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ON AN INVERSE PROBLEM IN ELASTICITY

1. Formulation of the problem

Let the simply connected plane domain D be bounded by the closed curve ℓ and $x = (x_1, x_2)$ be a point of the plane. Let every straight line $x_2 = c$, where $-\infty < \overset{(1)}{x}_1 < c < \overset{(2)}{x}_2 < +\infty$, have with the curve ℓ exactly two points in common. The straight lines $x_2 = \overset{(1)}{x}_2$ and $x_2 = \overset{(2)}{x}_2$ have with the curve ℓ the only one common point $M(\overset{(1)}{x}_1, \overset{(1)}{x}_2)$ and $N(\overset{(2)}{x}_1, \overset{(2)}{x}_2)$, respectively. The others straight lines $x_2 = c$ for $c < \overset{(1)}{x}_2$ and $c > \overset{(2)}{x}_2$ have no points in common with ℓ.

Let ℓ_1 be the part of ℓ lying between the points N and M in the positive direction on ℓ. Let the parametric equation of the curve be $y = y(s)$, i.e. $y_1 = y_1(s)$, $y_2 = y_2(s)$, $(y_1, y_2) \in \ell$ with s arc length of ℓ.

The problem consists in finding the vector $w(x_1, x_2) \in C^2(D) \cup C^1(\ell)$ and the vector $\eta(x_2) \in C_\alpha(D)$, $0 < \alpha < 1$, which satisfy the equation

$$(1) \qquad \mu \Delta w + (\lambda + \mu) \operatorname{grad} \operatorname{div} w = \eta(x_2), \qquad x = (x_1, x_2) \in D,$$

and the following boundary conditions

$$(2) \qquad w(x)\Big|_{\ell} = f(s),$$

(3) $$\left.\frac{\partial w}{\partial n}\right|_{\ell_1} = g(s_0),$$

where μ, λ are constants, $n = (n_1, n_2)$ is an external normal to ℓ_1 and s_0 is arc length of ℓ_1.

It is assumed that

1^0 The curve ℓ satisfies Lapunov conditions.

2^0 The vector functions f and g are of the class $C^1_\alpha(\ell)$ and $C'_\alpha(\ell_1)$, respectively.

The above stated problem is a generalization to the theory of elasticity of the problems considered for Laplace's equation in [2] and [3].

Let $\omega(x_2)$ be an arbitrary vector of the class $C^2_\alpha(D)$ such that

(4) $$\mu\,\Delta\omega + (\lambda+\mu)\text{ grad div }\omega = \eta(x_2).$$

Without any restriction we can assume that

(5) $$\overset{(1)}{\omega}(x_2) = \overset{(2)}{\omega}(x_2) = 0,$$

since, if the condition (5) would not be satisfied, then we would subtract still arbitrary vector $\omega_1(x)$ such that (5) is satisfied.

We seek the solution of the problem (1)-(3) in the form

(6) $$w(x) = u(x) + \omega(x_2),$$

where $u(x)$ and $\omega(x_2)$ are to be determined as a solution of the problem

(7) $$\mu\Delta u + (\lambda+\mu)\text{ grad div } u = 0,$$

(8) $$\left.u\right|_{\ell} = f(s) - \omega[y_2(s)],$$

(9) $$\frac{\partial u}{\partial n}\Big|_{\ell_1} = g(s_0) - \omega'\left[y_2(s_0)\right] \frac{dy_2}{dn}.$$

The uniqueness of the solution of the problem (7)-(9) follows if we repeat the arguments given in [2], since the maximum principle is valid for the equation (7) and, due to the assumption 1^0 and (5), the homogeneous problem ($f = 0$, $g = 0$) possesses only zero solution. Therefore if the solution exists, then it is unique.

2. The existence of the solution

From the potential methods in the theory of elasticity it is known [1] that each vector of the form

(10) $$u(\overset{0}{x}) = \frac{1}{2\pi}\int_{\ell} \overset{\varkappa}{T}\, G(\overset{0}{x},y(s)) \left\{f(s)-\omega[y_2(s)]\right\} ds, \quad \overset{0}{x} \in D,$$

satisfies the equation (7) and the boundary condition (8), where

(11) $$G(x,y) = \Gamma(x,y) + v(x,y)$$

is the Green tensor of the first boundary value problem in the domain D, $\overset{\varkappa}{T} = \left[\overset{\varkappa}{T}_{ij}\right]_{2\times 2}$ is the generalized stress operator

(12) $$\overset{\varkappa}{T}_{ij} = (\lambda+\mu)\, n_i \frac{\partial}{\partial x_j} + \varkappa\left(n_j \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_j}\right) + \delta_{ij}\mu \frac{\partial}{\partial n},$$

$\Gamma(x,y) = \left[\Gamma_{ij}(x,y)\right]_{2\times 2}$ is the fundamental solution of (7) with the components

(13) $$\Gamma_{ij}(x,y) = \delta_{ij}\, a \ln r - b\left(\frac{\partial r}{\partial x_i}\right)\left(\frac{\partial r}{\partial x_j}\right),$$

$$a = \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}, \qquad b = \frac{\lambda+\mu}{2\mu(\lambda+2\mu)},$$

and $v(x,y)$ is the regular part of the Green tensor.

