 = \left[|w_k|^2-1\right] \sum_{l=k+1}^{2k} \frac{|w_{l-1}|^2-1}{|w_k|^2-1}(l-1)^2|a_l|^2 \leqslant$$

$$\leqslant \left[|w_k|^2-1\right] \sum_{l=k+1}^{2k} (l-1)^2|a_l|^2 \leqslant \left[\left|B + \frac{A+B}{k}\right|^2 - 1\right]|A+B|^2.$$

In the second case, by Lemma 1 (2^0 with $l_0 = k$) we have

$$|A+lB|^2 - (l-1)^2 = (l-1)^2\left[|w_{l-1}|^2 - 1\right] \leqslant 0 \quad \text{for} \quad l \geqslant k+1.$$

It means that the left side of (28) is nonpositive as a sum of nonpositive numbers. Thus (28) holds in both cases and therefore (18) holds for $v = 1$.

Suppose now that inequalities (17) and (18) hold for $v = 1,2,\dots,q$. Then by Lemma 2, using (27) with $v = q+1$ and (18) with $v = q$, we have

$$\sum_{l=(q+1)k+1}^{(q+2)k} (l-1)^2|a_l|^2 \leqslant |A+B|^2 + \sum_{\mu=1}^{q_1}\left\{\left[\frac{1}{\mu!}\prod_{i=0}^{\mu-1}\left|iB+\frac{A+B}{k}\right|\right]^2 (k\mu)^2\left[\left|B+\frac{A+B}{k}\right|^2 - 1\right]\right\} =$$

$$= \left\{\frac{k}{q_1!}\prod_{\mu=0}^{q_1}\left|\mu B + \frac{A+B}{k}\right|\right\}^2, \quad \text{where} \quad q_1 = \min\{q, v_0\}.$$

It means that (17) holds also for $v = q+1$.

Now if $v_0 > q+1$, then by (17) with $v = q+1$ and by Lemma 1 (3^0 with $l_0 = (q+1)k$) we have

$$\sum_{l=(q+1)k+1}^{(q+2)k}\left[|A+lB|^2 - (l-1)^2\right]|a_l|^2 =$$

$$= \left[|w_{(q+1)k}|^2 - 1\right] \sum_{l=(q+1)k+1}^{(q+2)k} \frac{|w_{l-1}|^2 - 1}{|w_{(q+1)k}|^2 - 1}(l-1)^2|a_l|^2 \leqslant$$

$$\leqslant \left[|w_{(q+1)k}|^2 - 1\right] \sum_{l=(q+1)k+1}^{(q+2)k}(l-1)^2|a_l|^2 \left[\left|B + \frac{A+B}{(q+1)k}\right|^2 - 1\right]\left[\frac{k}{q!}\prod_{\mu=0}^{q}\left|\mu B + \frac{A+B}{k}\right|\right]^2 =$$

$$= \left[(q+1)k\right]^2\left[\left|B + \frac{A+B}{(q+1)k}\right|^2 - 1\right]\left[\frac{1}{(q+1)!}\prod_{\mu=0}^{q}\left|\mu B + \frac{A+B}{k}\right|\right]^2.$$

Thus (18) holds in this case for $v = q+1$.

If $v_0 \leqslant q+1$, then by Lemma 1 (2^0 with $l_0 = (q+1)k+1$) we have

$$|A+lB|^2-(l-1)^2 = (l-1)^2\left[|w_{l-1}|^2-1\right] \leqslant 0 \quad \text{for} \quad l \geqslant (q+1)k+1$$

and (18) holds because the left side is the sum of some non-positive numbers.

By induction argument the proof of Lemma 4 is completed.

