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TOPOLOGICAL PROPERTIES OF C-NETS

1. Introduction

The purpose of this paper is to study the topological and algebraic properties of c-nets, more general systems than nets (A.Blikle [3]). A c-net is a poset with zero, in which every directed subset has a least upper bound. In a c-net a monoid operation is defined. It is distributive with respect to the least upper bounds of directed sets.

A T_0-topology (called hereafter a topology induced by a partial order) is introduced in the c-net. (compare with D.Scott [4]). According to D.Scott a complete lattice L is continuous, if for each $y \in L : y = \bigvee\{x \in L : x \prec y\}$, where $x \prec y \iff y \in \operatorname{Int}\{z \in L : x \leqslant z\}$; the interior is in the sense of the topology induced by a partial order in L. D.Scott proved that the injective T_0-topological spaces are exactly continuous lattices. The following problem arises: does there exist a theory of continuous posets, that generalizes the theory of continuous lattices? In this paper D.Scott's conception of continuous lattice is generalized in a natural way and fundamental theoréms are proved.

2. Symbols and definitions

Let C_1, C_2 be posets and φ be a mapping from C_1 into C_2. We say that φ is monotonic if it preserves order i.e.: $a,b \in C_1$ and $a \leqslant b$ imply $\varphi(a) \leqslant \varphi(b)$. The mapping φ is continuous (in the algebraic sense) if: for every non-empty directed set $D \subseteq C_1$, if there exists the join $\bigvee D$ in C_1 then there also exists the join $\bigvee\{\varphi(d) : d \in D\}$ in C_2 and the following equation is satisfied: $\varphi(\bigvee D) = \bigvee\{\varphi(d) : d \in D\}$.

In this paper we shall consider only non-empty directed sets. Let $C_1 \times \dots \times C_n$ $(n \geq 1)$ denote the direct product of a family posets $C_1, \dots, C_n$.

R e m a r k 2.1. The mapping $\varphi : C_1 \times \dots \times C_n \to C$ is monotonic if and only if it is monotonic with respect to each variable separately.

T h e o r e m 2.2. Let $C_1, \dots, C_n, C$ be posets such that every directed subset of C_i $(i = 1, \dots, n)$, C has a least upper bound. Then any mapping $\varphi : C_1 \times \dots \times C_n \to C$ is continuous if and only if it is continuous with respect to each variable separately.

The proof is analogous to the one given in [4] th.2.6 by D.Scott. So we omit here the details.

D e f i n i t i o n 2.3. By a c-net we shall mean the relation system $\langle C; \leq, \circ, 0, e\rangle$ such that:

1. $\langle C; \leq, 0\rangle$ is a poset with least element 0, in which every directed subset has a least upper bound.

2. $\langle C; \circ, 0, e\rangle$ is a semigroup with zero 0 and unit e. (the binary operation "o" is called a composition).

3. the composition "o" is continuous.

R e m a r k 2.4. The composition "o" is monotonic.

E x a m p l e 2.5. C-net of continuous operations. Let P be a poset with least element 0, in which every directed subset has a least upper bound. Furthermore, let F denote the set of all continuous mappings of the set P to itself such that $f(0) = 0$ for each $f \in F$. We introduce the partial order in F in natural way: if $f_1, f_2 \in F$ then $f_1 \leq f_2 \Leftrightarrow f_1(x) \leq f_2(x)$ for every $x \in P$. Let us introduce a partial operation of the least upper bound in F: if $\{f_t\}_{t\in T}$ is a directed set then $\left(\bigvee_{t\in T} f_t\right)(x) = \bigvee_{t\in T} f_t(x)$ for every $x \in P$. The relation system $\langle F; \leq, \circ, \mathbf{0}, E\rangle$ is a c-net provided that $(f_1 \circ f_2)(x) = f_1(f_2(x))$ for every $x \in P$ and $\mathbf{0}(x) = 0$, $E(x) = x$ for every $x \in P$.

D e f i n i t i o n 2.6. A mapping $h : C_1 \longrightarrow C_2$ where $\langle C_1; \leqslant_1, \circ_1, O_1, e_1 \rangle$, $\langle C_2; \leqslant_2, \circ_2, O_2, e_2 \rangle$ are c-nets, is said to be a homomorphism if and only if the following conditions are satisfied:

(h_1) h is continuous,

(h_2) h preserves the composition, that means: $h(x \circ_1 y) = h(x) \circ_2 h(y)$ for each $x, y \in C_1$,

(h_3) $h(O_1) = O_2$; $h(e_1) = e_2$.

