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Janusz Zyskowski

A CONVEX HULL AND EXTREME POINTS OF SOME FAMILY OF HOLOMORPHIC FUNCTIONS

1. In the present paper the notions of the set, family and class of functions will be regarded as equivalent. By $\mathbf{C}$, U and U_0 we shall denote the complex plane, the disc $\{z\in\mathbf{C}: |z|<1\}$ and the ring $\{z\in\mathbf{C}: 0<|z|<1\}$, respectively.

Denote by $H(D)$ the set of all functions holomorphic in the domain $D\subset\mathbf{C}$. V.Paatero, (see [6]), introduced and examined the family V_k, $k\geqslant 2$, of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots \tag{1}$$

holomorphic in U, with a nonvanishing derivative in U, and such that

$$\lim_{r\to 1^-}\int_0^{2\pi}\left|1+\operatorname{Re}\left\{re^{it}\frac{f''(re^{it})}{f'(re^{it})}\right\}\right|dt\leqslant K\pi.$$

Let C_k, $k\geqslant 2$, be a subset of $H(U)$ of functions f of the form (1) such that, for each of them, there exist a function φ belonging to V_k and a real constant $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, such that, for each $z\in U$,

$$\left|\arg\frac{f'(z)}{e^{i\alpha}\varphi'(z)}\right|<\frac{1}{2}\pi.$$

In particular, C_2 is the well-known family of close-to-convex functions introduced by M.Biernacki in a geometrical way, ([2]), and by W.Kaplan in an analytical way, ([5]). The family C_k was investigated earlier by J.Szynal and J.Waniurski ([8]). In this paper we shall examine its further properties.

2. Let $\mathfrak{M}(X)$ stand for the set of probability measures defined on Borel subsets of the space $X = \{x \in \mathbb{C} : |x| = 1\}$. The subset of the family H(U), consisting of the functions

$$(2) \qquad p(z) = \int_X (1 + xz)(1 - xz)^{-1}\, d\mu(x),$$

where $\mu \in \mathfrak{M}(X)$, will be denoted by P. It is the well-known family of Caratheodory functions.

The family C_k, $k \geqslant 2$, is a compact subset of the locally convex linear topological space H(U) with the topology of almost uniform convergence in U. Let G be any nonempty subset of H(U). By $\overline{co}\, G$ and E_G we shall denote, respectively, the closed convex hull of the set G and the set of extreme points of this set (e.g., [7], p.172).

We shall prove

T h e o r e m 1. If k is an arbitrary fixed real number, $k \geqslant 4$, then $\overline{co}\, C_k$ is a compact set, and

$$(3) \quad \overline{co}\, C_k = \left\{ \int_{X^3} K(z;x,y,v)\, d(\mu_1(x) \times \mu_2(y) \times \mu_3(v)) : \mu_i \in \mathfrak{M}(X) \right\},$$

$$(4) \quad E_{\overline{co}\, C_k} \subset \left\{ K(z;x,y,v) : |x| = |y| = |v| = 1 \right\} \subset C_k,$$

where the function $K(z;x,y,v)$, defined on the set $U \times X^3$, is the solution to the equation

$$(5)\quad \frac{d}{dz}K(z;x,y,v) = (1-xz)^{\frac{1}{2}k-1}(1-vz)(1-yz)^{-(\frac{1}{2}k+1)}$$

with the initial condition $K(0;x,y,v) = 0$.