Theorem 2. If $f(z) = z+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+ \ldots$ $\ldots \in S_k(A,B)$, then

$$(29) \qquad |a_n| \leqslant \frac{k}{(n-1)(v_1-1)!}\prod_{\mu=0}^{v_1-1}\left|\mu B + \frac{A+B}{k}\right| =$$

$$= \begin{cases} \dfrac{k}{(n-1)(v-1)!}\displaystyle\prod_{\mu=0}^{v-1}\left|\mu B + \frac{A+B}{k}\right| & \text{for } v \leqslant v_0, \\ \dfrac{k}{(n-1)(v_0-1)!}\displaystyle\prod_{\mu=0}^{v_0-1}\left|\mu B + \frac{A+B}{k}\right| & \text{for } v > v_0, \end{cases}$$

where $v = \left[\frac{n-1}{k}\right]$ and v_1, v_0 are defined in Lemma 4.

If $v_0 = 1$, then the inequality (29) gives sharp results for every $n \geqslant k+1$. The extremal functions have the form (16). If $v_0 > 1$, then the inequality (29) gives sharp results for $n = vk+1$ where $v = 1,2,3,\ldots,v_0$. In this case the extremal functions also have the form (16) where $n-1$ is replaced by k.

Proof. The inequality (29) follows immediately from (24) and (17).

Remarks:

1. For $A = 1$, $B = m$, $k = 1$ the Theorem 2 coincides with Janowski's Theorem 8, [2].

2. For $A = (1+m)\lambda-1$, $B = m$, $k = 1$ it coincides with Plaskota's Theorem 1, [4].

3. For $A = B = 1$ and any arbitrary natural number k we obtain MacGregor's Theorem 1, [3].

3. If $A = 1$, $B = m$, k is arbitrary and $v \leqslant v_0$, then inequality (29) coincides with the estimate by Szynal's Theorem 7, [6], but for $v > v_0$ these two estimations are different. In the proof of Theorem 7 ([6] p.117) there is a mistake. Namely, for one factor of the product it gives an upper estimation, but the other factor may be negative and the estimation is false in this case. For some special values of m, k and n the estimation for a_n in Theorem 7, [6] is better than the modulus of the n-th coefficient of the function (16) which belongs to this class.

5. Theorem 8 in paper [6] is also false, as it can be shown for the function

$$F_\lambda(z) = \lambda f_{\varepsilon_1}(z) + (1-\lambda)\, f_{\varepsilon_2}(z),$$

where $k = 1$, $\lambda = \frac{1}{2}$, $\varepsilon_1 = -\varepsilon_2 = 1$, $m = \frac{1}{2}$ with functions $f_\varepsilon(z)$ defined in [6] p.118. It seems that the author of paper [6] wanted to prove some other theorem which for the class $S_k(A,B)$ may be written in the following form.

T h e o r e m 3. If $f,g \in S_k(A,B)$ and G_λ ($\lambda \in [0,1]$) is defined as follows

$$(30)\qquad G_\lambda(z) = z\left[\frac{f(z)}{z}\right]^\lambda \left[\frac{g(z)}{z}\right]^{1-\lambda} \quad (\log G_\lambda = \lambda \log f + (1-\lambda)\log g)$$

then for every $\lambda \in [0, 1]$ the function G_λ also belongs to $S_k(A,B)$.

P r o o f . To prove this theorem we need an obvious lemma.

L e m m a 5. Let $H(z)$ be a convex, univalent function in U. If h_1, h_2 are holomorphic in U and $h_1 \prec H$, $h_2 \prec H$, then for every $\lambda \in [0, 1]$ we have

$$\lambda h_1 + (1 - \lambda) h_2 \prec H.$$

From the definition of the class $S_k(A,B)$ we have

$$h_1(z) = zf'(z)/f(z) \prec (1+Az)/(1-Bz) = H(z),$$

$$h_2(z) = zg'(z)/g(z) \prec (1+Az)/(1-Bz) = H(z).$$

Thus for G_λ defined by (30) we have

$$zG_\lambda'(z)/G_\lambda(z) = \lambda zf'(z)/f(z) + (1-\lambda)zg'(z)/g(z) \prec (1+Az)/(1-Bz)$$

and since $G_\lambda(z) = t+(\lambda a_{k+1}+(1-\lambda)b_{k+1})z^{k+1}+ \ldots$, then $G_\lambda \in S_k(A,B)$.

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