A homomorphism $h : C_1 \longrightarrow C_2$ is called a full one if the following condition is also satisfied:

(h_0) if $\{h(a_t)\}_{t \in T}$ is a directed set and $\bigvee_{t \in T} h(a_t) = h(a)$ then there exist: a directed set $\{b_t\}_{t \in T}$ and $b \in C_1$ such that $h(b_t) = h(a_t)$ for each $t \in T$, $h(b) = h(a)$ and $\bigvee_{t \in T} b_t = b$.

Any full homomorphism $h : C_1 \longrightarrow C_2$ which is "one-one" and "onto" is called an isomorphism.

T h e o r e m 2.7. Every c-net C can be embedded in the c-net F of continuous operations.

Outline of a proof: Since C is a c-net, it is a poset with the least element 0, in which each directed subset has a least upper bound. Let F be the set of all continuous mappings $f : C \longrightarrow C$. For each $a \in C$ we define a continuous mapping $f_a : C \longrightarrow C$ as follows

$$f_a(x) = a \circ x \quad \text{for each} \quad x \in C.$$

Let now $m : C \longrightarrow F$ be a mapping such that $m(a) = f_a$ for $a \in C$. It is easy to verify that "m" is an isomorphism (compare def.2.6).

3. Topology in a c-net

We define the open sets in a c-net as follows:

D e f i n i t i o n 3.1. Let $U \subseteq C$; U is an open set if and only if it satisfies the following conditions:

(O_1) If $x \in U$ and $x \leqslant y$ then $y \in U$.

(O_2) Whenever $D \subseteq C$ is a directed set and $\bigvee D \in U$ then $D \cap U \neq \emptyset$.

The sets satisfying (O_1) and (O_2) form the topology induced by the partial order in a c-net C. Therefore C becomes a topological T_0-space.

Now we can generalize the theorem which was proved for the complete lattices by D.Scott ([4], th.2.5).

T h e o r e m 3.2. If C, C' are posets with their topologies induced by the partial order and in which each directed subset has a least upper bound then a mapping $f: C \longrightarrow C'$ is continuous in the topological sense if and only if for each directed subset $D \subseteq C$ there exists $\bigvee\{f(d) : d \in D\}$ in C' and the following equation is satisfied:

$$f(\bigvee D) = \bigvee\{f(d) : d \in D\}. \tag{a}$$

P r o o f . Let us assume that for each directed set $D \subseteq C$ there exists $\bigvee\{f(d) : d \in D\}$ and the condition (a) is satisfied. Then let U' be an open set in C' and $U = \{x \in C : f(x) \in U'\}$. We shall prove that U is open in C. Since f is monotonic and U' is an open set in C', if $x \in U$ and $x \leqslant y$, we have: $f(x) \leqslant f(y) \in U'$. Therefore $y \in U$, so the set U satisfies the condition (O_1) from the definition 3.1. Let now $\bigvee D \in U$ for a directed set $D \subseteq C$, so $f(\bigvee D) \in U'$. Hence $\bigvee\{f(d) : d \in D\} \in U'$ and U' is open then there exists $d \in D$ such that $f(d) \in U'$ i.e. $d \in U$ thus $D \cap U \neq \emptyset$.

Conversely: First we shall show that a mapping $f : C \longrightarrow C'$ continuous in the topological sense is monotonic. Let us suppose that $x, y \in C$ and $x \leqslant y$. If $f(x) \nleqslant f(y)$ then $f(x) \in U' = \{z : z \nleqslant f(y)\}$ and U' is an open set in C'. Consequently $x \in f^{-1}(U') \subseteq C$. But $x \leqslant y$, therefore $y \in f^{-1}(U')$ so $f(y) \in U'$, which is a contradiction to the definition of U'.

Let now D be a directed subset in C, then $\{f(d) : d \in D\}$ is a directed set in C', so there exists the least upper bound $\bigvee\{f(d) : d \in D\}$. Let us consider any open set U' in C'. If $f(\bigvee D) \in U'$ then $\bigvee D \in U = f^{-1}(U')$ and according to the definition of an open set there exists an element $d \in D$ such that $d \in U = f^{-1}(U')$. Hence $f(d) \in U'$ and $\bigvee\{f(d) : d \in D\} \in U'$. On the other hand, if $\bigvee\{f(d) : d \in D\} \in U'$ there exists an element $d \in D$ such that $f(d) \in U'$ i.e. $d \in U = f^{-1}(U')$. But $d \leqslant \bigvee D$, consequently: $\bigvee D \in U = f^{-1}(U')$, thus $f(\bigvee D) \in U'$. We conclude that $\bigvee\{f(d) : d \in D\} \in U' \Longleftrightarrow f(\bigvee D) \in U'$. C is known to be a topological T_0-space which means the open sets distinguish points. This implies that the condition (a) is fulfilled.