P r o o f . Let f be any fixed function of the family C_k, $k \geqslant 4$. Then, according to the definition of the family C_k and formula (2), there exist: a function $\varphi \in V_k$, a constant $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and a measure $\gamma \in \mathfrak{M}(X)$, such that

$$(6)\qquad f'(z) = \varphi'(z)\int_X (1+cvz)(1-vz)^{-1}\, d\gamma(v), \quad c = e^{2i\alpha}.$$

In paper [3] the closed convex hull of the family V_k was determined in the case $k \geqslant 4$. From the result obtained there (Theorem 4.1) we have that, for some measure $\mu_1 \times \mu_2$, $\mu_i \in \mathfrak{M}(X)$,

$$\varphi'(z) = \int_{X^2} (1-xz)^{\frac{1}{2}k-1}(1-uz)^{-\left(\frac{1}{2}k+1\right)} d(\mu_1(x)\times\mu_2(u)).$$

From this and formula (6), by taking account of the Fubini theorem, we shall get that

$$(7)\quad f'(z) = \int_{X^3} (1-xz)^{\frac{1}{2}k-1}(1-uz)^{-\left(\frac{1}{2}k+1\right)}(1+cvz)(1-vz)^{-1}\cdot$$

$$d(\mu_1(x)\times\mu_2(u)\times\gamma(v))$$

for some probatility measures μ_1, μ_2, γ.

Since the function $g(z) := \log(1 - z)$, $\log 1 = 0$, maps U onto the convex domain $g(U)$, therefore

$$\frac{k+2}{k+4}\log(1 - uz) + \frac{2}{k+4}\log(1 - vz) \in g(U).$$

So, there exists a function $h \in H(U)$, $h(0) = 0$, $|h(z)| < 1$, (e.g., [7], p.2), for which

$$(1 - uz)^{-\frac{k+2}{k+4}}(1 - vz)^{-\frac{2}{k+4}} = (1 - h(z))^{-1},$$

which means that

(8) $$(1 - uz)^{-\frac{k+2}{k+4}}(1 - vz)^{-\frac{2}{k+4}} \prec (1 - z)^{-1},$$

where the symbol $\prec$ stands for the domain subordination ([7], p.2).

It is known, ([3]), that, if $F = \{f \in H(U): f \prec (1+dz)(1-z)^{-1}\}$ for a fixed $d \in \overline{U}$, then, for any fixed $a \geqslant 1$,

$$\overline{co}\,\{f^a : f \in F\} = \left\{\int_X \left[(1 + dyz)(1 - yz)^{-1}\right]^a d\sigma(y);\ \sigma \in \mathfrak{M}(X)\right\}.$$

Hence, by adopting $d = 0$ and $a = \frac{1}{2}k+2$, and from relation (8) it follows that there exists a measure $\sigma \in \mathfrak{M}(X)$ for which

$$(1 - uz)^{-\frac{1}{2}k+1}(1 - vz)^{-1} = \int_X (1 - yz)^{-\frac{1}{2}k+2}\, d\sigma(y).$$

Taking account of this equality and relation (7), we shall obtain the following form of the function f'

$$f'(z) = \int_{X^3} (1-xz)^{\frac{1}{2}k-1} (1-yz)^{-\left(\frac{1}{2}k+2\right)} (1+cvz)\cdot d(\mu_1(x)\times\sigma(y)\times\tau(v)),$$

where μ_1, σ, τ are some probability measures on X. Consequently, we have shown that f' belongs to the set

$$B' := \left\{ \int_{X^3} \frac{d}{dz} K(z;x,y,v)\ d(\mu_1(x)\times\mu_2(y)\times\mu_3(v));\quad \mu_i\in\mathfrak{M}(X) \right\},$$

the function $K(z;x,y,v)$ being the solution to equation (5).

The set $\mathfrak{M}(X)$ is a compact set in the weak-$*$-topology, (e.g., [9], p.93), and a convex one.
Taking this into consideration, in a simple way one can prove the convexity and compactness of the set B' and of the set

$$B := \left\{ \int_{X^3} K(z;x,y,v)\ d(\mu_1(x)\times\mu_2(y)\times\mu_3(v));\ \mu_i\in\mathfrak{M}(X) \right\}.$$

Of course, the function f belongs to the set B. Since f is any function of the family C_k, and the set B is convex and compact, therefore $\overline{co}\ C_k \subset B$.