Differentiating (10) over $\overset{\circ}{x}_1$ and $\overset{\circ}{x}_2$ and next integrating by parts the expressions obtained for $\frac{\partial u}{\partial \overset{\circ}{x}_1}$, $\frac{\partial u}{\partial \overset{\circ}{x}_2}$, we then substitute them into the condition (9) and get for $(\overset{\circ}{x}_1, \overset{\circ}{x}_2) \to (\overset{\circ}{y}_1, \overset{\circ}{y}_2)$ the following equation

$$(14) \quad -\frac{y_1'(s_0)E}{2\pi}\int_l \frac{\overset{\circ}{y}_1-y_1}{r^2}\psi'(s)ds - \frac{y_2'(s_0)E}{2\pi}\int_l \frac{\overset{\circ}{y}_2-y_2}{r^2}\psi'(s)ds +$$

$$+\frac{1}{2}\left[(\varkappa+\mu)a-1\right]E\mu(\psi) +$$

$$+\int_l \frac{\partial}{\partial n}\left[(\varkappa+\mu)\Gamma(\overset{\circ}{y},y) - E\ln r\right]\mu(\psi)ds -$$

$$-\frac{y_2'(s_0)}{2\pi}\int_l P(\overset{\circ}{y},y)\psi'(s)ds + \frac{y_1'(s_0)}{2\pi}\int_l Q(\overset{\circ}{y},y)\psi'(s)ds =$$

$$= g(s_0) - y_1'(s_0)\,\omega'\left[y_2(s_0)\right],$$

where $\psi(s) = f(s) - \omega[y_2(s)]$, $P(\overset{\circ}{x},y) = \int_l \frac{\partial^2 v}{\partial \overset{\circ}{x}_1 \partial n}ds$, $Q(\overset{\circ}{x},y) = \int_l \frac{\partial^2 v}{\partial \overset{\circ}{x}_2 \partial n}ds$, E is the unit marix and μ is the operator with the components

$$\mu_{jk} = n_j\frac{\partial}{\partial y_k} - n_k\frac{\partial}{\partial y_j} \qquad k,j = 1,2.$$

Clearly $y_1'(s) = \cos(n,\overset{\circ}{y}_2)$, $y_2'(s) = -\cos(n,\overset{\circ}{y}_1)$.

The constant $\varkappa$ being arbitrary, let us now fix it in such a way, that $(\varkappa+\mu)a - 1 = 0$, i.e.

$$(15) \qquad \varkappa = \frac{\mu(\lambda+\mu)}{\lambda+3\mu}.$$

From the classical potential theory it is known that on the Lapunov curve l the last three integrals in (14) have a weak singularity of order $O\left(\frac{1}{r^{1-\alpha}}\right)$. Finally we obtain the following system of integral equations with respect to $\omega'[y_2(s_0)]$

$$(16)\quad E\, y_1'(s_0)\omega'[y_2(s_0)] - \frac{y_1'(s_0)E}{2\pi}\int_l \frac{y_1(s_0)-y_1(s)}{r^2}\, y_2'(s)\omega'[y_2(s)]ds -$$

$$- \frac{y_2'(s_0)E}{2\pi}\int_l \frac{y_2(s_0)-y_2(s)}{r^2}\, y_2'(s)\omega'[y_2(s)]\, ds +$$

$$+ \frac{1}{2\pi}\int_l K(\overset{o}{y},y)\, y_2'(s)\omega'[y_2(s)]\, ds = \tilde{g}(s_0),$$

where

$$K(\overset{o}{y},y) = \frac{\partial}{\partial n}\left[(\varkappa+\mu)\,\Gamma(\overset{o}{y},y) - E\ln r\right] - y_2'(s_0)P(\overset{o}{y},y) + y_1'(s_0)Q(\overset{o}{y},y),$$

$$\tilde{g}(s_0) = g(s_0) - \frac{E}{2\pi}\int_l \frac{y_1'(s_0)[y_1(s_0)-y_1(s)] + y_2'(s_0)[y_2(s_0)-y_2(s)]}{r^2}\, f'(s)ds -$$

$$- \frac{1}{2\pi}\int_l \left[y_2'(s_0)\, P(\overset{o}{y},y) - y_1'(s_0)\, Q(\overset{o}{y},y)\right] f'(s)ds.$$

The above system of integral equations consists of two singular integral equations with the main part being the same as in the integral equation considered in [2] and so it is solvable. Due to the condition (5), we get from (16) the vector $\omega(x_2)$, from (10) the vector $u(x)$, from (6) the vector $w(x)$, and finally from (4) the vector $\eta(x_2)$. Note that the choice (15) of $\varkappa$ is essential, since the kernel of the potential (10) is in that case weakly singular on l, [1].

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