4. Theorem concerning topology in a direct product of c-nets.

The direct product of a family $\{C_\alpha\}_{\alpha\in\Sigma}$ of c-nets is the c-net $\mathcal{H} = \langle H; \leqslant, \circ, 0, E\rangle$, where: $H = \prod_{\alpha\in\Sigma} C_\alpha$ is the cartesian product of a family $\{C_\alpha\}_{\alpha\in\Sigma}$. We introduce the partial order "by the components" i.e.: if $h_1, h_2 \in H$ then $h_1 \leqslant h_2 \Leftrightarrow h_1(\alpha) \leqslant h_2(\alpha)$ for each $\alpha \in \Sigma$. We define the operations as follows: if $\{h_t\}_{t\in T}$ is a directed set then $\left(\bigvee_{t\in T} h_t\right)(\alpha) = \bigvee_{t\in T} h_t(\alpha)$ for each $\alpha \in \Sigma$; $(h_1 \circ h_2)(\alpha) = h_1(\alpha) \circ h_2(\alpha)$ for each $\alpha \in \Sigma$. The functions 0, E are the distinguished elements of H, where $0(\alpha) = 0_{C_\alpha}$, $E(\alpha) = e_{C_\alpha}$ for each $\alpha \in \Sigma$; 0_{C_α}, e_{C_α} are zero and unit in the c-net C_α, correspondingly. If $C_\alpha = C$ for every $\alpha \in \Sigma$ then we write C^Σ instead of $\prod_{\alpha\in\Sigma} C_\alpha$ and C^Σ is called the direct power of a c-net C.

Let C be a c-net with the topology induced by the partial order. We define a binary relation "$\prec$" in C as follows:

D e f i n i t i o n 4.2. (D. Scott [4]). For $x, y \in C$: $x \prec y \Longleftrightarrow y \in \operatorname{Int}\{z : x \leqslant z\}$.

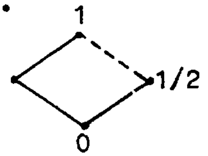
D e f i n i t i o n 4.3. A c-net C is continuous if and only if for every $y \in C$: the set $\{x : x \prec y\}$ is directed and $y = \bigvee \{x : x \prec y\}$.

L e m m a 4.4. Let C be a c-net. For every $y \in C$ the following conditions are equivalent:

1. A set $\{x : x \prec y\}$ is directed and $y = \bigvee \{x : x \prec y\}$.
2. For each open set U in C the following condition holds: if $y \in U$ then there exists $x \in U$ such that $x \prec y$.

P r o o f . Suppose, condition 1 is satisfied. If $y = \bigvee \{x : x \prec y\}$ and $y \in U$ then by the definition of an open set there exists $x \in U$ such that $x \prec y$. If condition 2 is satisfied, we shall prove first that $Y = \{x : x \prec y\}$ is a directed set. Let $x_1, x_2 \in Y$. Then $y \in \mathrm{Int}\{z : x_1 \leqslant z\}$ and $y \in \mathrm{Int}\{z; x_2 \leqslant z\}$. Consequently $y \in \mathrm{Int}\{z : x_1 < z\} \cap \cap \mathrm{Int}\{z : x_2 \leqslant z\} = \mathrm{Int}\{z : x_1 < z \text{ and } x_2 \leqslant z\}$. By virtue of condition 2, there exists $x \in \mathrm{Int}\{z : x_1 \leqslant z \text{ and } x_2 \leqslant z\}$ such that $x \prec y$. Thus $x_1 \leqslant x$, $x_2 < x$, $x \prec y$ and Y is a directed set. Obviously $\bigvee\{x : x \prec y\} \leqslant y$. If $a = \bigvee\{x : x \prec y\} \not\leqslant y$ then $y \in U = \{z : z \not\leqslant a\}$ and there exists $x \in U$ such that $x \prec y$. But then $x \prec y$ implies $x \leqslant a$, a contradiction.