The extreme points of the set B are generated by the measures μ_i, $i = 1,2,3$, cumulated at one point, (e.g., [4], Theorem 1). Consequently, each function g, being an extreme point of the set B, is of the form $g(z) = K(z;x,y,v)$ for some $|x| = |y| = |v| = 1$. Besides, the function $K(z;x,y,v)$ belongs to the family C_k for each value of x,y,v, $|x| = |y| = |v| = 1$. So we have shown the following relations of inclusion

(9) $$E_B \subset \{K(z;x,y,v) : |x| = |y| = |v| = 1\} \subset C_k.$$

For the set B, as convex and compact, the equality $B = \overline{co}\, B$ is true. It follows from the Krein-Millman theorem, (e.g., [7], p.172), that $B = \overline{co}\, B = \overline{co}\, E_B$, which, in view of relation (9), means that $B \subset \overline{co}\, C_k$. From this and the relation $\overline{co}\, C_k \subset B$ proved earlier we obtain the equality $\overline{co}\, C_k = B$, which concludes the proof of (3).

Inclusion (4) follows from (9). Equality does not hold in this case because the function $K(z;x,x,x) = z(1-x\dot{z})^{-1}$ is not an extreme point of the set $\overline{co}\, S^*$ contained in $\overline{co}\, C_k$, where S^* denotes the well-known family of starlike functions, (e.g., [7], p.13). Theorem 1 has thus been proved in full.

In the case $k = 2$ the theorem was proved in paper [4]. It remains open to determine $\overline{co}\, C_k$ for $2 < k < 4$.

3. We shall now make use of the information on the extreme points of the set $\overline{co}\, C_k$, $k \geq 4$, included in Theorem 1, in order to solve a certain extremal problem.

Define on the family C_k, $k \geq 4$, a functional

$$J(f) := \left\{ \int_{-\pi}^{\pi} \left| f'(re^{it}) \right|^p dt \right\}^{\frac{1}{p}},$$

where $r \in (0,1)$ and $p \in [1,\infty)$ are fixed. Since it is a convex and continuous functional defined on a compact set, therefore

(10) $$\max_{f \in C_k} J(f) = \max_{f \in \overline{co}\, C_k} J(f) = \max_{f \in E_{\overline{co}\, C_k}} J(f).$$

On the basis of property (10), we shall prove the following

Theorem 2. If $k \geqslant 4$, $p \in [1,\infty)$ and $r \in (0,1)$ are any fixed numbers, then

$$\max_{f \in C_k} J(f) = J(K(\cdot;-1,1,-1)),$$

where the function K is given by equation (5).

Proof. By remark (10) and Theorem 1, it is sufficient to confine our considerations only to the function $K(z;x,y,v)$, $|x| = |y| = |v| = 1$. We then have that

$$J(f) = \left\{ \int_{-\pi}^{\pi} u_1(t-t_1)\, u_2(t-t_2)\, u_3(t-t_3)\, dt \right\}^p ,$$

where

$$u_1(t) = \left|1 + re^{it}\right|^{p(\frac{1}{2}k-1)},$$

$$u_2(t) = \left|1 + re^{it}\right|^p,$$

$$u_3(t) = \left|1 - re^{it}\right|^{-p(\frac{1}{2}k+2)},$$

and t_1, t_2, t_3 are some real numbers such that $e^{-it_1} = -x$, $e^{-it_2} = -v$, $e^{-it_3} = y$. Making now use of the result obtained in paper ([1], Lemma 5.1), we shall get

$$J(f) \leqslant \left\{ \int_{-\pi}^{\pi} u_1(t)u_2(t)u_3(t)\, dt \right\}^p = \left\{ \int_{-\pi}^{\pi} \left| \frac{d}{dz} K(z;-1,1,-1) \right|^p_{z=re^{it}} dt \right\}^p =$$

$$= J(K(\cdot;-1,1,-1)),$$

which completes the proof of the theorem.

Taking account of the above Theorem 2 in the case $p=1$, one can obtain an exact estimation of the length of the curve being the image of the circle $|z| = r$ under the mapping through a function of the family C_k, $k \geq 4$.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ
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