E x a m p l e 4.5. The c-net, which is not continuous.

C: 

The c-net C consists of the closed interval of the real numbers $\langle 1/2, 1\rangle$, the elements 0 and e, as on a diagram.

We define the composition as follows:

$0 \circ x = x \circ 0 = 0$,

$e \circ x = x \circ e = x$ for each $x \in C$,

$x \circ y = \min(x,y)$ for x,y $\langle 1/2, 1\rangle$.

Let us take an open set $U = \langle 1/2, 1\rangle \cup \{e\} \subseteq C$. We shall show that there does not exist $x \in U$ such that $x \prec e$. If $x \prec e$ then $x \leqslant e$, consequently $x = e$. But $e \prec e \Leftrightarrow$ $\Leftrightarrow e \in \mathrm{Int}\{z : e \leqslant z\} \Leftrightarrow B = \{z : e \leqslant z\}$ is the open set. The set B is not open, because the least upper bound of the set $\langle 1/2,1)$ belongs to B, but the elements of the set $\langle 1/2,1)$ don't belong to B.

T h e o r e m 4.6. The direct product $\mathcal{H} = \langle H; \leqslant, \circ, 0, E \rangle$ of a family $\{C_\alpha\}_{\alpha \in \Sigma}$ of the continuous c-nets is a continuous c-net. Moreover the topology induced by the partial order in H coincides with the product topology.

Before we prove Theorem 4.6. We shall prove the following lemmas.

L e m m a 4.7. For every finite sequence of indices $\alpha_1, \dots, \alpha_n \in \Sigma$, a set of the form $X = \{h \in H : h(\alpha_i) \in U_{\alpha_i}, i = 1, \dots, n\}$, where U_{α_i} are the open sets in C_{α_i} for $i = 1, \dots, n$, is open in the topology induced by the partial order in H.

P r o o f . If $h \in H$ and $h \leqslant h_1$ then $h(\alpha_i) \leqslant h_1(\alpha_i)$, $i = 1, \dots, n$. Since U_{α_i} is an open set for $i = 1, \dots, n$ then $h_1(\alpha_i) \in U_{\alpha_i}$ and $h_1 \in X$. If $\bigvee_{t \in T} h_t \in X$, where $\{h_t\}_{t \in T}$ is directed in H then $\bigvee_{t \in T} h_t(\alpha_i) = \left(\bigvee_{t \in T} h_t\right)(\alpha_i) \in U_{\alpha_i}$. U_{α_i} is the open set, therefore there exists $t_i \in T$ such that $h_{t_i}(\alpha_i) \in U_{\alpha_i}$, $i = 1, \dots, n$. Consequently there is $h_{t_0} \in \{h_t\}_{t \in T}$ such that $h_{t_i} \leqslant h_{t_0}$ for $i = 1, \dots, n$. Since $h_{t_i}(\alpha_i) \in U_{\alpha_i}$ and U_{α_i} are the open sets, $h_{t_0}(\alpha_i) \in U_{\alpha_i}$ for $i = 1, \dots, n$. Thus $h_{t_0} \in X$ and $\{h_t\}_{t \in T} \cap X \neq \emptyset$.

L e m m a 4.8. If $U \subseteq H$ is an open set in the topology induced by the partial order then for every $\alpha \in \Sigma : p_\alpha(U) = \{h(\alpha) : h \in U\}$ is an open set in the topology induced by the partial order in C_α (i.e. the projections are open mappings in the topology induced by the partial order).

P r o o f . If $h(\alpha) \in p_\alpha(U)$ and $h(\alpha) \leqslant c \in C_\alpha$ then let $h_1(\alpha) = c$ and $h_1(\alpha') = h(\alpha')$ for $\alpha' \neq \alpha$. So we have: $h_1 \in U$ and $h_1(\alpha) = c \in p_\alpha(U)$ because $h \leqslant h_1$ and $h \in U$. If $\{c_t\}_{t \in T}$ is directed in C_α and $\bigvee_{t \in T} c_t \in p_\alpha(U)$ then there exists $h \in U$ such that $h(\alpha) = \bigvee_{t \in T} c_t$. Let now

$h_t(\alpha) = c_t$ and $h_t(\alpha') = h(\alpha')$ for $\alpha' \neq \alpha$. The set $\{h_t\}_{t\in T}$ is directed and $h = \bigvee_{t\in T} h_t \in U$. Hence there exists $t_o \in T$ such that $h_{t_o} \in U$ and $h_{t_o}(\alpha) = c_{t_o} \in p_\alpha(U)$. Consequently $p_\alpha(U)$ is open in the topology induced by the partial order in C.

Lemma 4.9. Let $\prod_{\alpha\in\Sigma} W_\alpha \subseteq H = \prod_{\alpha\in\Sigma} C_\alpha$, where $W_\alpha = C_\alpha$ for all except finite number of $\alpha \in \Sigma$. Then $\mathrm{Int} \prod_{\alpha\in\Sigma} W_\alpha = \prod_{\alpha\in\Sigma} \mathrm{Int}\, W_\alpha$ (in the topology induced by the partial order in H).

Proof. An element h belongs to $\mathrm{Int} \prod_{\alpha\in\Sigma} W_\alpha$ if and only if there exists an open set U such that $h \in U \subseteq \subseteq \prod_{\alpha\in\Sigma} W_\alpha$. Hence for each $\alpha \in \Sigma$: $h(\alpha) \in p_\alpha(U) \subseteq W_\alpha$. By Lemma 4.8 we have that $p_\alpha(U)$ are open for each $\alpha \in \Sigma$, so $h(\alpha) \in \mathrm{Int}\, W_\alpha$ and $h \in \prod_{\alpha\in\Sigma} \mathrm{Int}\, W_\alpha$. Conversely: if $h \in \prod_{\alpha\in\Sigma} \mathrm{Int}\, W_\alpha$ then $h(\alpha) \in \mathrm{Int}\, W_\alpha$ for each $\alpha \in \Sigma$. Hence there exists an open set $U_\alpha \subseteq C_\alpha$ such that $h(\alpha) \in \in U_\alpha \subseteq W_\alpha$ for each $\alpha \in \Sigma$ and $U_\alpha = W_\alpha$ for all except finite number of $\alpha \in \Sigma$. Consequently $h \in \prod_{\alpha\in\Sigma} U_\alpha \subseteq \prod_{\alpha\in\Sigma} W_\alpha$ and $\prod_{\alpha\in\Sigma} U_\alpha$ is open in the topology induced by the partial order in H. Hence $h \in U \subseteq \prod_{\alpha\in\Sigma} W_\alpha$ and $h \in \mathrm{Int} \prod_{\alpha\in\Sigma} W_\alpha$.

Lemma 4.10. If Φ is a finite subset of the set Σ and $h_\Phi \in H$ such that:

$$h_\Phi(\alpha) = \begin{cases} c_\alpha & \text{for } \alpha \in \Phi \\ 0 & \text{for } \alpha \notin \Phi \end{cases}$$

then: $h \prec h_\Phi \iff h(\alpha) \prec h_\Phi(\alpha)$ for every $\alpha \in \Sigma$ (the relation "$\prec$" is in the sense of the topology induced by the partial order).

Proof. $h \prec h_\Phi \iff h_\Phi \in \mathrm{Int}\{k : h \leqslant k\} \iff h_\Phi \in \in \mathrm{Int} \prod_{\alpha\in\Sigma} \{k(\alpha) : h(\alpha) \leqslant k(\alpha)\} = \prod_{\alpha\in\Sigma} \mathrm{Int}\{k(\alpha) : h(\alpha) < k(\alpha)\}$

(by Lemma 4.9). That way we see that $h \prec h_{\Phi}$ if and only if $h_{\Phi}(\alpha) \in \operatorname{Int}\{k(\alpha) : h(\alpha) \leqslant k(\alpha)\}$ for every $\alpha \in \Sigma \Leftrightarrow h(\alpha) \prec h_{\Phi}(\alpha)$ for every $\alpha \in \Sigma$.

P r o o f of theorem 4.6. Any element $h \in H$ can be written in the following form: $h = \bigvee_{\Phi \in Z} h_{\Phi}$, where Z is a family of all finite subsets of the set Σ and for each $\Phi \in Z$ there is

$$h_{\Phi}(\alpha) = \begin{cases} h(\alpha) & \text{for } \alpha \in \Phi \\ 0 & \text{for } \alpha \notin \Phi . \end{cases}$$

The set $\{h_{\Phi}\}_{\Phi \in Z}$ is directed. First we shall prove that the c-net H is continuous. Let $h = \bigvee_{\Phi \in Z} h_{\Phi} \in H$ and $P_h = \{k \in H : k \prec h\}$. We shall show that the set P_h is directed. If $k_1, k_2 \in P_h$ then $k_1 \prec h$, $k_2 \prec h$ and $k_1 \prec \bigvee_{\Phi \in Z} h_{\Phi}$, $k_2 \prec \bigvee_{\Phi \in Z} h_{\Phi}$. Consequently $\bigvee_{\Phi \in Z} h_{\Phi} \in \operatorname{Int}\{h : k_1 \leqslant h\}$ and $\bigvee_{\Phi \in Z} h_{\Phi} \in \operatorname{Int}\{h : k_2 \leqslant h\}$. Hence there exist $\Phi_1, \Phi_2 \in Z$ such that $k_1 \prec h_{\Phi_1}$, $k_2 \prec h_{\Phi_2}$. Then by Lemma 4.10

$$\begin{cases} k_1(\alpha) \prec h_{\Phi_1}(\alpha) & \text{for } \alpha \in \Phi_1 \\ k_1(\alpha) = 0 & \text{for } \alpha \notin \Phi_1 \end{cases}$$

$$\begin{cases} k_2(\alpha) \prec h_{\Phi_2}(\alpha) & \text{for } \alpha \in \Phi_2 \\ k_2(\alpha) = 0 & \text{for } \alpha \notin \Phi_2 . \end{cases}$$

If $\alpha \in \Phi_1 \cap \Phi_2$ then $k_1(\alpha) \prec h_{\Phi_1}(\alpha) = h(\alpha)$, $k_2(\alpha) \prec h_{\Phi_2}(\alpha) = h(\alpha)$ but the c-net C_{α} is continuous so there exists $c_{\alpha} \in C_{\alpha}$ such that

$$k_1(\alpha) \leqslant c_{\alpha} \prec h(\alpha) = h_{\Phi_1}(\alpha)$$

$$k_2(\alpha) \leqslant c_{\alpha} \prec h(\alpha) = h_{\Phi_2}(\alpha) .$$

Let now

$$k(\alpha) = \begin{cases} k_1(\alpha) & \text{for} \quad \alpha \in \Phi_1 - \Phi_2 \\ k_2(\alpha) & \text{for} \quad \alpha \in \Phi_2 - \Phi_1 \\ c_\alpha & \text{for} \quad \alpha \in \Phi_1 \cap \Phi_2 \\ 0 & \text{for} \quad \alpha \notin \Phi_1 \cup \Phi_2. \end{cases}$$

Obviously $k_1 \leqslant k$, $k_2 \leqslant k$ and $k \prec h$. Then $P_h = \{k : k \prec h\}$ is directed. Moreover $P_h = \{k : k \prec h\} = \{k \in H : k \prec h_\Phi$ for some $\Phi \in Z\} = \bigcup_{\Phi \in Z} \{k \in H : k \prec h_\Phi\} = \bigcup_{\Phi \in Z} P_{h\Phi}$, where $P_{h\Phi} = \{k \in H : k \prec h_\Phi\}$ is the directed set, for each $\Phi \in Z$; so

$$\left(\bigvee P_{h\Phi}\right)(\alpha) = \begin{cases} \bigvee_{k(\alpha) \prec h_\Phi(\alpha)} k(\alpha) & \text{for} \quad \alpha \in \Phi \\ 0 & \text{for} \quad \alpha \notin \Phi. \end{cases}$$

Consequently $\left(\bigvee P_{h\Phi}\right)(\alpha) = h_\Phi(\alpha)$ because the c-net C is continuous and $\bigvee P_{h\Phi} = h_\Phi$, for each $\Phi \in Z$. Thus

$$\bigvee P_h = \bigvee\left(\bigcup_{\Phi \in Z} P_{h\Phi}\right) = \bigvee_{\Phi \in Z}\left(\bigvee P_{h\Phi}\right) = \bigvee_{\Phi \in Z} h_\Phi = h.$$

In this way we have proved that the c-net H is continuous. Finally we shall prove that the product topology in the c-net H coincides with the topology induced by the partial order. Let U be an open set in the induced topology in H and let $h = \bigvee_{\Phi \in Z} h_\Phi \in U$. Thus, there exists $\Phi_0 \in Z$ such that $h_{\Phi_0} \in U$ and

$$h_{\Phi_0}(\alpha) = \begin{cases} h(\alpha) & \text{for} \quad \alpha \in \Phi_0 \\ 0 & \text{for} \quad \alpha \notin \Phi_0. \end{cases}$$

Since H is continuous, Lemma 4.4 implies existing $h' \in U$ such that $h' \prec h_{\Phi_0}$. Consequently $h_{\Phi_0} \in \operatorname{Int}\{k : h' \leqslant k\} \subseteq U$. Moreover

$$\operatorname{Int}\{k : h' \leqslant k\} = \operatorname{Int} \prod_{\alpha \in \Sigma} \{k(\alpha) : h'(\alpha) \leqslant k(\alpha)\} =$$

$$= \prod_{\alpha \in \Sigma} \operatorname{Int}\{k(\alpha) : h'(\alpha) \leqslant k(\alpha)\} = \prod_{\alpha \in \Sigma} W_\alpha,$$

where

$$W_\alpha = \begin{cases} \operatorname{Int}\{k(\alpha) : h'(\alpha) \leqslant k(\alpha)\} & \text{for } \alpha \in \Phi_0 \\ C_\alpha & \text{for } \alpha \notin \Phi_0 . \end{cases}$$

We have $h \in \prod_{\alpha \in \Sigma} W_\alpha \subseteq U$, because $h_{\Phi_0} \leqslant h$. Lemma 4.7 implies that $\prod_{\alpha \in \Sigma} W_\alpha$ are open in the induced topology, consequently they form a basis for the open sets in this topology. On the other hand it is known that $\prod_{\alpha \in \Sigma} W_\alpha$ form a basis for the product topology.

R e m a r k 4.11. If the direct product $\mathcal{H} = \langle \prod_{\alpha \in \Sigma} C_\alpha ; \leqslant, {}^\circ, 0, E \rangle$ of a family $\{C_\alpha\}_{\alpha \in \Sigma}$ of the c-nets is continuous c-net then C_α is continuous, for each $\alpha \in \Sigma$.

P r o o f . Let $c \in C_\alpha$. We shall show that a set $P_\alpha = \{c_\alpha : c_\alpha \prec c\}$ is directed and $c = \bigvee P_\alpha$. This results from the fact that $P = \{k : k \prec h\}$ is directed and $\bigvee P = h$ for $h \in H$ such that $h(\alpha) = c$ and $h(\alpha') = 0$ for $\alpha' \neq \alpha$.

D e f i n i t i o n 4.12. A T_0-space T is injective if and only if for any spaces X and Y such that X is a subspace of Y, every continuous function $f : X \longrightarrow T$ can be extended to a continuous function $\bar{f} : Y \longrightarrow T$. Following diagram illustrates the above definition

$$\begin{array}{ccc} X & \subseteq & Y \\ & f \searrow \quad \swarrow \bar{f} & \\ & T & \end{array} \qquad \bar{f}|_X = f.$$

E x a m p l e 4.13. The two-element Boolean algebra $A = \{0, e\}$ $(0 \leqslant e)$ is a c-net if we mean the operation of the greatest lower bound as a composition. In this c-net the partial order induces the T_0-topology. The topological space defined in this way is injective. Note, that the c-net A^{Σ} which is a direct power of the c-net A is an injective space in the product topology (D.Scott [4] th.1.3). Finally, since the c-net A is a continuous lattice, the product topology in A^{Σ} coincides with the topology induced by the partial order (D.Scott [4] th.2.8, th.2.9).

5. Topological retract of a c-net

Let C be a c-net with the topology induced by the partial order. Following theorem gives a sufficient condition for the topological retract of the c-net C to be a c-net too.

T h e o r e m 5.1. If a mapping $j : C \to C$ is a retraction such that (*) $j(x) \circ y = x \circ j(y)$ for each $x, y \in C$ then the topological retract $j(C) = T$ is a c-net with respect to the restrictions of the partial order, the least upper bound of the directed sets and the restriction of the composition in C. Moreover the subspace topology coincides with the topology induced by the partial order in the retract T.

P r o o f . The partial order "$\leqslant$" in the set T is the restriction of the partial order "$\leqslant$" because the mapping j is monotonic. Since the mapping j is continuous and $j(\bigvee D) = \bigvee\{j(d) : d \in D\} = \bigvee D$ in T for any directed set $D \subseteq T$, the set T is closed with respect to the least upper bounds of the directed sets. If $x, y \in T$ then $j(x \circ y) = j(x \circ y) \circ e = (x \circ y) \circ j(e) = (x \circ j(y)) \circ e = x \circ y$. Thus the set T is closed with respect to the composition. The element 0 is the least element in T because $j(0) = j(0) \circ e = 0 \circ j(e) = 0$ and the mapping j is monotonic. The element $e_1 = j(e)$ is unit in T because $e_1 \circ x = j(e) \circ x = e \circ j(x) = e \circ x = x$ and similarly $x \circ e_1 = x$ for each $x \in T$. The set T is a semigroup with zero 0 and unit $e_1 = j(e)$ and "$\circ$" is the composition just like in the c-net C. Moreover, if $x \in T$ and $D \subseteq T$ is a directed set then $x \circ \bigvee D = \bigvee \{x \circ d : d \in D\}$ and $\bigvee D \circ x = \bigvee \{d \circ x : d \in D\}$. In this way we see that the set T is the c-net.

Next we'll show that the subspace topology coincides with the topology induced by the partial order in T. Since the partial order in T is the same as in C then, of course, every open set in the subspace topology is open in the topology induced by the partial order in the space T too. On the other hand: if a set U is open in $T = \{x : x = j(x)\}$ then we'll show that $U = A \cap T$, where $A = \{c \in C : j(c) \in U\}$ is open in C. First we'll verify the conditions of the definition 3.1. for the set A. (O_1) If $x \in A$ and $x \leqslant y$ then $j(x) \leqslant j(y)$ and $j(x) \in U$. The set U is open in T so $j(y) \in U$ and consequently $y \in A$. (O_2) If $\bigvee_{t \in T} c_t \in A$ for a directed set $\{c_t\}_{t \in T}$ then $j\left(\bigvee_{t \in T} c_t\right) \in U$ and $\bigvee_{t \in T} j(c_t) \in U$. Hence there exists $t_o \in T$ such that $j(c_{t_o}) \in U$ and $c_{t_o} \in A$. The set U is contained in the set A because $U \subseteq T$. It is evident that $U \subseteq A \cap T$. Conversely, if $x \in A$ and $x \in T$ then $j(x) \in U$ and $j(x) = x$. Hence $x \in U$ then $U = A \cap T$.

The contrary theorem (i.e. if a topological retract of a c-net is a c-net and the subspace topology coincides with the induced topology then the retraction - a mapping j satisfies the condition (*)) is not true. The following example shows it:

E x a m p l e 5.2. Let C be an arbitrary c-net $(|C| > 2)$ and $A = \{0, e\}$ be the c-net of the example 3.3. A is a topological subspace of the space C. The subspace topology coincides with the topology induced by the partial order in A. We define the mapping $j : C \longrightarrow A$ as follows:

$$j(0) = 0; \quad j(x) = e \quad \text{for} \quad x \neq 0.$$

Obviously $A = \{x \in C : x = j(x)\}$. The mapping j is continuous because for a directed set $\{a_t\}_{t \in T}$ in C we have:

$$j\left(\bigvee_{t\in T} a_t\right) = \begin{cases} e & \text{if there exists } t\in T \text{ such that } a_t \neq 0 \\ 0 & \text{if } a_t = 0 \text{ for each } t\in T \end{cases}$$

$$\bigvee_{t\in T} j(a_t) = \begin{cases} e & \text{if there exists } t\in T \text{ such that } a_t \neq 0 \\ 0 & \text{if } a_t = 0 \text{ for each } t\in T. \end{cases}$$

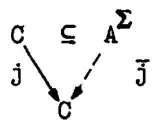
Of course A is the c-net with the same operations as in Example 3.3. The condition (*) of Theorem 5.1 is not satisfied, because for $y = e$ we have: $j(x) \circ e = x \circ j(e)$ that is $j(x) = x$ for each $x \in C$.

T h e o r e m 5.3 (P.S.Aleksandrow [2]). Every T_0-space can be embedded in an injective space, in fact, in a cartesian power of the 2-element Sierpinski Space.

C o r o l l a r y 5.4. Every c-net can be embedded as a topological subspace in the c-net A^{Σ} (the direct power of the 2-element c-net A).

C o r o l l a r y 5.5. If a c-net C is injective in the topology induced by the partial order then it is the topological retract of the c-net A^{Σ}.

P r o o f . If C is injective then it is (homeomorphism too) a subspace of the space A^{Σ}. But, since C is injective, the identity mapping on the subspace to itself can be extended to the whole space A^{Σ} resulting in the required retraction



$\bar{j}(c) = c$ for $c \in C$.

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